CHAPTER 4
Curve Sketching

Review of Prerequisite Skills, pp. 162–163

1. a. $2y^2 + y - 3 = 0$

$(2y + 3)(y - 1) = 0$

$y = -\frac{3}{2}$ or $y = 1$

b. $x^2 - 5x + 3 = 17$

$x^2 - 5x - 14 = 0$

$(x - 7)(x + 2) = 0$

$x = 7$ or $x = -2$

c. $4x^2 + 20x + 25 = 0$

$(2x + 5)(2x + 5) = 0$

$x = -\frac{5}{2}$

d. $y^3 + 4y^2 + y - 6 = 0$

$y = 1$ is a zero, so $y - 1$ is a factor. After synthetic division, the polynomial factors to $(y - 1)(y^2 + 5y + 6)$.

So $(y - 1)(y + 3)(y + 2) = 0$.

$y = 1$ or $y = -3$ or $y = -2$

2. a. $3x + 9 < 2$

$3x < -7$

$x < -\frac{7}{3}$

b. $5(3 - x) \geq 3x - 1$

$15 - 5x \geq 3x - 1$

$16 \geq 8x$

$8x \leq 16$

$x \leq 2$

c. $t^2 - 2t < 3$

$t^2 - 2t - 3 < 0$

$(t - 3)(t + 1) < 0$

Consider $t = 3$ and $t = -1$.

t values	$t < -1$	$-1 < t < 3$	$t > 3$
$(t + 1)$	−	+	+
$(t - 3)$	−	−	+
$(t - 3)(t + 1)$	+	−	+

The solution is $-1 < t < 3$.

d. $x^2 + 3x - 4 > 0$

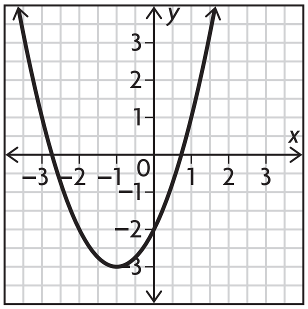
$(x + 4)(x - 1) > 0$

Consider $x = -4$ and $x = 1$.

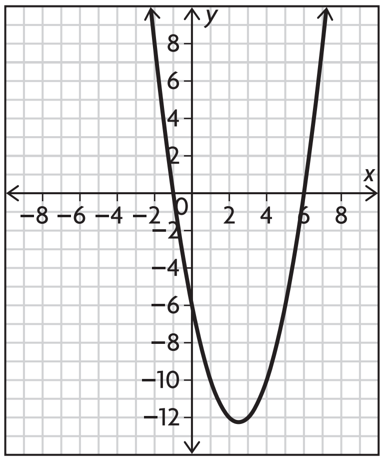
x values	$x < -4$	$-4 < x < 1$	$x > 1$
$(x + 4)$	−	+	+
$(x - 1)$	−	−	+
$(x + 4)(x - 1)$	+	−	+

The solution is $x < -4$ or $x > 1$.

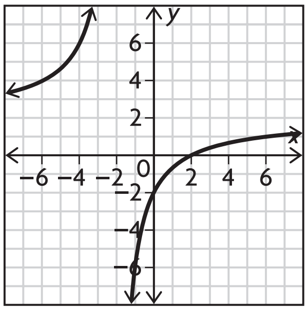
3. a.

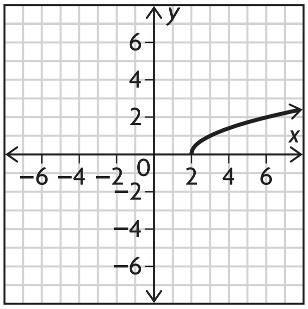


b.



c.



d. 

4. a. $\lim_{x\to 2^-} (x^2 - 4) = 2^2 - 4$
$= 0$

b. $\lim_{x\to 2} \frac{x^2 + 3x - 10}{x - 2}$
$= \lim_{x\to 2} \frac{(x + 5)(x - 2)}{x - 2}$
$= \lim_{x\to 2} (x + 5)$
$= 7$

c. $\lim_{x\to 3} \frac{x^3 - 27}{x - 3}$
$= \lim_{x\to 3} \frac{(x - 3)(x^2 + 3x + 9)}{x - 3}$
$= \lim_{x\to 3} (x^2 + 3x + 9)$
$= 3^2 + 3 \times 3 + 9$
$= 27$

d. $\lim_{x\to 4^+} \sqrt{2x + 1}$
$= \sqrt{2 \times 4 + 1}$
$= 3$

5. a. $f(x) = \frac{1}{4}x^4 + 2x^3 - \frac{1}{x}$
$= \frac{1}{4}x^4 + 2x^3 - x^{-1}$
$f'(x) = x^3 + 6x^2 + x^{-2}$

b. $f(x) = \frac{x + 1}{x^2 - 3}$
$f'(x) = \frac{(x^2 - 3)(1) - (x + 1)(2x)}{(x^2 - 3)^2}$
$= \frac{x^2 - 3 - 2x^2 - 2x}{(x^2 - 3)^2}$
$= \frac{-x^2 - 2x - 3}{(x^2 - 3)^2}$
$= -\frac{x^2 + 2x + 3}{(x^2 - 3)^2}$

c. $f(x) = (3x^2 - 6x)^2$
$f'(x) = 2(3x^2 - 6x)(6x - 6)$

d. $f(t) = \frac{2t}{\sqrt{t - 4}}$
$f'(t) = \frac{2\sqrt{t - 4} - \frac{2t}{2\sqrt{t - 4}}}{t - 4}$
$f'(t) = \frac{\frac{4(t - 4)}{2\sqrt{t - 4}} - \frac{2t}{2\sqrt{t - 4}}}{t - 4}$
$f'(t) = \frac{4(t - 4) - 2t}{2(t - 4)^{\frac{3}{2}}}$
$= \frac{2t - 16}{2(t - 4)^{\frac{3}{2}}}$
$= \frac{t - 8}{(t - 4)^{\frac{3}{2}}}$

6. a.
$$\begin{array}{r} x - 8 \\ x + 3\overline{)x^2 - 5x + 4} \\ \underline{x^2 + 3x} \\ -8x + 4 \\ \underline{-8x - 24} \\ 28 \end{array}$$
$(x^2 - 5x - 4) \div (x + 3) = x - 8 + \frac{28}{x + 3}$

b.
$$\begin{array}{r} x + 7 \\ x - 1\overline{)x^2 + 6x - 9} \\ \underline{x^2 - x} \\ 7x - 9 \\ \underline{7x - 7} \\ -2 \end{array}$$
$(x^2 - 6x - 9) \div (x - 1) = x + 7 - \frac{2}{x - 1}$

7. $f(x) = x^3 + 0.5x^2 - 2x + 3$
$f'(x) = 3x^2 + x - 2$
Let $f'(x) = 0$:
$3x^2 + x - 2 = 0$
$(3x - 2)(x + 1) = 0$
$x = \frac{2}{3}$ or $x = -1$
The points are $\left(\frac{2}{3}, 2.19\right)$ and $(-1, 4.5)$.

8. a. If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$.
b. If $f(x) = k$, where k is a constant, then $f'(x) = 0$.
c. If $k(x) = f(x)g(x)$, then
$k'(x) = f'(x)g(x) - f(x)g'(x)$
d. If $h(x) = \frac{f(x)}{g(x)}$, then $h'(x)$
$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$, $g(x) \neq 0$.

e. If f and g are functions that have derivatives, then the composite function $h(x) = f(g(x))$ has a derivative given by $h'(x) = f'(g(x))g'(x)$.

f. If u is a function of x, and n is a positive integer, then $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$

9. a. $\lim_{x\to\infty} 2x^2 - 3x + 4 = \infty$

$\lim_{x\to-\infty} 2x^2 - 3x + 4 = \infty$

b. $\lim_{x\to\infty} 2x^3 + 4x - 1 = \infty$

$\lim_{x\to-\infty} 2x^3 + 4x - 1 = -\infty$

c. $\lim_{x\to\infty} -5x^4 + 2x^3 - 6x^2 + 7x - 1 = -\infty$

$\lim_{x\to-\infty} -5x^4 + 2x^3 - 6x^2 + 7x - 1 = -\infty$

10. a. $\frac{1}{f(x)} = \frac{1}{2x}$

Let $2x = 0$

$x = 0$, so the graph has a vertical asymptote at $x = 0$.

b. $\frac{1}{f(x)} = \frac{1}{-x + 3}$

Let $-x + 3 = 0$

$x = 3$, so the graph has a vertical asymptote at $x = 3$.

c. $\frac{1}{f(x)} = \frac{1}{(x + 4)^2 + 1}$

Let $(x + 4)^2 + 1 = 0$

There is no solution, so the graph has no vertical asymptotes.

d. $\frac{1}{f(x)} = \frac{1}{(x + 3)^2}$

Let $(x + 3)^2 = 0$

$x = -3$, so the graph has a vertical asymptote at $x = -3$.

11. a. $\lim_{x\to\infty}\frac{5}{x + 1} = 0$, so the horizontal asymptote is $y = 0$.

b. $\lim_{x\to\infty}\frac{4x}{x - 2} = 4$, so the horizontal asymptote is $y = 4$.

c. $\lim_{x\to\infty}\frac{3x - 5}{6x - 3} = \frac{1}{2}$, so the horizontal asymptote is $y = \frac{1}{2}$.

d. $\lim_{x\to\infty}\frac{10x - 4}{5x} = 2$, so the horizontal asymptote is $y = 2$.

12. a. i. $y = \frac{5}{x + 1}$

To find the x-intercept, let $y = 0$.

$\frac{5}{x + 1} = 0$

There is no solution, so there is no x-intercept.

The y-intercept is $y = \frac{5}{0 + 1} = 5$.

ii. $y = \frac{4x}{x - 2}$

To find the x-intercept, let $y = 0$.

$\frac{4x}{x - 2} = 0$

$x = 0$

The y-intercept is $y = \frac{0}{0 - 2} = 0$.

iii. $y = \frac{3x - 5}{6x - 3}$

To find the x-intercept, let $y = 0$:

$\frac{3x - 5}{6x - 3} = 0$

Therefore, $3x - 5 = 0$

$x = \frac{5}{3}$

The y-intercept is $y = \frac{0 - 5}{0 - 3} = \frac{5}{3}$.

iv. $y = \frac{10x - 4}{5x}$

To find the x-intercept, let $y = 0$.

$\frac{10x - 4}{5x} = 0$

Therefore, $10x - 4 = 0$

$x = \frac{2}{5}$

The y-intercept is $y = \frac{0 - 4}{0}$, which is undefined, so there is no y-intercept.

b. i. $y = \frac{5}{x + 1}$

Domain: $\{x \in \mathbf{R} | x \neq -1\}$

Range: $\{y \in \mathbf{R} | y \neq 0\}$

ii. $y = \frac{4x}{x - 2}$

Domain: $\{x \in \mathbf{R} | x \neq 2\}$

Range: $\{y \in \mathbf{R} | y \neq 4\}$

iii. $y = \frac{3x - 5}{6x - 3}$

Domain: $\left\{x \in \mathbf{R} | x \neq \frac{1}{2}\right\}$

Range: $\left\{y \in \mathbf{R} | y \neq \frac{1}{2}\right\}$

iv. $y = \frac{10x - 4}{5x}$

Domain: $\{x \in \mathbf{R} | x \neq 0\}$

Range: $\{y \in \mathbf{R} | y \neq 2\}$

4.1 Increasing and Decreasing Functions, pp. 169–171

1. a. $f(x) = x^3 + 6x^2 + 1$

$f'(x) = 3x^2 + 12x$

Let $f'(x) = 0$: $3x(x + 4) = 0$

$x = 0$ or $x = -4$

The points are $(0, 1)$ and $(-4, 33)$.

b. $f(x) = \sqrt{x^2 + 4}$

$= (x^2 + 4)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(x^2 + 4)^{-\frac{1}{2}}(2x)$

$= \frac{x}{\sqrt{x^2 + 4}}$

Let $f'(x) = 0$:

$\frac{x}{\sqrt{x^2 + 4}} = 0$

So $x = 0$.

The point is $(0, 2)$.

c. $f(x) = (2x - 1)^2(x^2 - 9)$

$f'(x) = 2(2x - 1)(2)(x^2 - 9) + 2x(2x - 1)^2$

Let $f'(x) = 0$:

$2(2x - 1)(2(x^2 - 9) + x(2x - 1)) = 0$

$2(2x - 1)(4x^2 - x - 18) = 0$

$2(2x - 1)(4x - 9)(x + 2) = 0$

$x = \frac{1}{2}$ or $x = \frac{9}{4}$ or $x = -2$.

This points are $\left(\frac{1}{2}, 0\right)$, $(2.25, -48.2)$ and $(-2, -125)$.

d. $f(x) = \frac{5x}{x^2 + 1}$

$f'(x) = \frac{5(x^2 + 1) - 5x(2x)}{(x^2 + 1)^2} = \frac{5(1 - x^2)}{(x^2 + 1)^2}$

Let $f'(x) = 0$:

$\frac{5(1 - x^2)}{(x^2 + 1)^2} = 0$

Therefore, $5(1 - x^2) = 0$

$(1 - x)(1 + x) = 0$

$x = \pm 1$

The points are $\left(1, \frac{5}{2}\right)$ and $\left(-1, -\frac{5}{2}\right)$.

2. A function is increasing when $f'(x) > 0$ and is decreasing when $f'(x) < 0$.

3. a. i. $x < -1, x > 2$

ii. $-1 < x < 2$

iii. $(-1, 4), (2, -1)$

b. i. $-1 < x < 1$

ii. $x < -1, x > 1$

iii. $(-1, 2), (2, 4)$

c. i. $x < -2$

ii. $-2 < x < 2, 2 < x$

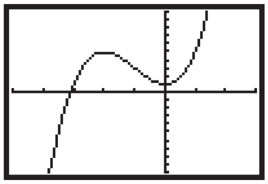
iii. none

d. i. $-1 < x < 2, 3 < x$

ii. $x < -1, 2 < x < 3$

iii. $(2, 3)$

4.



a. $f(x) = x^3 + 3x^2 + 1$

$f'(x) = 3x^2 + 6x$

Let $f'(x) = 0$

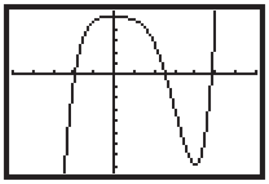
$3x^2 + 6x = 0$

$3x(x + 2) = 0$

$x = 0$ or $x = -2$

x	$x < -2$	-2	$-2 < x < 0$	0	$x > 0$
$f'(x)$	+	0	−	0	+
Graph	Increasing		Decreasing		Increasing

b.



$f(x) = x^5 - 5x^4 + 100$

$f'(x) = 5x^4 - 20x^3$

Let $f'(x) = 0$:

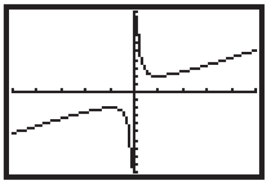
$5x^4 - 20x^3 = 0$

$5x^3(x - 4) = 0$

$x = 0$ or $x = 4$.

x	$x < 0$	0	$0 < x < 4$	4	$x > 4$
$f'(x)$	+	0	−	0	+
Graph	Increasing		Decreasing		Increasing

c.



$$f(x) = x + \frac{1}{x}$$

$$f'(x) = 1 - \frac{1}{x^2}$$

Let $f'(x) = 0$

$$1 - \frac{1}{x^2} = 0$$

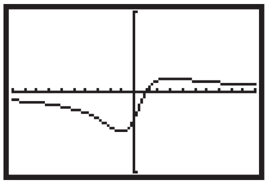
$$x^2 - 1 = 0$$

$$x = -1 \text{ or } x = 1$$

Also note that $f(x)$ is undefined for $x = 0$.

x	$x < -1$	-1	$-1 < x < 0$	0	$0 < x < 1$	1	$x > 1$
$f'(x)$	+	0	−	undefined	−	0	+
Graph	Increasing		Decreasing		Decreasing		Increasing

d.



$$f(x) = \frac{x - 1}{x^2 + 3}$$

$$f'(x) = \frac{x^2 + 3 - 2x(x - 1)}{(x^2 + 3)^2}$$

Let $f'(x) = 0$, therefore, $-x^2 + 2x + 3 = 0$.

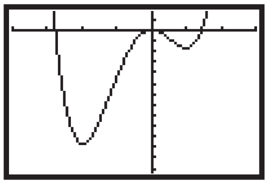
Or $x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3$ or $x = -1$

x	$x < -1$	-1	$-1 < x < 3$	3	$x > 3$
$f'(x)$	−	0	+	0	−
Graph	Decreasing		Increasing		Decreasing

e.



$y = 3x^4 + 4x^3 - 12x^2$

$y' = 12x^3 + 12x^2 - 24x$

Intervals of increasing:

$12x^3 + 12x^2 - 24x > 0$

$x(x^2 + x - 2) > 0$

$x(x - 1)(x + 2) > 0$

Intervals of decreasing:

$12x^3 + 12x^2 - 24x < 0$

$x(x^2 + x - 2) < 0$

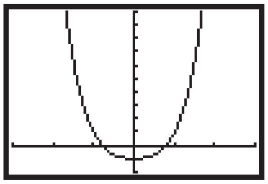
$x(x - 1)(x + 2) < 0$

	$x < -2$	$-2 < x < 0$	$0 < x < 1$	$x < 1$
x	−	−	+	+
$x - 1$	+	−	−	+
$x + 2$	−	+	+	+
y'	+	+	−	+

Intervals of increasing: $-2 < x < 0, x > 1$

Intervals of decreasing: $x < -2, 0 < x < 1$

f.



$y = x^4 + x^2 - 1$

$y' = 4x^3 + 2x$

Interval of increasing:

$4x^3 + 2x > 0$

$x(2x^2 + 1) > 0$

Interval of decreasing:

$4x^3 + 2x < 0$

$x(2x^2 + 1) < 0$

But $2x^2 + 1$ is always positive.

Interval of increasing: $x > 0$

Interval of decreasing: $x < 0$

5. $f'(x) = (x - 1)(x + 2)(x + 3)$

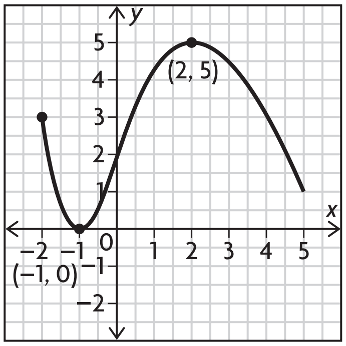
Let $f'(x) = 0$:

Then $(x - 1)(x + 2)(x + 3) = 0$

$x = 1$ or $x = -2$ or $x = -3$.

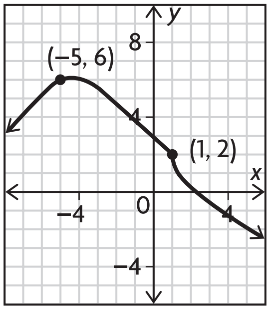
x	$x < -3$	-3	$-3 < x < -3$	-2	$-2 < x < 1$	1	$x > 1$
$f'(x)$	−	0	+	0	−	0	+
Graph	Decreasing		Increasing		Decreasing		Increasing

6.

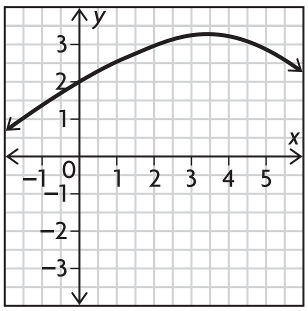


7. $f(x) = x^3 + ax^2 + bx + c$
$f'(x) = 3x^2 + 2ax + b$
Since $f(x)$ increases to $(-3, 18)$ and then decreases, $f'(3) = 0$.
Therefore, $27 - 6a + b = 0$ or $6a - b = 27$. (1)
Since $f(x)$ decreases to the point $(1, -14)$ and then increases $f'(1) = 0$.
Therefore, $3 + 2ab + b = 0$ or $2a + b = -3$. (2)
Add (1) to (2) $8a = 24$ and $a = 3$.
When $a = 3$, $b = 6 + b = -3$ or $b = -9$.
Since $(1, -14)$ is on the curve and $a = 3$, $b = -9$, then $-14 = 1 + 3 - 9 + c$
$c = -9$.
The function is $f(x) = x^3 + 3x^2 - 9x - 9$.

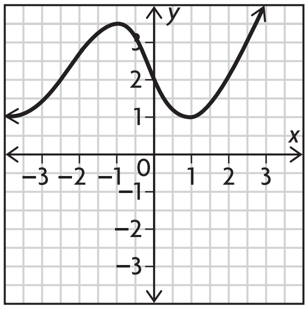
8.



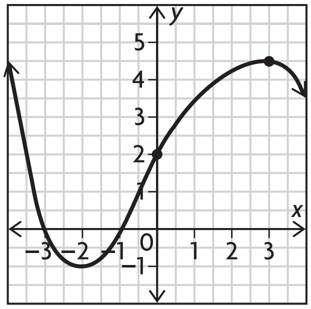
9. a. i. $x < 4$
ii. $x > 4$
iii. $x = 4$



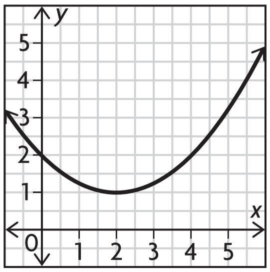
b. i. $x < -1, x > 1$
ii. $-1 < x < 1$
iii. $x = -1, x = 1$



c. i. $-2 < x < 3$
ii. $x < -2, x > 3$
iii. $x = -2, x = 3$



d. i. $x > 2$
ii. $x < 2$
iii. $x = 2$



10. $f(x) = ax^2 + bx + c$
$f'(x) = 2ax + b$

Let $f'(x) = 0$, then $x = \frac{-b}{2a}$.

If $x < \frac{-b}{2a}$, $f'(x) < 0$, therefore the function is decreasing.

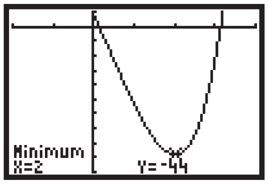
If $x > \frac{-b}{2a}$, $f'(x) > 0$, therefore the function is increasing.

11. $f(x) = x^4 - 32x + 4$
$f'(x) = 4x^3 - 32$
Let $f'(x) = 0$:
$4x^3 - 32 = 0$
$4x^3 = 32$

$x^3 = 8$
$x = 2$

x	$x < 2$	2	$x > 2$
$f(x)$	$-$	0	$+$
Graph	Dec.	Local Min	Inc

Therefore the function is decreasing for $x < 2$ and increasing for $x > 2$. The function has a local minimum at the point $(2, -44)$.



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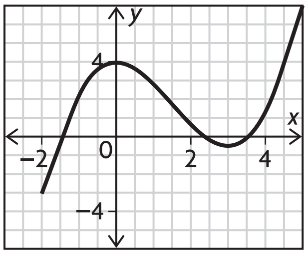
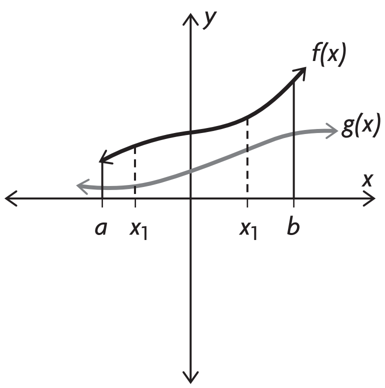


13. Let $y = f(x)$ and $u = g(x)$.
Let x_1 and x_2 be any two values in the interval $a \leq x \leq b$ so that $x_1 < x_2$.
Since $x_1 < x_2$, both functions are increasing:
$f(x_2) > f(x_1) \quad (1)$
$g(x_2) > g(x_1) \quad (2)$
$yu = f(x) \cdot g(x)$.
$(1) \times (2)$ results in $f(x_2) \cdot g(x_2) > f(x_1)g(x_1)$.
The function yu or $f(x) \cdot g(x)$ is strictly increasing.



14. Let x_1, x_2 be in the interval $a \leq x \leq b$, such that $x_1 < x_2$. Therefore, $f(x_2) > f(x_1)$, and $g(x_2) > g(x_1)$. In this case, $f(x_1), f(x_2), g(x_1)$, and $g(x_2) < 0$. Multiplying an inequality by a negative will reverse its sign.
Therefore, $f(x_2) \cdot g(x_2) < f(x_1) \cdot g(x_1)$.
But LS > 0 and RS > 0.
Therefore, the function fg is strictly decreasing.

4.2 Critical Points, Relative Maxima, and Relative Minima, pp. 178–180

1. Finding the critical points means determining the points on the graph of the function for which the derivative of the function at the x-coordinate is 0.

2. a. Take the derivative of the function. Set the derivative equal to 0. Solve for x. Evaluate the original function for the values of x. The (x, y) pairs are the critical points.

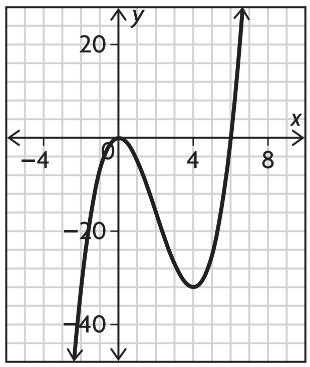
b. $y = x^3 - 6x^2$

$$\frac{dy}{dx} = 3x^2 - 12x$$
$$= 3x(x - 4)$$

Let $\frac{dy}{dx} = 0$.

$3x(x - 4) = 0$
$x = 0, 4$

The critical points are $(0, 0)$ and $(4, -32)$.



3. a. $y = x^4 - 8x^2$

$$\frac{dy}{dx} = 4x^3 - 16x = 4x(x^2 - 4)$$
$$= 4x(x + 2)(x - 2)$$

Let $\frac{dy}{dx} = 0$

$4x(x + 2)(x - 2) = 0$
$x = 0, \pm 2.$

The critical points are $(0, 0)$, $(-2, 16)$, and $(2, -16)$.

x	$x < -2$	-2	$-2 < x < 0$	0	$0 < x < 2$	2	$x < 2$
$\frac{dy}{dx}$	$-$	0	$+$	0	$-$	0	$+$
Graph	Dec.	Local Min	Inc.	Local Max	Dec.	Local Min	Inc.

Local minima at $(-2, -16)$ and $(2, -16)$
Local maximum at $(0, 0)$

b. $f(x) = \dfrac{2x}{x^2 + 9}$

$$f'(x) = \frac{2(x^2 + 9) - 2x(2x)}{(x^2 + 9)^2}$$
$$= \frac{18 - 2x^2}{(x^2 + 9)^2}$$

Let $f'(x) = 0$
Therefore, $18 - 2x^2 = 0$
$x^2 = 9$
$x = \pm 3.$

x	$x < -3$	-3	$-3 < x < 3$	3	$x > 3$
$f'(x)$	$-$	0	+	0	$-$
Graph	Decreasing	Local Min	Increasing	Local Max	Decreasing

Local minimum at $(-3, -0.3)$ and local maximum at $(3, 0.3)$.

c. $y = x^3 + 3x^2 + 1$

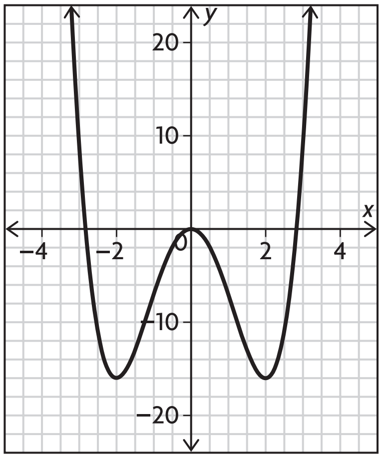
$$\frac{dy}{dx} = 3x^2 + 6x = 3x(x + 2)$$

Let $\dfrac{dy}{dx} = 0$
$3x(x + 2) = 0$
$x = 0, -2$
The critical points are $(0, 1)$ and $(-2, 5)$.

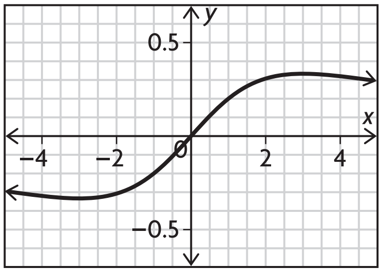
x	$x < -2$	-2	$-2 < x < 0$	0	$x < 0$
$\frac{dy}{dx}$	+	0	$-$	0	+
Graph	Inc.	Local Min		Local Max	Inc.

Local maximum at $(-2, 5)$
Local minimum at $(0, 1)$

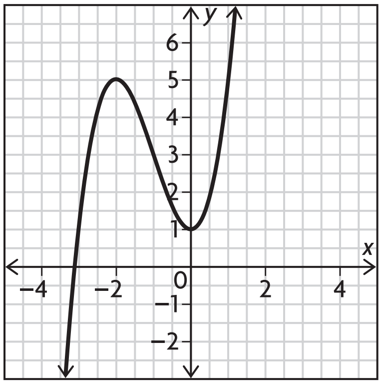
4. a. $y = x^4 - 8x^2$
To find the x-intercepts, let $y = 0$.
$x^4 - 8x^2 = 0$
$x^2(x^2 - 8) = 0$
$x = 0, \pm\sqrt{8}$
To find the y-intercepts, let $x = 0$.
$y = 0$



b. $f(x) = \dfrac{2x}{x^2 + 9}$
To find the x-intercepts, let $y = 0$.
$\dfrac{2x}{x^2 + 9} = 0$
Therefore, $2x = 0$
$x = 0$
To find the y-intercepts, let $x = 0$.
$y = \dfrac{0}{9} = 0$



c. $y = x^3 + 3x^2 + 1$
To find the x-intercepts, let $y = 0$.
$0 = x^3 + 3x^2 + 1$
The x-intercept cannot be easily obtained algebraically. Since the function has a local maximum when $x = -2$, it must have an x-intercept prior to this x-value. Since $f(-3) = 1$ and $f(-4) = -15$, an estimate for the x-intercept is about -3.1.
To find the y-intercepts, let $x = 0$.
$y = 1$



5. a. $h(x) = -6x^3 + 18x^2 + 3$

$h'(x) = -18x^2 + 36x$

Let $h'(x) = 0$:

$-18x^2 + 36x = 0$

$18x(2 - x) = 0$

$x = 0$ or $x = 2$

The critical points are (0, 3) and (2, 27).

Local minimum at (0, 3)

Local maximum at (2, 27)

Since the derivative is 0 at both points, the tangent is parallel to the horizontal axis for both.

b. $g(t) = t^5 + t^3$

$g'(t) = 5t^4 + 3t^2$

Let $g'(t) = 0$:

$5t^4 + 3t^2 = 0$

$t^2(5t^2 + 3) = 0$

$t = 0$

x	$x < 0$	0	$0 < x < 2$	0	$x > 2$
$\frac{dy}{dx}$	+	0	−	0	+
Graph	Inc.	Local Min	Dec.	Local Max	Inc.

The critical point is (0, 0).

t	$t < 0$	0	$t > 0$
$g'(x)$	+	0	+
Graph	Inc.	Local Min	Inc.

(0, 0) is neither a maximum nor a minimum

Since the derivative at (0, 0) is 0, the tangent is parallel to the horizontal axis there.

c. $y = (x - 5)^{\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{1}{3}(x - 5)^{\frac{-2}{3}}$$

$$= \frac{1}{3(x - 5)^{\frac{2}{3}}}$$

$$\frac{dy}{dx} \neq 0$$

The critical point is at (5, 0), but is neither a maximum or minimum. The tangent is not parallel to the x-axis.

d. $f(x) = (x^2 - 1)^{\frac{1}{3}}$

$$f'(x) = \frac{1}{3}(x^2 - 1)^{-\frac{2}{3}}(2x)$$

Let $f'(x) = 0$:

$$\frac{1}{3}(x^2 - 1)^{-\frac{2}{3}}(2x) = 0$$

$$x = 0$$

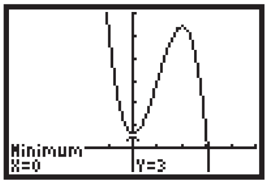
There is a critical point at $(0, -1)$. Since the derivative is undefined for $x = \pm 1$, (1, 0) and $(-1, 0)$ are also critical points.

x	$x < -1$	-1	$-1 < x < 0$	0	$0 < x < 1$	0	$x < 1$
$\frac{dy}{dx}$	−	DNE	−	0	+	DNE	+
Graph	Dec.		Dec.	Local Min	Inc.		Inc.

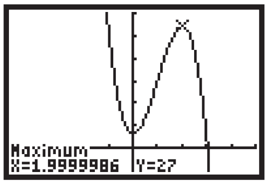
Local minimum at $(0, -1)$

The tangent is parallel to the horizontal axis at $(0, -1)$ because the derivative is 0 there. Since the derivative is undefined at $(-1, 0)$ and (1, 0), the tangent is not parallel to the horizontal axis at either point.

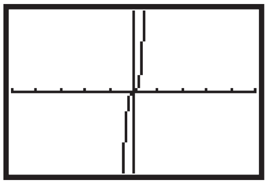
6. a.



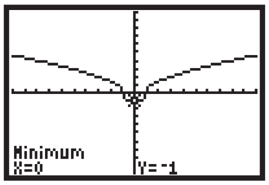
b.



c.



d.



7. a. $f(x) = -2x^2 + 8x + 13$

$f'(x) = -4x + 8$

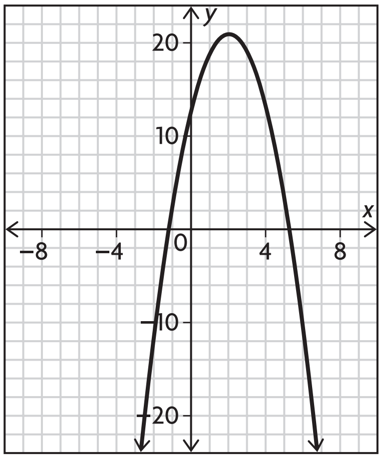
Let $f'(x) = 0$:

$-4x + 8 = 0$

$x = 2$

The critical point is (2, 21).
Local maximum at (2, 21)

x	$x < 2$	2	$x > 2$
$f'(x)$	+	0	−
Graph	Inc.	Local Max.	Dec.



b. $f(x) = \frac{1}{3}x^3 - 9x + 2$

$f'(x) = x^2 - 9$

Let $f'(x) = 0$:

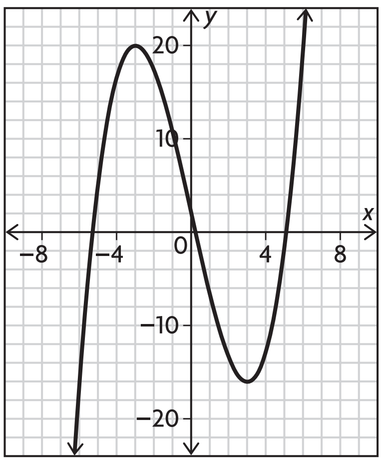
$x^2 - 9 = 0$

$x^2 = 9$

$x = \pm 3$

The critical points are $(-3, 20)$ and $(3, -16)$
Local maximum at $(-3, 20)$
Local minimum at $(3, -16)$

x	$x < -3$	-3	$-3 < x < 3$	3	$x > 3$
$\frac{dy}{dx}$	+	0	−	0	+
Graph	Inc.	Local Max	Dec.	Local Min	Inc.



c. $f(x) = 2x^3 + 9x^2 + 12x$

$f'(x) = 6x^2 + 18x + 12$

Let $f'(x) = 0$:

$6x^2 + 18x + 12 = 0$

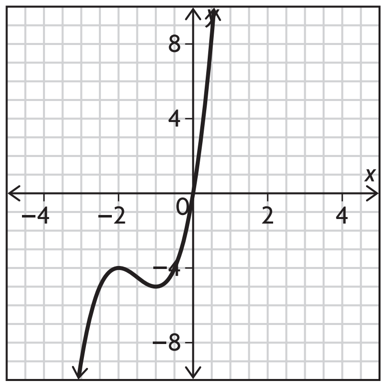
$6(x + 2)(x + 1) = 0$

$x = -2$ or $x = -1$

The critical points are $(-2, -4)$ and $(-1, -5)$.

x	$x < -2$	-2	$-2 < x < -1$	-1	$x > -1$
$\frac{dy}{dx}$	+	0	−	0	+
Graph	Inc.	Local Max	Dec.	Local Min	Inc.

Local maximum at $(-2, -4)$
Local minimum at $(-1, -5)$



d. $f(x) = -3x^3 - 5x$

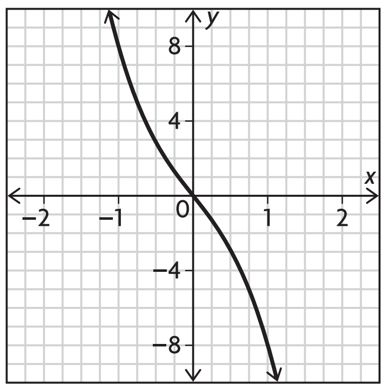
$f'(x) = -9x^2 - 5$

Let $f'(x) = 0$:

$-9x^2 - 5 = 0$

$x^2 = -\frac{5}{9}$

This equation has no solution, so there are no critical points.



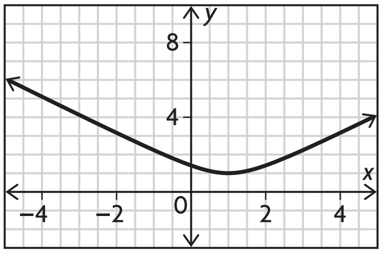
e. $f(x) = \sqrt{x^2 - 2x + 2}$

$f'(x) = \frac{2x - 2}{2\sqrt{x^2 - 2x + 2}} = \frac{x - 1}{\sqrt{x^2 - 2x + 2}}$

Let $f'(x) = 0$:
Therefore, $x - 1 = 0$
$x = 1$
The critical point is (1, 1).
$\sqrt{x^2 - 2x + 2}$ is never undefined or equal to zero, so (1, 1) is the only critical point.

x	$x < 1$	1	$x > 1$
$f'(x)$	$-$	0	$+$
Graph	Dec.	Local Min	Inc.

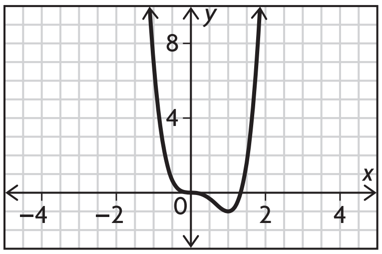
Local minimum at (1, 1)



f. $f(x) = 3x^4 - 4x^3$
$f'(x) = 12x^3 - 12x^2$
Let $f'(x) = 0$:
$12x^3 - 12x^2 = 0$
$12x^2(x - 1) = 0$
$x = 0$ or $x = 1$

x	$x < 0$	0	$0 < x < 1$	1	$x > 1$
$\frac{dy}{dx}$	$-$	0	$-$	0	$+$
Graph	Dec.		Dec.	Local Min	Inc.

There are critical points at (0, 0) and $(1, -1)$.
Neither local minimum nor local maximum at (0, 0)
Local minimum at $(1, -1)$

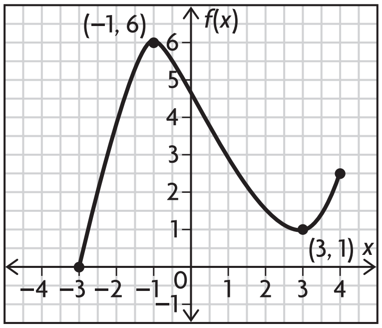


8. $f'(x) = (x + 1)(x - 2)(x + 6)$
Let $f'(x) = 0$:
$(x + 1)(x - 2)(x + 6) = 0$
$x = -6$ or $x = -1$ or $x = 2$
The critical numbers are -6, -1, and 2.

x	$x < -6$	-6	$-6 < x < -1$	-1	$-1 < x < 2$	2	$x < 2$
$\frac{dy}{dx}$	$-$	0	$+$	0	$-$	0	$+$
Graph	Dec.	Local Min	Inc.	Local Max	Dec.	Local Min	Inc.

Local minima at $x = -6$ and $x = 2$
Local maximum at $x = -1$

9.



10. $y = ax^2 + bx + c$
$\frac{dy}{dx} = 2ax + b$
Since a relative maximum occurs at $x = 3$, then $2ax + b = 0$ at $x = 3$. Or, $6a + b = 0$. Also, at $(0, 1)$, $1 = 0 + 0 + c$ or $c = 1$. Therefore, $y = ax^2 + bx + 1$. Since $(3, 12)$ lies on the curve,
$12 = 9a + 3b + 1$
$9a + 3b = 11$
$6a + b = 0.$
Since $b = -6a$,
Then $9a - 18a = 11$
or $a = -\frac{11}{9}$
$b = \frac{22}{3}.$

The equation is $y = -\frac{11}{9}x^2 + \frac{22}{3}x + 1.$

11. $f(x) = x^2 + px + q$
$f'(x) = 2x + p$
In order for 1 to be an extremum, $f'(1)$ must equal 0.
$2(1) + p = 0$
$p = -2$
To find q, substitute the known values for p and x into the original equation and set it equal to 5.

x	$x < 1$	1	$x > 1$
$f'(x)$	$-$	0	$+$
Graph	Dec.	Local Min	Inc.

$(1)^2 + (1)(-2) + q = 5$
$q = 6$
This extremum is a minimum value.

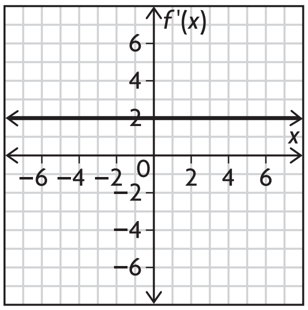
12. a. $f(x) = x^3 - kx$
$f'(x) = 3x^2 - k$
In order for f to have no critical numbers, $f'(x) = 0$ must have no solutions. Therefore, $3x^2 = k$ must have no solutions, so $k < 0$.

b. $f(x) = x^3 - kx$
$f'(x) = 3x^2 - k$
In order for f to have one critical numbers, $f'(x) = 0$ must have exactly one solution. Therefore, $3x^2 = k$ must have one solution, which occurs when $k = 0$.

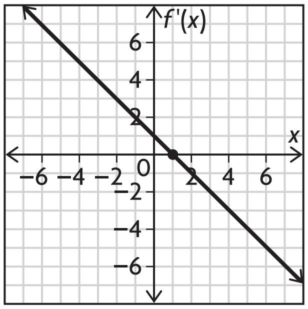
c. $f(x) = x^3 - kx$
$f'(x) = 3x^2 - k$
In order for f to have two critical numbers, $f'(x) = 0$ must have two solutions. Therefore, $3x^2 = k$ must have two solutions, which occurs when $k > 0$.

13. $g(x) = ax^3 + bx^2 + cx + d$
$g'(x) = 3ax^2 + 2bx + c$
Since there are local extrema at $x = 0$ and $x = 2$,
$0a + 0b + c = 0$ and $12a + 4b + c = 0$
Therefore, $c = 0$ and $12a + 4b = 0$
Going back to the original equation, we have the points (2, 4) and (0, 0). Substitute these values of x in the original function to get two more equations: $8a + 4b + 2c + d = 4$ and $d = 0$. We now know that $c = 0$ and $d = 0$. We are left with two equations to find a and b:
$12a + 4b = 0$
$8a + 4b = 4$
Subtract the second equation from the first to get $4a = -4$. Therefore $a = -1$, and $b = 3$.

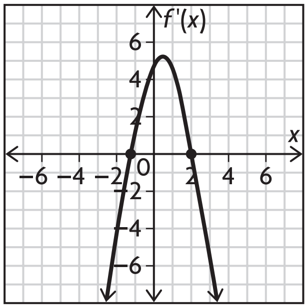
14. a.



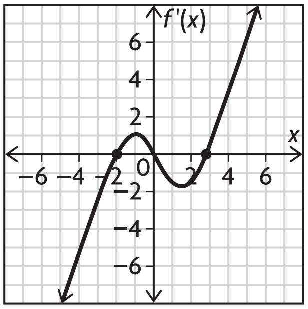
b.



c.



d.



15. $f(x) = 3x^4 + ax^3 + bx^2 + cx + d$

a. $f'(x) = 12x^3 + 3ax^2 + 2bx + c$
At $x = 0$, $f'(0) = 0$, then $f'(0) = 0 + 0 + 0 + c$ or $c = 0$.
At $x = -2$, $f'(-2) = 0$,
$-96 + 12a - 4b = 0.$ (1)
Since $(0, -9)$ lies on the curve,
$-9 = 0 + 0 + 0 + 0 + d$ or $d = -9$.
Since $(-2, -73)$ lies on the curve,
$-73 = 48 - 8a + 4b + 0 - 9$
$-8a + 4b = -112$
or $2a - b = 28$ (2)
Also, from (1): $3a - b = 24$
$2a - b = -28$
$a = -4$
$b = -36.$

The function is $f(x) = 3x^4 - 4x^3 - 36x^2 - 9$.

b. $f'(x) = 12x^3 - 12x^2 - 72x$

Let $f'(x) = 0$:

$x^3 - x^2 - 6x = 0$

$x(x - 3)(x + 2) = 0$.

Third point occurs at $x = 3$,

$f(3) = -198$.

c.

Local minimum is at $(-2, -73)$ and $(3, -198)$.

x	$x < -2$	-2	$-2 < x < 0$	0	$0 < x < 3$	3	$x > 3$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
Graph	Decreasing	Local Min	Increasing	Local Max	Decreasing	Local Min	Increasing

Local maximum is at $(0, -9)$.

16. a. $y = 4 - 3x^2 - x^4$

$\frac{dy}{dx} = -6x - 4x^3$

Let $\frac{dy}{dx} = 0$:

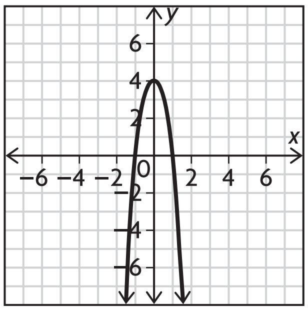
$-6x - 4x^3 = 0$

$-2x(2x^2 + 3) = 0$

$x = 0$ or $x^2 = -\frac{3}{2}$; inadmissible

x	$x < 0$	0	$x > 0$
$\frac{dy}{dx}$	$+$	0	$-$
Graph	Increasing	Local Max	Decreasing

Local maximum is at $(0, 4)$.



b. $y = 3x^5 - 5x^3 - 30x$

$\frac{dy}{dx} = 15x^4 - 15x^2 - 30$

Let $\frac{dy}{dx} = 0$:

$15x^4 - 15x^2 - 30 = 0$

$x^4 - x^2 - 2 = 0$

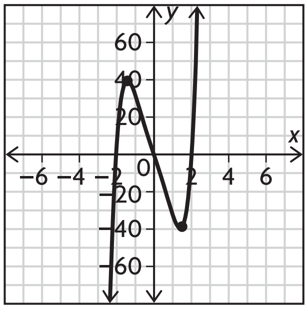
$(x^2 - 2)(x^2 + 1) = 0$

$x^2 = 2$ or $x^2 = -1$

$x = \pm\sqrt{2}$; inadmissible

At $x = 100$, $\frac{dy}{dx} > 0$.

Therefore, function is increasing into quadrant one, local minimum is at $(1.41, -39.6)$ and local maximum is at $(-1.41, 39.6)$.



17. $h(x) = \frac{f(x)}{g(x)}$

Since $f(x)$ has local maximum at $x = c$, then $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$.

Since $g(x)$ has a local minimum at $x = c$, then $g'(x) < 0$ for $x < c$ and $g'(x) > 0$ for $x > c$.

$h(x) = \frac{f(x)}{g(x)}$

$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$

If $x < c$, $f'(x) > 0$ and $g'(x) < 0$, then $h'(x) > 0$.

If $x > c$, $f'(x) < 0$ and $g'(x) > 0$, then $h'(x) < 0$.

Since for $x < c$, $h'(x) > 0$ and for $x > c$, $h'(x) < 0$.

Therefore, $h(x)$ has a local maximum at $x = c$.

4.3 Vertical and Horizontal Asymptotes, pp. 193–195

1. a. vertical asymptotes at $x = -2$ and $x = 2$; horizontal asymptote at $y = 1$

b. vertical asymptote at $x = 0$; horizontal asymptote at $y = 0$

2. $f(x) = \frac{g(x)}{h(x)}$

Conditions for a vertical asymptote: $h(x) = 0$ must have at least one solution s, and $\lim_{x\to\infty} f(x) = \infty$.

Conditions for a horizontal asymptote: $\lim_{x\to\infty} f(x) = k$, where $k \in \mathbf{R}$,

or $\lim_{x\to-\infty} f(x) = k$ where $k \in \mathbf{R}$.

Condition for an oblique asymptote is that the highest power of $g(x)$ must be one more than the highest power of $h(x)$.

3. a. $$\lim_{x\to\infty}\frac{2x+3}{x-1}=\lim_{x\to\infty}\frac{x\left(2+\frac{3}{x}\right)}{x\left(x-\frac{1}{x}\right)}$$
$$=\lim_{x\to\infty}\frac{1+\frac{3}{2x}}{1-\frac{1}{x}}$$
$$=\frac{\lim_{x\to\infty}\left(2+\frac{3}{x}\right)}{\lim_{x\to\infty}\left(1-\frac{1}{x}\right)}$$
$$=\frac{2+0}{1-0}$$
$$=2$$

Similarly, $\lim_{x\to-\infty}\frac{2x+3}{x-1}=2$.

b. $$\lim_{x\to\infty}\frac{5x^2-3}{x^2+2}=\lim_{x\to\infty}\frac{x^2\left(5-\frac{3}{x^2}\right)}{x^2\left(1+\frac{2}{x^2}\right)}$$
$$=\lim_{x\to\infty}\frac{5-\frac{3}{x^2}}{1+\frac{2}{x^2}}$$
$$=\frac{\lim_{x\to\infty}\left(5-\frac{3}{x^2}\right)}{\lim_{x\to\infty}\left(1+\frac{2}{x^2}\right)}$$
$$=\frac{5-0}{1+0}$$
$$=5$$

Similarly, $\lim_{x\to-\infty}\frac{5x^2-3}{x^2+2}=5$.

c. $$\lim_{x\to\infty}\frac{-5x^2+3x}{2x^2-5}=\lim_{x\to\infty}\frac{x^2\left(-5+\frac{3}{x}\right)}{x^2\left(2-\frac{5}{x^2}\right)}$$
$$=\lim_{x\to\infty}\frac{-5+\frac{3}{x}}{2-\frac{5}{x^2}}$$
$$=\frac{\lim_{x\to\infty}\left(-5+\frac{3}{x}\right)}{\lim_{x\to\infty}\left(2-\frac{5}{x^2}\right)}$$
$$=\frac{-5+0}{2-0}$$
$$=-\frac{5}{2}$$

Similarly, $\lim_{x\to-\infty}\frac{-5x^2+3x}{2x^2-5}=-\frac{5}{2}$.

d. $$\lim_{x\to\infty}\frac{2x^5-3x^2+5}{3x^4+5x-4}=\lim_{x\to\infty}\frac{x^5\left(2-\frac{3}{x^3}+\frac{5}{x^5}\right)}{x^4\left(3+\frac{5}{x^3}-\frac{4}{x^4}\right)}$$
$$=\lim_{x\to\infty}\frac{x\left(2-\frac{3}{x^3}+\frac{5}{x^5}\right)}{3+\frac{5}{x^3}-\frac{4}{x^4}}$$
$$=\frac{\lim_{x\to\infty}\left(x\left(2-\frac{3}{x^3}+\frac{5}{x^5}\right)\right)}{\lim_{x\to\infty}\left(3+\frac{5}{x^3}-\frac{4}{x^4}\right)}$$
$$=\frac{\lim_{x\to\infty}(x)\times\lim_{x\to\infty}\left(2-\frac{3}{x^3}+\frac{5}{x^5}\right)}{\lim_{x\to\infty}\left(3+\frac{5}{x^3}-\frac{4}{x^4}\right)}$$
$$=\lim_{x\to\infty}(x)\times\frac{2-0+0}{3+0-0}$$
$$=\infty$$

Similarly, $\lim_{x\to-\infty}\frac{2x^5-3x^2+5}{3x^4+5x-4}=\lim_{x\to\infty}(x)=-\infty$.

4. a. This function is discontinuous at $x=-5$. The numerator is non-zero at this point, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

x-values	x	$x+5$	y	$\lim_{x\to c} y$
$x\to-5^-$	<0	<0	>0	$+\infty$
$x\to-5^+$	<0	>0	<0	$-\infty$

b. This function is discontinuous at $x=2$. The numerator is non-zero at this point, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

x-values	$x+2$	$x-2$	$f(x)$	$\lim_{x\to c} f(x)$
$x\to2^-$	<0	<0	<0	$-\infty$
$x\to2^+$	>0	>0	>0	$+\infty$

c. This function is discontinuous at $t=3$. The numerator is non-zero at this point, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

t-values	1	$(t-3)^2$	s	$\lim_{t\to c} s$
$x\to3^-$	>0	>0	>0	$+\infty$
$x\to3^+$	>0	>0	>0	$+\infty$

d. This function is discontinuous at $x = 3$. However, the numerator also has value 0 there, since $3^2 - 3 - 6 = 0$, so this function has no vertical asymptotes.

e. The denominator of the function has value 0 when

$(x + 3)(x - 1) = 0$

$x = -3$ or $x = 1$. The numerator is non-zero at these points, so the function has vertical asymptotes there. The behaviour of the function near the asymptotes is:

x-values	6	*x* + 3	*x* − 1	*f*(*x*)	$\lim_{x\to c} f(x)$
$x \to -3^-$	> 0	< 0	< 0	> 0	$+\infty$
$x \to -3^+$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 1^-$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 1^+$	> 0	> 0	> 0	> 0	$+\infty$

f. This function is discontinuous when

$$x^2 - 1 = 0$$
$$(x + 1)(x - 1) = 0$$

$x = -1$ or $x = 1$. The numerator is non-zero at these points, so the function has vertical asymptotes there. The behaviour of the function near the asymptotes is:

x-values	x^2	*x* + 1	*x* − 1	*y*	$\lim_{x\to c} y$
$x \to -1^-$	> 0	< 0	< 0	> 0	$+\infty$
$x \to -1^+$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 1^-$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 1^+$	> 0	> 0	> 0	> 0	$+\infty$

5. a.
$$\lim_{x\to\infty} \frac{x}{x+4} = \lim_{x\to\infty} \frac{x}{x\left(1 + \frac{4}{x}\right)}$$
$$= \lim_{x\to\infty} \frac{1}{1 + \frac{4}{x}}$$
$$= \frac{\lim_{x\to\infty}(1)}{\lim_{x\to\infty}\left(1 + \frac{4}{x}\right)}$$
$$= \frac{1}{1 + 0}$$
$$= 1$$

Similarly, $\lim_{x\to-\infty} \frac{x}{x+4} = 1$, so $y = 1$ is a horizontal asymptote of the function.

At a point x, the difference between the function $y = \frac{x}{x+4}$ and its asymptote $y = 1$ is

$$\frac{x}{x+4} - 1 = \frac{x - (x + 4)}{x + 4}$$
$$= -\frac{4}{x+4}.$$

When x is large and positive, this difference is negative, which means that the curve approaches the asymptote from below. When x is large and negative, this difference is positive, which means that the curve approaches the asymptote from above.

b.
$$\lim_{x\to\infty} \frac{2x}{x^2 - 1} = \lim_{x\to\infty} \frac{2x}{x^2\left(1 - \frac{1}{x^2}\right)}$$
$$= \lim_{x\to\infty} \frac{2}{x\left(1 - \frac{1}{x^2}\right)}$$
$$= \frac{\lim_{x\to\infty}(2)}{\lim_{x\to\infty} x\left(1 - \frac{1}{x^2}\right)}$$
$$= \frac{\lim_{x\to\infty}(2)}{\lim_{x\to\infty}(x) \times \lim_{x\to\infty}\left(1 - \frac{1}{x^2}\right)}$$
$$= \lim_{x\to\infty} \frac{1}{x} \times \frac{1}{1 - 0}$$
$$= 0$$

Similarly, $\lim_{x\to-\infty} \frac{2x}{x^2 - 1} = 0$, so $y = 0$ is a horizontal asymptote of the function.

At a point x, the difference between the function $f(x) = \frac{2x}{x^2 - 1}$ and its asymptote $y = 0$ is $\frac{2x}{x^2 - 1}$. When x is large and positive, this difference is positive, which means that the curve approaches the asymptote from above. When x is large and negative, this difference is negative, which means that the curve approaches the asymptote from below.

c.
$$\lim_{x\to\infty} \frac{3t^2 + 4}{t^2 - 1} = \lim_{x\to\infty} \frac{t^2\left(3 + \frac{4}{t^2}\right)}{t^2\left(1 - \frac{1}{t^2}\right)}$$
$$= \lim_{x\to\infty} \frac{3 + \frac{4}{t^2}}{1 - \frac{1}{t^2}}$$
$$= \frac{\lim_{x\to\infty}\left(3 + \frac{4}{t^2}\right)}{\lim_{x\to\infty}\left(1 - \frac{1}{t^2}\right)}$$

$$= \frac{3 + 0}{1 - 0}$$
$$= 3$$

Similarly, $\lim_{x \to -\infty} \frac{3t^2 + 4}{t^2 - 1} = 3$, so $y = 3$ is a horizontal asymptote of the function.

At a point x, the difference between the function $g(t) = \frac{3t^2 + 4}{t^2 - 1}$ and its asymptote $y = 3$ is

$$\frac{3t^2 + 4}{t^2 - 1} - 3 = \frac{3t^2 + 4 - 3(t^2 - 1)}{t^2 - 1}$$
$$= \frac{7}{t^2 - 1}.$$

When x is large and positive, this difference is positive, which means that the curve approaches the asymptote from above. When x is large and negative, this difference is positive, which means that the curve approaches the asymptote from above.

d. $$\lim_{x \to \infty} \frac{3x^2 - 8x - 7}{x - 4} = \lim_{x \to \infty} \frac{x^2\left(3 - \frac{8}{x} - \frac{7}{x^2}\right)}{x\left(1 - \frac{4}{x}\right)}$$
$$= \lim_{x \to \infty} \frac{x\left(3 - \frac{8}{x} - \frac{7}{x^2}\right)}{1 - \frac{4}{x}}$$
$$= \frac{\lim_{x \to \infty}\left(x\left(3 - \frac{8}{x} - \frac{7}{x^2}\right)\right)}{\lim_{x \to \infty}\left(1 - \frac{4}{x}\right)}$$
$$= \frac{\lim_{x \to \infty}(x) \times \lim_{x \to \infty}\left(3 - \frac{8}{x} - \frac{7}{x^2}\right)}{\lim_{x \to \infty}\left(1 - \frac{4}{x}\right)}$$
$$= \lim_{x \to \infty}(x) \times \frac{3 - 0 - 0}{1 - 0}$$
$$= \infty$$

Similarly, $\lim_{x \to \infty} \frac{3x^2 - 8x - 7}{x - 4} = \lim_{x \to \infty}(x) = -\infty$, so this function has no horizontal asymptotes.

6. a. This function is discontinuous at $x = -5$. Since the numerator is not equal to 0 there, the function has a vertical asymptote at this point. The behaviour of the function near the asymptote is:

***x*-values**	***x* − 3**	***x* + 5**	***y***	$\lim_{x \to c} y$
$x \to -5^-$	< 0	< 0	> 0	$+\infty$
$x \to -5^+$	< 0	> 0	< 0	$-\infty$

To check for a horizontal asymptote:

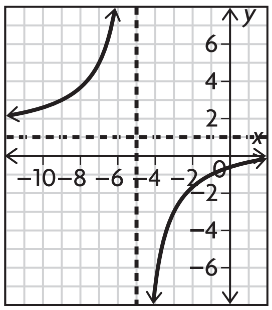
$$\lim_{x \to \infty} \frac{x - 3}{x + 5} = \lim_{x \to \infty} \frac{x\left(1 - \frac{3}{x}\right)}{x\left(1 + \frac{5}{x}\right)}$$
$$= \lim_{x \to \infty} \frac{1 - \frac{3}{x}}{1 + \frac{5}{x}}$$
$$= \frac{\lim_{x \to \infty}\left(1 - \frac{3}{x}\right)}{\lim_{x \to \infty}\left(1 + \frac{5}{x}\right)}$$
$$= \frac{1 - 0}{1 + 0}$$
$$= 1$$

Similarly, $\lim_{x \to -\infty} \frac{x - 3}{x + 5} = 1$, so $y = 1$ is a horizontal asymptote of the function.

At a point x, the difference between the function $y = \frac{x - 3}{x + 5}$ and its asymptote $y = 1$ is

$$\frac{x - 3}{x + 5} - 1 = \frac{x - 3 - (x + 5)}{x + 5} = -\frac{8}{x + 5}.$$

When x is large and positive, this difference is negative, which means that the curve approaches the asymptote from below. When x is large and negative, this difference is positive, which means that the curve approaches the asymptote from above.



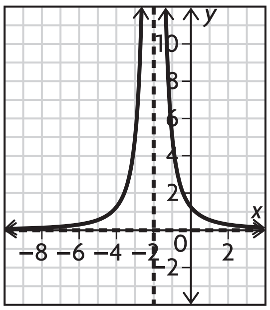
b. This function is discontinuous at $x = -2$. Since the numerator is non-zero there, the function has a vertical asymptote at this point. The behaviour of the function near the asymptote is:

***x*-values**	**5**	**$(x + 2)^2$**	***f*(*x*)**	$\lim_{x \to c} f(x)$
$x \to 2^-$	> 0	> 0	> 0	$+\infty$
$x \to 2^+$	> 0	> 0	> 0	$+\infty$

To check for a horizontal asymptote:

$$\lim_{x\to\infty}\frac{5}{(x+2)^2} = \lim_{x\to\infty}\frac{5}{x^2+4x+4}$$
$$= \lim_{x\to\infty}\frac{5}{x^2\left(1+\frac{4}{x}+\frac{4}{x^2}\right)}$$
$$= \frac{\lim_{x\to\infty}(5)}{\lim_{x\to\infty}\left(x^2\left(1+\frac{4}{x}+\frac{4}{x^2}\right)\right)}$$
$$= \frac{\lim_{x\to\infty}(5)}{\lim_{x\to\infty}(x^2)\times\lim_{x\to\infty}\left(1+\frac{4}{x}+\frac{4}{x^2}\right)}$$
$$= \lim_{x\to\infty}\frac{1}{x^2}\times\frac{5}{1+0+0}$$
$$= 0$$

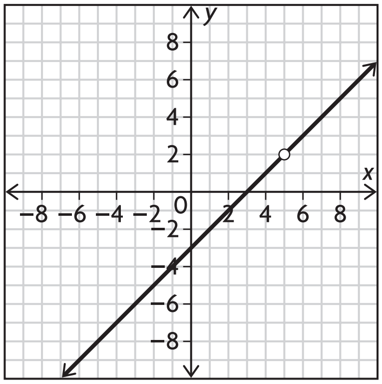
Similarly, $\lim_{x\to-\infty}\frac{5}{(x+2)^2} = 0$, so $y = 0$ is a horizontal asymptote of the function.
At a point x, the difference between the function $f(x) = \frac{5}{(x+2)^2}$ and its asymptote $y = 0$ is $\frac{5}{(x+2)^2}$. When x is large and positive, this difference is positive, which means that the curve approaches the asymptote from above. When x is large and negative, this difference is positive, which means that the curve approaches the asymptote from above.



c. This function is discontinuous at $t = 5$. However, the numerator is equal to zero there, since $5^2 - 2(5) - 15 = 0$, so this function has no vertical asymptote.
To check for an oblique asymptote:

$$\begin{array}{r} t - 3 \\ t-5\overline{)t^2 - 2t - 15} \\ t^2 - 5t \\ 0 + 3t - 15 \\ \underline{0 + 3t - 15} \\ 0 + 0 + 0 \end{array}$$

So $g(t)$ can be written in the form
$g(t) = t - 3$



d. This function is discontinuous when
$x^2 - 3x = 0$
$x(x - 3) = 0$
$x = 0$ or $x = 3$
The numerator is non-zero at these points, so the function has vertical asymptotes at both of them. The behaviour of the function near these asymptotes is:

x-values	$2 + x$	$3 - 2x$	x	$x - 3$	y	$\lim_{x\to c} y$
$x \to 0^-$	>0	>0	<0	<0	>0	$+\infty$
$x \to 0^+$	>0	>0	>0	<0	<0	$-\infty$
$x \to 3^-$	>0	<0	>0	<0	>0	$+\infty$
$x \to 3^+$	>0	<0	>0	>0	<0	$-\infty$

To check for horizontal asymptotes:

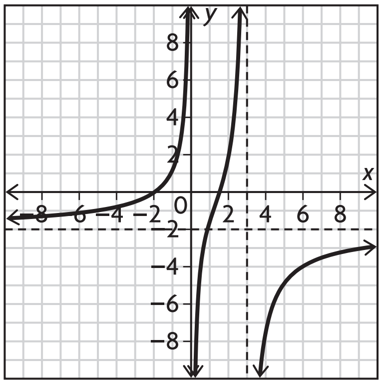
$$\lim_{x\to\infty}\frac{(2+x)(3-2x)}{x^2-3x} = \lim_{x\to\infty}\frac{-2x^2-x+6}{x^2-3x}$$
$$= \lim_{x\to\infty}\frac{x^2\left(-2-\frac{1}{x}+\frac{6}{x^2}\right)}{x^2\left(1-\frac{3}{x}\right)}$$
$$= \lim_{x\to\infty}\frac{-2-\frac{1}{x}+\frac{6}{x^2}}{1-\frac{3}{x}}$$
$$= \frac{\lim_{x\to\infty}\left(-2-\frac{1}{x}+\frac{6}{x^2}\right)}{\lim_{x\to\infty}\left(1-\frac{3}{x}\right)}$$
$$= \frac{-2-0+0}{1-0}$$
$$= -2$$

Similarly, $\lim_{x\to-\infty}\frac{(2+x)(3-2x)}{x^2-3x} = -2$, so $y = -2$ is a horizontal asymptote of the function.

At a point x, the difference between the function $y = \frac{-2x^2 - x + 6}{x^2 - 3x}$ and its asymptote $y = -2$ is

$$\frac{-2x^2 - x + 6}{x^2 - 3x} + 2 = \frac{-2x^2 - x + 6 + 2(x^2 - 3x)}{x^2 - 3x}$$
$$= \frac{-7x + 6}{x^2 - 3x}.$$

When x is large and positive, this difference is negative, which means that the curve approaches the asymptote from below. When x is large and negative, this difference is positive, which means that the curve approaches the asymptote from above.



7. a.

$$\begin{array}{r}
3x - 7 \\
x - 3 \overline{)\,3x^2 - 2x - 17} \\
\underline{3x^2 - 9x} \\
7x - 17 \\
\underline{7x - 21} \\
4
\end{array}$$

So $f(x)$ can be written in the form $f(x) = 3x - 7 + \frac{4}{x - 3}$. Since $\lim_{x\to\infty} \frac{4}{x - 3} = 0$ and $\lim_{x\to\infty} \frac{4}{x - 3} = 0$, the line $y = 3x - 7$ is an asymptote to the function $f(x)$.

b.

$$\begin{array}{r}
x + 3 \\
2x + 3 \overline{)\,2x^2 + 9x + 2} \\
\underline{2x^2 + 3x} \\
6x + 2 \\
\underline{6x + 9} \\
-7
\end{array}$$

So $f(x)$ can be written in the form $f(x) = x + 3 - \frac{7}{2x + 3}$. Since $\lim_{x\to\infty} \frac{7}{2x + 3} = 0$ and $\lim_{x\to-\infty} \frac{7}{2x + 3} = 0$, the line $y = x + 3$ is an asymptote to the function $f(x)$.

c.

$$\begin{array}{r}
x - 2 \\
x^2 + 2x \overline{)\,x^3 + 0x^2 + 0x - 1} \\
\underline{x^3 + 2x^2} \\
-2x^2 + 0x - 1 \\
\underline{-2x^2 - 4x} \\
4x - 1
\end{array}$$

So $f(x)$ can be written in the form $f(x) = x - 2 + \frac{4x - 1}{x^2 + 2x}$. Since

$$\lim_{x\to\infty} \frac{4x - 1}{x^2 + 2x} = \lim_{x\to\infty} \frac{x\left(4 - \frac{1}{x}\right)}{x^2\left(1 + \frac{2}{x}\right)}$$
$$= \lim_{x\to\infty} \frac{4 - \frac{1}{x}}{x\left(1 + \frac{2}{x}\right)}$$
$$= \frac{\lim_{x\to\infty}\left(4 - \frac{1}{x}\right)}{\lim_{x\to\infty}\left(x\left(1 + \frac{2}{x}\right)\right)}$$
$$= \frac{\lim_{x\to\infty}\left(4 - \frac{1}{x}\right)}{\lim_{x\to\infty}(x) \times \lim_{x\to\infty}\left(1 + \frac{2}{x}\right)}$$
$$= \lim_{x\to\infty}\left(\frac{1}{x}\right) \times \frac{4 - 0}{1 + 0}$$
$$= 0,$$

and similarly $\lim_{x\to-\infty} \frac{4x - 1}{x^2 + 2x} = 0$, the line $y = x - 2$ is an asymptote to the function $f(x)$.

d.

$$\begin{array}{r}
x + 3 \\
x^2 - 4x + 3 \overline{)\,x^3 - x^2 - 9x + 15} \\
\underline{x^3 - 4x^2 + 3x} \\
3x^2 - 12x + 15 \\
\underline{3x^2 - 12x + 9} \\
6
\end{array}$$

So $f(x)$ can be written in the form $f(x) = x + 3 + \frac{6}{x^2 - 4x + 3}$. Since $\lim_{x\to\infty} \frac{6}{x^2 - 4x + 3}$ and $\lim_{x\to-\infty} \frac{6}{x^2 - 4x + 3} = 0$, the line $y = x + 3$ is an asymptote to the function $f(x)$.

8. a. At a point x, the difference between the function $f(x) = f(x) = 3x - 7 + \frac{4}{x - 3}$ and its oblique asymptote $y = 3x - 7$ is $3x - 7 + \frac{4}{x - 3} - (3x - 7) = \frac{4}{x - 3}$. When x is large and positive, this difference is positive, which means that the curve approaches the asymptote from above. When x is large and negative, this difference is negative, which means that the curve approaches the asymptote from below.

b. At a point x, the difference between the function $f(x) = x + 3 - \frac{7}{2x + 3}$ and its oblique asymptote $y = x + 3$ is $x + 3 - \frac{7}{2x + 3} - (x + 3) = -\frac{7}{2x + 3}$. When x is large and positive, this difference is negative, which means that the curve approaches the asymptote from below. When x is large and negative, this difference is positive, which means that the curve approaches the asymptote from above.

9. a. This function is discontinuous at $x = -5$. The numerator is non-zero at this point, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

x-values	$3x - 1$	$x + 5$	$f(x)$	$\lim_{x \to c} f(x)$
$x \to -5^-$	< 0	< 0	> 0	$+\infty$
$x \to -5^+$	< 0	> 0	< 0	$-\infty$

To check for a horizontal asymptote:

$$\lim_{x \to \infty} \frac{3x - 1}{x + 5} = \lim_{x \to \infty} \frac{x\left(3 - \frac{1}{x}\right)}{x\left(1 + \frac{5}{x}\right)}$$

$$= \lim_{x \to \infty} \frac{3 - \frac{1}{x}}{1 + \frac{5}{x}}$$

$$= \frac{\lim_{x \to \infty}\left(3 - \frac{1}{x}\right)}{\lim_{x \to \infty}\left(1 + \frac{5}{x}\right)}$$

$$= \frac{3 - 0}{1 + 0}$$

$$= 1$$

Similarly, $\lim_{x \to \infty} \frac{3x - 1}{x + 5} = 3$, so $y = 3$ is a horizontal asymptote of the function.

b. This function is discontinuous at $x = 1$. The numerator is non-zero at this point, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

To check for a horizontal asymptote:

x-values	$x^2 + 3x - 2$	$(x - 1)^2$	$g(x)$	$\lim_{x \to c} g(x)$
$x \to 1^-$	> 0	> 0	> 0	$+\infty$
$x \to 1^+$	> 0	> 0	> 0	$+\infty$

$$\lim_{x \to \infty} \frac{x^2 + 3x - 2}{(x - 1)^2} = \lim_{x \to \infty} \frac{x^2 + 3x - 2}{x^2 - 2x + 1}$$

$$= \lim_{x \to \infty} \frac{x^2\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{x^2\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}$$

$$= \lim_{x \to \infty} \frac{1 + \frac{3}{x} - \frac{2}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}}$$

$$= \frac{\lim_{x \to \infty}\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{\lim_{x \to \infty}\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}$$

$$= \frac{1 + 0 - 0}{1 - 0 + 0}$$

$$= 1$$

Similarly, $\lim_{x \to \infty} \frac{x^2 + 3x - 2}{(x - 1)^2} = 1$, so $y = 1$ is a horizontal asymptote of the function.

c. This function is discontinuous when

$x^2 - 4 = 0$

$x^2 = 4$

$x = \pm 2.$

At $x = 2$ the numerator is 0, since $2^2 + 2 - 6 = 0$, so the function has no vertical asymptote there. At $x = -2$, however, the numerator is non-zero, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

x-values	$x^2 + x - 6$	$x^2 - 4$	$h(x)$	$\lim_{x \to c} h(x)$
$x \to -2^-$	< 0	> 0	< 0	$-\infty$
$x \to -2^+$	< 0	< 0	> 0	$+\infty$

To check for a horizontal asymptote:

$$\lim_{x \to \infty} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to \infty} \frac{x^2\left(1 + \frac{1}{x} - \frac{6}{x^2}\right)}{x^2\left(1 - \frac{4}{x^2}\right)}$$

$$= \lim_{x \to \infty} \frac{1 + \frac{1}{x} - \frac{6}{x^2}}{1 - \frac{4}{x^2}}$$

$$= \frac{\lim_{x \to \infty}\left(1 + \frac{1}{x} - \frac{6}{x^2}\right)}{\lim_{x \to \infty}\left(1 - \frac{4}{x^2}\right)}$$

$$= \frac{1 + 0 - 0}{1 - 0}$$
$$= 1$$

Similarly, $\lim_{x \to -\infty} \frac{x^2 + x - 6}{x^2 - 4} = 1$, so $y = 1$ is a horizontal asymptote of the function.

d. This function is discontinuous at $x = 2$. The numerator is non-zero at this point, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

x-values	$5x^2 - 3x + 2$	$x - 2$	$m(x)$	$\lim_{x \to c} m(x)$
$x \to 2^-$	> 0	< 0	< 0	$-\infty$
$x \to 2^+$	> 0	> 0	> 0	$+\infty$

To check for a horizontal asymptote:

$$\lim_{x \to \infty} \frac{x^2 + 3x - 2}{(x - 1)^2} = \lim_{x \to \infty} \frac{x^2 + 3x - 2}{x^2 - 2x + 1}$$
$$= \lim_{x \to \infty} \frac{x^2\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{x^2\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}$$
$$= \lim_{x \to \infty} \frac{1 + \frac{3}{x} - \frac{2}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}}$$
$$= \frac{\lim_{x \to \infty}\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{\lim_{x \to \infty}\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}$$
$$= \frac{1 + 0 - 0}{1 - 0 + 0}$$
$$= 1$$

Similarly, $\lim_{x \to \infty} \frac{x^2 + 3x - 2}{(x - 1)^2} = 1$, so $y = 1$ is a horizontal asymptote of the function.

10. a. $f(x) = \frac{3 - x}{2x + 5}$

Discontinuity is at $x = -2.5$.

$$\lim_{x \to -25^-} \frac{3 - x}{2x + 5} = -\infty$$

$$\lim_{x \to -25^+} \frac{3 - x}{2x + 5} = +\infty$$

Vertical asymptote is at $x = -2.5$.

Horizontal asymptote:

$$\lim_{x \to \infty} \frac{3 - x}{2x + 5} = -\frac{1}{2},$$

$$\lim_{x \to -\infty} \frac{3 - x}{2x + 5} = -\frac{1}{2}.$$

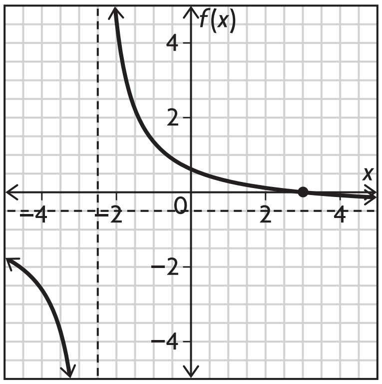
Horizontal asymptote is at $y = -\frac{1}{2}$.

$$f'(x) = \frac{-(2x + 5) - 2(3 - x)}{(2x + 5)^2} = \frac{-11}{(2x + 5)^2}$$

Since $f'(x) \neq 0$, there are no maximum or minimum points.

y-intercept, let $x = 0$, $y = \frac{3}{5} = 0.6$

x-intercept, let $y = 0$, $\frac{3 - x}{2x + 5} = 0$, $x = 3$



b. This function is a polynomial, so it is continuous for every real number. It has no horizontal, vertical, or oblique asymptotes.

The y-intercept can be found by letting $t = 0$, which gives $y = -10$.

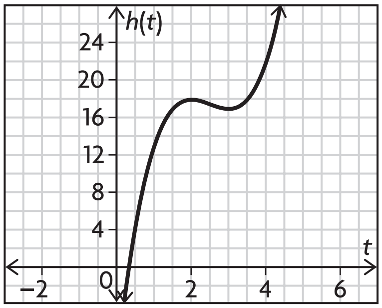
$h'(t) = 6t^2 - 30t + 36$

Set $h'(t) = 0$ and solve for t to determine the critical points.

$$6t^2 - 30t + 36 = 0$$
$$t^2 - 5t + 6 = 0$$
$$(t - 2)(t - 3) = 0$$
$$t = 2 \text{ or } t = 3$$

t	$t < 2$	$t = 2$	$2 < t < 3$	$t = 3$	$t > 3$
$h'(t)$	+	0	−	0	+
Graph	Inc.	Local Max	Dec.	Local Min	Inc.

The x-intercept cannot be easily obtained algebraically. Since the polynomial function has a local maximum when $x = 2$, it must have an x-intercept prior to this x-value. Since $f(0) = -10$ and $f(1) = 13$, an estimate for the x-intercept is about 0.3.



c. This function is discontinuous when

$x^2 + 4 = 0$

$x^2 = -4$

This equation has no real solutions, however, so the function is continuous everywhere.

To check for a horizontal asymptote:

$$\lim_{x\to\infty}\frac{20}{x^2+4} = \lim_{x\to\infty}\frac{20}{x^2\left(1+\frac{4}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty}(20)}{\lim_{x\to\infty}\left(x^2\left(1+\frac{4}{x^2}\right)\right)}$$

$$= \frac{\lim_{x\to\infty}(20)}{\lim_{x\to\infty}(x^2)\times\lim_{x\to\infty}\left(1+\frac{4}{x^2}\right)}$$

$$= \lim_{x\to\infty}\frac{1}{x^2}\times\frac{20}{1+0}$$

$$= 0$$

Similarly, $\lim_{x\to-\infty}\frac{20}{x^2+4} = 0$, so $y = 0$ is a horizontal asymptote of the function.

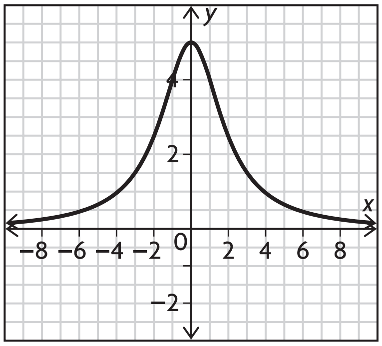
The y-intercept of this function can be found by letting $x = 0$, which gives $y = \frac{20}{0^2+4} = 5$. Since the numerator of this function is never 0, it has no x-intercept. The derivative can be found by rewriting the function as $y = 20(x^2+4)^{-1}$, then

$$y' = -20(x^2+4)^{-1}(2x)$$

$$= -\frac{40x}{(x^2+4)^2}$$

Letting $y' = 0$ shows that $x = 0$ is a critical point of the function.

x	$x < 0$	$x = 0$	$x > 0$
y′	+	0	−
Graph	Inc.	Local Max	Dec.



d. $s(t) = t + \frac{1}{t}$

Discontinuity is at $t = 0$.

$$\lim_{t\to0^+}\left(t+\frac{1}{t}\right) = +\infty$$

$$\lim_{t\to0^-}\left(t+\frac{1}{t}\right) = -\infty$$

Oblique asymptote is at $s(t) = t$.

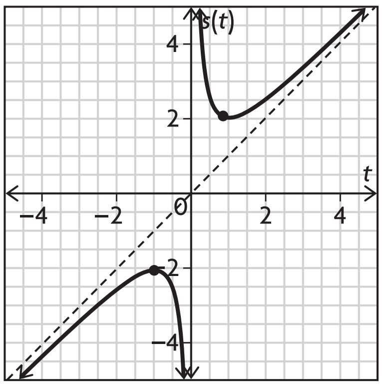
$$s'(t) = 1 - \frac{1}{t^2}$$

Let $s'(t) = 0$, $t^2 = 1$

$t = \pm1$.

Local maximum is at $(-1, -2)$ and local minimum is at $(1, 2)$.

t	$t < -1$	$t = -1$	$-1 < t < 0$	$0 < t < 1$	$t = 1$	$t > 1$
s′(t)	+	0	−	−	0	+
Graph	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing



e. $g(x) = \dfrac{2x^2 + 5x + 2}{x + 3}$

Discontinuity is at $x = -3$.

$\dfrac{2x^2 + 5x + 2}{x + 3} = 2x - 1 + \dfrac{5}{x + 3}$

Oblique asymptote is at $y = 2x - 1$.

$\lim\limits_{x \to -3^+} g(x) = +\infty, \ \lim\limits_{x \to -3^-} g(x) = -\infty$

$$g'(x) = \frac{(4x + 5)(x + 3) - (2x^2 + 5x + 2)}{(x + 3)^2}$$
$$= \frac{2x^2 + 12x + 13}{(x + 3)^2}$$

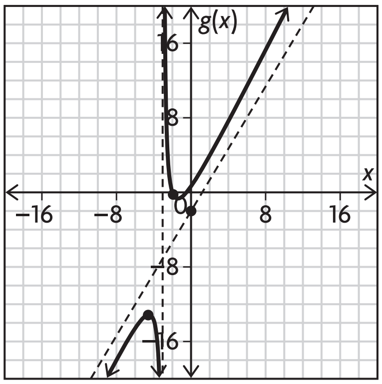
Let $g'(x) = 0$, therefore, $2x^2 + 12x + 13 = 0$:

$$x = \frac{-12 \pm \sqrt{144 - 104}}{4}$$

$x = -1.4$ or $x = -4.6$.

t	$x < -4.6$	-4.6	$-4.6 <$ $x < -3$	-3	$-3 <$ $x < -1.4$	$x = 1.4$	$x > -1.4$
$s'(t)$	+	0	−	Undefined	−	0	+
Graph	Increasing	Local Max	Decreasing	Vertical Asymptote	Decreasing	Local Min	Increasing

Local maximum is at $(-4.6, -10.9)$ and local minimum is at $(-1.4, -0.7)$.



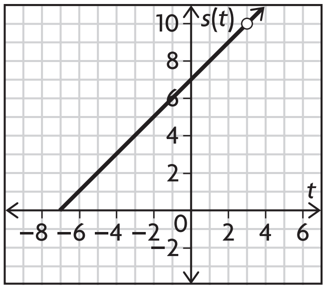
f. $s(t) = \dfrac{t^2 + 4t - 21}{t - 3}, t \geq -7$

$$= \frac{(t + 7)(t - 3)}{(t - 3)}$$

Discontinuity is at $t = 3$.

$$\lim_{x \to 3^+} \frac{(t + 7)(t - 3)}{(t - 3)} = \lim_{x \to 3^+} (t + 7)$$
$$= 10$$
$$\lim_{x \to 3^-} (t + 7) = 10$$

There is no vertical asymptote. The function is the straight line $s = t + 7, t \geq -7$.

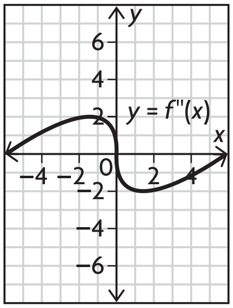


11. a. The horizontal asymptote occurs at $y = \frac{a}{c}$.

b. The vertical asymptote occurs when $cx + d = 0$ or $x = -\frac{d}{c}$.

12. a. Since f is defined for all values of x, f' and f'' are also defined for all values of x. f has a horizontal asymptote at $y = 0$. So f' and f'' will have horizontal asymptotes there. f has a local maximum at (0, 1) so f' will be 0 when $x = 0$. f has a point of inflection at $(-0.7, 0.6)$ and $(0.7, 0.6)$, so f'' will be 0 at $x = \pm 0.7$. At $x = 0.7$, f changes from concave up to concave down, so the sign of f'' changes from positive to negative. At $x = 0.7$, f changes from concave down to concave up, so the sign of f'' changes from negative to positive there. f is increasing for $x < 0$, so f' will be positive. f is decreasing for $x > 0$, so f' will be negative. The graph of f is concave up for $x < -0.7$ and $x > 0.7$, so f'' is positive for $x < -0.7$ and $x > 0.7$. The graph of f is concave down for $-0.7 < x < 0.7$, so f'' is negative for $-0.7 < x < 0.7$.

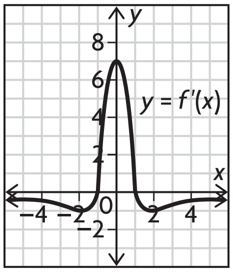
Also, since f'' is 0 at $x = \pm 0.7$, the graph of f' will have a local minimum or local maximum at these points. Since the sign of f'' changes from negative to positive at $x = 0.7$, it must be a local minimum point. Since the sign of f'' changes from positive to negative at $x = -0.7$, it must be a local maximum point.



b. Since f is defined for all values of x, f' and f'' are also defined for all values of x. f has a horizontal asymptote at $y = 0$ so f' and f'' will have a horizontal asymptote there. f has a local maximum at $(1, 3.5)$ so f' will be 0 when $x = 1$. f has a local minimum at $(-1, -3.5)$ so f' will be 0 when $x = -1$. f has a point of inflection at $(-1.7, -3)$, $(1.7, 3)$ and $(90, 0)$ so f'' will be 0 at $x = \pm 1.7$ and $x = 0$. At $x = 0$, f changes from concave up to concave down, so the sign of f'' changes from positive to negative.

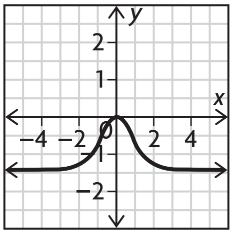
At $x = -1.7$, f changes from concave down to concave up, so the sign of f'' changes from negative to positive. At $x = 1.7$, f changes from concave down to concave up, so the sign of f'' changes from negative to positive. f is decreasing for $x < -1$ and $x > 1$, so f' will be negative. The graph of f is concave up for $-1.7 < x < 0$ and $x > 1.7$, so f'' is positive for $-1.7 < x < 0$ and $x > 1.7$. The graph of f is concave down for $x < -1.7$ and $0 < x < 1.7$, so f'' is negative for $x < -1.7$ and $0 < x < 1.7$.

Also, since f'' is 0 when $x = 0$ and $x = \pm 1.7$, the graph of f' will have a local maximum or minimum at these points. Since the sign of f'' changes from negative to positive at $x = -1.7$, f' has a local minimum at $x = -1.7$. Since the sign of f'' changes from positive to negative at $x = 0$, it must be a local maximum point. Since the sign of f'' changes from negative to positive at $x = 1.7$, it must be a local minimum point.

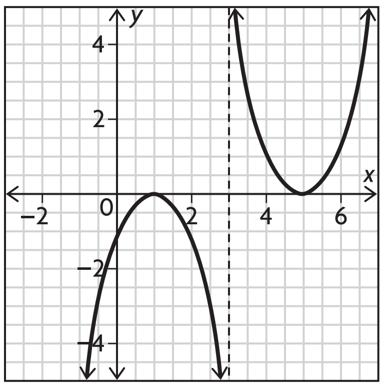


13. a. If $f(x)$ is increasing, then $f'(x) > 0$. From the graph of f', $f'(x) > 0$ for $x < 0$. If $f(x)$ is decreasing, then $f'(x) < 0$. From the graph of f', $f'(x) < 0$ for $x > 0$. At a stationary point, $f'(x) = 0$. From the graph, the zero of $f'(x)$ occurs at $(0, 0)$. At $x = 0$, The graph changes from positive to negative, so f has a local maximum there. If the graph of f is concave up, $f''(x)$ is positive. From the slope of f', the graph of f is concave up for $x < -0.6$ and $x > 0.6$. If the graph of f is concave down, $f''(x)$ is negative and concave down for $-0.6 < x < 0.6$. Graphs will vary slightly.

An example showing the shape of the curve is illustrated.



b. If $f(x)$ is increasing, then $f'(x) > 0$. From the graph of f', $f''(x) > 0$ for $x < 1$ and $x > 5$. If $f(x)$ is decreasing, then $f'(x) < 0$. From the graph of f', $f'(x) < 0$ for $1 < x < 3$ and $3 < x < 5$. At a stationary point, $f'(x) = 0$. From the graph, the zeros of $f'(x)$ occur at $x = 1$ and $x = 5$. At $x = 1$, the graph changes from positive to negative, so f has a local maximum there. At $x = 5$, the graph changes from negative to positive, so f has a local minimum there. If the graph of f is concave up, $f''(x)$ is positive. From the slope of f', the graph is concave up for $x > 3$. If the graph of f is concave down, $f''(x)$ is negative. From the slope of f', the graph of f is concave down for $x < 3$. There is a vertical asymptote at $x = 3$ since f' is not defined there. Graphs will vary slightly. An example showing the shape of the curve is illustrated.



14. a. $f(x)$ and $r(x)$: $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} r(x)$ exist.

b. $h(x)$: the highest degree of x in the numerator is exactly one degree higher than the highest degree of x in the denominator.

c. $h(x)$: the denominator is defined for all $x \in \mathbf{R}$.

$f(x) = \dfrac{-x-3}{(x-7)(x+2)}$ has vertical asymptotes at $x = 7$ and $x = -2$. $f(-2.001) = -110.99$ so as $x \to -2^-$, $f(x) \to -\infty$

$f(-1.999) = 111.23$ so as $x \to -2^+$, $f(x) \to \infty$

$f(6.999) = 111.12$ so as $x \to 7^-$, $f(x) \to \infty$

$f(7.001) = -111.10$ so as $x \to 7^+$, $f(x) \to -\infty$

$f(x)$ has a horizontal asymptote at $y = 0$.

$g(x)$ has a vertical asymptote at $x = 3$.

$g(2.999) = 23\,974.009$ so as $x \to 3^-$, $g(x) \to \infty$

$g(3.001) = -24\,026.009$ so as $x \to 3^+$, $g(x) \to -\infty$

By long division, $h(x) = x + \left(\dfrac{-4x-1}{x^2+1}\right)$ so $y = x$ is an oblique asymptote.

$r(x) = \dfrac{(x+3)(x-2)}{(x-4)(x+4)}$ has vertical asymptotes at $x = -4$ and $x = 4$.

$r(-4.001) = 750.78$ so as $x \to -4^-$, $r(x) \to \infty$

$r(-3.999) = -749.22$ so as $x \to -4^+$, $r(x) \to -\infty$

$r(3.999) = -1749.09$ so as $x \to 4^-$, $r(x) \to -\infty$

$r(4.001) = 1750.91$ so as $x \to 4^+$, $r(x) \to \infty$

$r(x)$ has a horizontal asymptote at $y = 1$.

15. $f(x) = \dfrac{ax+5}{3-bx}$

Vertical asymptote is at $x = -4$.

Therefore, $3 - bx = 0$ at $x = -5$.

That is, $3 - b(-5) = 0$

$$b = \frac{3}{5}.$$

Horizontal asymptote is at $y = -3$.

$$\lim_{x\to\infty}\left(\frac{ax+5}{3-bx}\right) = -3$$

$$\lim_{x\to\infty}\left(\frac{ax+5}{3-bx}\right) = \lim_{x\to\infty}\left(\frac{a+\frac{5}{x}}{\frac{3}{x}-b}\right) = \frac{-a}{b}$$

But $-\frac{a}{b} = -3$ or $a = 3b$.

But $b = \frac{3}{5}$, then $a = \frac{9}{5}$.

16. a.
$$\lim_{x\to\infty}\frac{x^2+1}{x+1} = \lim_{x\to\infty}\frac{x+\frac{1}{x}}{1+\frac{1}{x}} = \infty$$

$$\lim_{x\to\infty}\frac{x^2+2x+1}{x+1} = \lim_{x\to\infty}\frac{(x+1)(x+1)}{(x+1)} = \lim_{x\to\infty}(x+1) = \infty$$

b.
$$\lim_{x\to\infty}\left[\frac{x^2+1}{x+1} - \frac{x^2+2x+1}{x+1}\right] = \lim_{x\to\infty}\frac{x^2+1-x^2-2x-1}{x+1} = \lim_{x\to\infty}\frac{-2x}{x+1} = \lim_{x\to\infty}\frac{-2}{1+\frac{1}{x}} = -2$$

17. $f(x) = \dfrac{2x^2-2x}{x^2-9}$

Discontinuity is at $x^2 - 9 = 0$ or $x = \pm 3$.

$\lim_{x\to 3^+} f(x) = +\infty$

$\lim_{x\to 3^-} f(x) = -\infty$

$\lim_{x\to -3^+} f(x) = -\infty$

$\lim_{x\to -3^-} f(x) = +\infty$

Vertical asymptotes are at $x = 3$ and $x = -3$.

Horizontal asymptote:

$\lim_{x\to\infty} f(x) = 2$ (from below)

$\lim_{x\to\infty} f(x) = 2$ (from above)

Horizontal asymptote is at $y = 2$.

$$f'(x) = \frac{(4x-2)(x^2-9) - 2x(2x^2-2x)}{(x^2-9)^2} = \frac{4x^3 - 2x^2 - 36x + 18 - 4x^3 + 4x^2}{(x^2-9)^2} = \frac{2x^2 - 36x + 18}{(x^2-9)^2}$$

Let $f'(x) = 0$,

$2x^2 - 36x + 18 = 0$ or $x^2 - 18x + 9 = 0$.

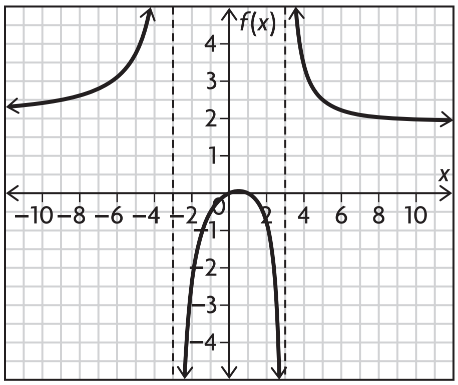
$$x = \frac{18 \pm \sqrt{18^2 - 36}}{2}$$

$x = 0.51$ or $x = 17.5$

$y = 0.057$ or $y = 1.83$.

Local maximum is at $(0.51, 0.057)$ and local minimum is at $(17.5, 1.83)$.

t	$-3 < x < 0.51$	0.51	$0.51 < x < 3$	$3 < x < 17.5$	17.5	$x > 17.5$
$s'(t)$	+	0	−	−	0	+
Graph	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing



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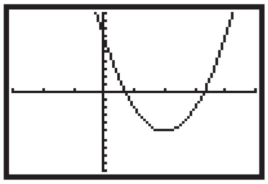
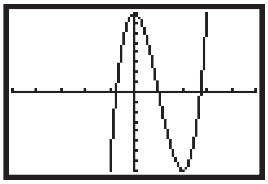
1. a.



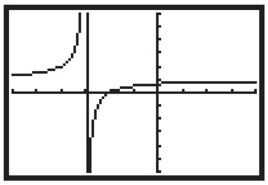
The function appears to be decreasing on $(-\infty, 2)$ and increasing on $(2, \infty)$.

b.



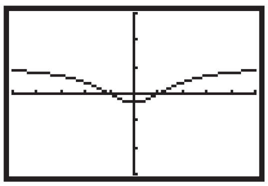
The function appears to be increasing on $(-\infty, 0)$ and $(2, \infty)$ and decreasing on $(0, 2)$.

c.



The function is increasing on $(-\infty, -3)$ and $(-3, \infty)$.

d.



The function appears to be decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

2. The slope of a general tangent to the graph $g(x) = 2x^3 - 3x^2 - 12x + 15$ is given by $\frac{dg}{dx} = 6x^2 - 6x - 12$. We first determine values of x for which $\frac{dg}{dx} = 0$.

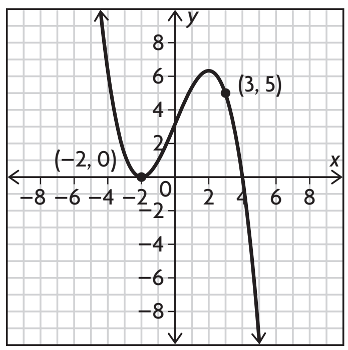
So $6x^2 - 6x - 12 = 0$

$$6(x^2 - x - 2) = 0$$
$$6(x + 1)(x - 2) = 0$$
$$x = -1, x = 2$$

Since $\frac{dg}{dx}$ is defined for all values of x, and since $\frac{dg}{dx} = 0$ only at $x = -1$ and $x = 2$, it must be either positive or negative for all other values of x. Consider the intervals between $x < -1$, $-1 < x < 2$, and $x > 2$.

Value of x	$x < -1$	$-1 < x < 2$	$x > 2$
Value of $\frac{dg}{dx} = 6x^2 - 6x - 12$	$\frac{dg}{dx} > 0$	$\frac{dg}{dx} < 0$	$\frac{dg}{dx} > 0$
Slope of Tangents	positive	negative	positive
y-values Increasing or Decreasing	increasing	decreasing	increasing

3.



4. The critical numbers can be found when $\frac{dy}{dx} = 0$.

a. $\frac{dy}{dx} = -4x + 16$. When $\frac{dy}{dx} = 0$,

$$\frac{dy}{dx} = -4(x + 4) = 0$$
$$x = -4$$

	$x < 1$	$1 < x < 2$	$x > 2$
$x - 1$	−	+	+
$x - 2$	−	−	+
$(x - 1)(x - 2)$	$(-)(-) = +$	$(+)(-) = -$	$(+)(+) = +$
$\frac{dy}{dx}$	> 0	< 0	> 0
$g(x) = 2x^3 - 9x^2 + 12x$	increasing	decreasing	increasing

b. $\frac{dy}{dx} = x^3 - 27x$. When $\frac{dy}{dx} = 0$,

$\frac{dy}{dx} = x(x^2 - 27) = 0$

$x = 0, x = \pm 3\sqrt{3}$

c. $\frac{dy}{dx} = 4x^3 - 8x$. When $\frac{dy}{dx} = 0$,

$\frac{dy}{dx} = 4x(x^2 - 2) = 0$

$x = 0, x = \pm\sqrt{2}$

d. $\frac{dy}{dx} = 15x^4 - 75x^2 + 60$. When $\frac{dy}{dx} = 0$,

$\frac{dy}{dx} = 15(x^4 - 5x^2 + 4) = 0$

$\frac{dy}{dx} = 15(x^2 - 1)(x^2 - 4) = 0$

$x = \pm 1, x = \pm 2$

e. $\frac{dy}{dx} = \frac{2x(x^2 + 1) - (x^2 - 1)(2x)}{(x^2 + 1)^2}$. When $\frac{dy}{dx} = 0$, the numerator equals 0. So $\frac{dy}{dx} = 2x(x^2 + 1) - (x^2 - 1)(2x) = 0$. After simplifying, $\frac{dy}{dx} = 4x = 0$.

$x = 0$

f. $\frac{dy}{dx} = \frac{(x^2 + 2) - x(2x)}{(x^2 + 2)^2}$. When $\frac{dy}{dx} = 0$, the numerator equals 0. So after simplifying,

$\frac{dy}{dx} = -x^2 + 2 = 0$.

$x = \pm\sqrt{2}$

5. a. $\frac{dg}{dx} = 6x^2 - 18x + 12$

To find the critical numbers, set $\frac{dg}{dx} = 0$. So

$6x^2 - 18x + 12 = 0$

$6(x - 1)(x - 2) = 0$

$x = 1, x = 2$

From the table above, $x = 1$ is the local maximum and $x = 2$ is the local minimum.

b. $\frac{dg}{dx} = 3x^2 - 4x - 4$

To find the critical numbers, set $\frac{dg}{dx} = 0$.

$3x^2 - 4x - 4 = 0$

$(3x + 2)(x - 2) = 0$

$x = -\frac{2}{3}$ or $x = 2$

	$x < -\frac{2}{3}$	$-\frac{2}{3} < x < 2$	$x > 2$
$3x + 2$	−	+	+
$x - 2$	−	−	+
$\frac{dg}{dx}$	+	−	+
$g(x)$	increasing	decreasing	increasing

The function has a local maximum at $x = -\frac{2}{3}$ and a local minimum at $x = 2$

6. $\frac{df}{dx} = 2x + k$

To have a local minimum value, $\frac{df}{dx} = 0$. This occurs when $x = -\frac{k}{2}$. So $f\left(-\frac{k}{2}\right) = 1$.

$\frac{k^2}{4} - \frac{k^2}{2} + 2 = 1$

$-\frac{k^2}{4} + 2 = 1$

$-\frac{k^2}{4} = -1$

$k^2 = 4$

$k = \pm 2$

7. $f'(x) = 4x^3 - 32$

To find the critical numbers, set $f'(x) = 0$.

$4x^3 - 32 = 0$

$4(x^3 - 8) = 0$

$x = 2$

	$x < 2$	$x > 2$
$f'(x) = 4x^3 - 32$	−	+
$f(x)$	decreasing	increasing

The function has a local minimum at $x = 2$.

8. a. Since $x + 2 = 0$ for $x = -2$, $x = -2$ is a vertical asymptote. Large and positive to left of asymptote, large and negative to right of asymptote.

b. Since $9 - x^2 = 0$ for $x = \pm 3$, $x = -3$ and $x = 3$ are vertical asymptotes. For $x = -3$: large and negative to left of asymptote, large and positive to right of asymptote.

c. Since $3x + 9 = 0$ for $x = -3$, $x = -3$ is a vertical asymptote. Large and negative to left of asymptote, large and positive to right of asymptote.

d. Since $3x^2 - 13x - 10 = 0$ when $x = -\frac{2}{3}$ and $x = 5$, $x = -\frac{2}{3}$ and $x = 5$ are vertical asymptotes. For $x = -\frac{2}{3}$ large and positive to left of asymptote, large and negative to right of asymptote. For $x = 3$: large and positive to left of asymptote, large and negative to right of asymptote.

9. a. $f(x) = \dfrac{3x - 1}{x + 5} = \dfrac{3x\left(1 - \frac{1}{3x}\right)}{x\left(1 + \frac{5}{x}\right)}$

$$= \frac{3\left(1 - \frac{1}{3x}\right)}{1 + \frac{5}{x}}$$

$$\lim_{x\to+\infty} f(x) = \frac{3\left[\lim_{x\to-\infty}\left(1 - \frac{1}{3x}\right)\right]}{\lim_{x\to-\infty}\left(1 + \frac{5}{x}\right)}$$

$$= \frac{3(1 - 0)}{(1 + 0)}$$

$$= 3$$

So the horizontal asymptote is $y = 3$. Similarly, we can prove $\lim_{x\to-\infty} f(x) = 3$. If x is large and positive, for example, if $x = 1000$, $f(x) = \frac{2999}{1005}$, which is smaller than 3. If x is large and negative, for example, if $x = -1000$, $f(x) = \frac{-3001}{-995}$, which is larger than 3. So $f(x)$ approaches $y = 3$ from below when x is large and positive and approached $y = 3$ from above when x is large and negative.

b. $f(x) = \dfrac{x^2 + 3x - 2}{(x - 1)^2} = \dfrac{x^2\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{x^2\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}$

$$= \frac{\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}$$

$$\lim_{x\to+\infty} \frac{\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)} = \frac{(1 + 0 - 0)}{(1 - 0 + 0)}$$

$$= 1$$

So the horizontal asymptote is 1. Similarly, we can prove $\lim_{x\to-\infty} f(x) = 1$. If x is large and positive, for example, $x = 1000$, $f(x) = \frac{1000^2 + 3(1000) - 2}{(1000 - 1)^2} = \frac{9\,96\,998}{10\,02\,001}$, which is greater than 1. If x is large and negative, for example, $x = -1000$, $f(x) = \frac{(-1000)^2 + 3(-1000) - 2}{(-1000 - 1)^2} = \frac{9\,96\,998}{10\,02\,001}$, which is less than 1. So $f(x)$ approaches $y = 1$ from above when x is large and positive and approaches $y = 1$ from below when x is large and negative.

10. a. Since $(x - 5)^2 = 0$ when $x = 5$, $x = 5$ is a vertical asymptote.

$$f(x) = \frac{x}{(x - 5)^2} = \frac{x}{x^2\left(1 - \frac{10}{x} + \frac{25}{x^2}\right)}$$

$$= \frac{1}{x\left(1 - \frac{10}{x} + \frac{25}{x^2}\right)}$$

$$\lim_{x\to+5} f(x) = \frac{\lim_{x\to+5}(1)}{\lim_{x\to+5}\left(x\left(1 - \frac{10}{x} + \frac{25}{x^2}\right)\right)} = +\infty$$

This limit gets larger as it approaches 5 from the right. Similarly, we can prove that the limit goes to $+\infty$ as it approaches 5 from the left. For example, if $x = 1000$ $f(x) = \dfrac{1}{1000\left(1 - \frac{10}{1000} + \frac{25}{1000^2}\right)}$, which gets larger as x gets larger. Thus, $f(x)$ approaches $+\infty$ on both sides of $x = 5$.

b. There are no discontinuities because $x^2 + 9$ never equals zero.

c. Using the quadratic formula, we find that $x^2 - 12x + 12 = 0$ when $x = 6 \pm 2\sqrt{6}$. So $x = 6 \pm 2\sqrt{6}$ are vertical asymptotes.

$$f(x) = \frac{x - 2}{x^2 - 12x + 12} = \frac{x\left(1 - \frac{2}{x}\right)}{x^2\left(1 - \frac{12}{x} + \frac{12}{x^2}\right)}$$

$$\lim_{x \to +6+2\sqrt{6}} f(x) = \frac{\lim_{x \to +6+2\sqrt{6}} x\left(1 - \frac{2}{x}\right)}{\lim_{x \to +6+2\sqrt{6}} x^2\left(1 - \frac{12}{x} + \frac{12}{x^2}\right)}.$$

This limit gets smaller as it approaches $6 + 2\sqrt{6}$ from the right and get larger as it approaches $6 + 2\sqrt{6}$ from the left. Similarly, we can show that the limit gets smaller as it approaches $6 - 2\sqrt{6}$ from the left and gets larger as it approaches from the right.

11. a. $f'(x) > 0$ implies that $f(x)$ is increasing.

b. $f'(x) < 0$ implies that $f(x)$ is decreasing.

12. a. $h(t) = -4.9t^2 + 9.5t + 2.2$

Note that $h(0) = 2.2 < 3$ because when the diver dives, the board is curved down.

$h'(t) = -9.8t + 9.5$

Set $h'(t) = 0$

$0 = -9.8t + 9.5$

$t \doteq 0.97$

	$0 < t < 0.97$	$t > 0.97$
$-9.8t + 9.5$	+	−
Sign of $h'(t)$	+	−
Behaviour of $h(t)$	increasing	decreasing

b. $h'(t) = v(t)$

$v(t) = -9.8t + 9.5$

$v'(t) = -9.8 < 0$

The velocity is decreasing all the time.

13. $C(t) = \frac{t}{4} + 2t^{-2}$

$C'(t) = \frac{1}{4} - 4t^{-3}$

Set $C'(t) = 0$

$0 = \frac{1}{4} - 4t^{-3}$

$\frac{1}{4} = 4t^{-3}$

$t^3 = 16$

$t \doteq 2.5198$

	$t < 2.5198$	$t > 2.5198$
$\frac{1}{4} - 4t^{-3}$	−	+
Sign of $C'(t)$	−	+
Behaviour of $C(t)$	decreasing	increasing

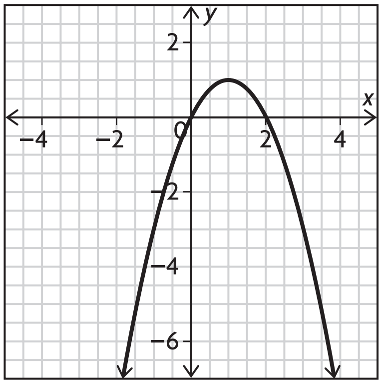
14. For $f(x)$ the derivative function $f'(0) = 0$ and $f'(2) = 0$.

Therefore, $f'(x)$ passes through (0, 0) and (2, 0).

When $x < 0$, $f(x)$ is decreasing, therefore, $f'(x) < 0$.

When $0 < x < 2$, $f(x)$ is increasing, therefore, $f'(x) > 0$.

When $x > 2$, $f(x)$ is decreasing, therefore, $f'(x) < 0$.



15. a. $f(x) = x^2 - 7x - 18$

i. $f'(x) = 2x - 7$

Set $f'(x) = 0$

$0 = 2x - 7$

$x = \frac{7}{2}$

ii.

	$x < \frac{7}{2}$	$x > \frac{7}{2}$
$2x - 7$	−	+
Sign of $f'(x)$	−	+
Behaviour of $f(x)$	decreasing	increasing

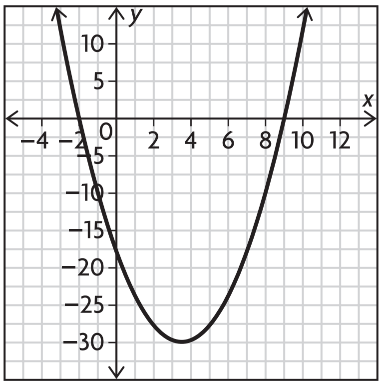
iii. From ii., there is a minimum at $x = \frac{7}{2}$.

$f\left(\frac{7}{2}\right) = \left(\frac{7}{2}\right)^2 - 7\left(\frac{7}{2}\right) - 18$

$f\left(\frac{7}{2}\right) = \frac{49}{4} - \frac{49}{2} - 18$

$f\left(\frac{7}{2}\right) = -\frac{121}{4}$

iv.



b. $f(x) = -2x^3 + 9x^2 + 3$

i. $f'(x) = -6x^2 + 18x$

Set $f'(x) = 0$

$0 = -6x^2 + 18x$

$0 = -6x(x - 3)$

$x = 0$ or $x = 3$

ii.

	$x < 0$	$0 < x < 3$	$x > 3$
$-6x$	+	−	−
$x - 3$	−	−	+
Sign of $f'(x)$	$(+)(-) = -$	$(-)(-) = +$	$(-)(+) = -$
Behaviour of $f(x)$	decreasing	increasing	decreasing

iii. From ii., there is a minimum at $x = 0$ and a maximum at $x = 3$.

$f(0) = -2(0)^3 + 9(0)^2 + 3$

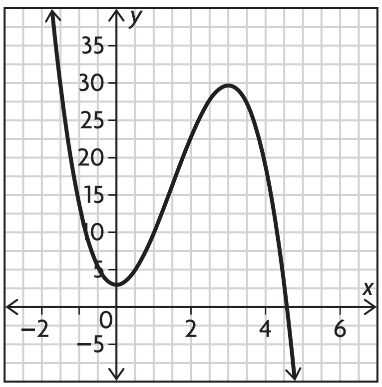
$f(0) = 3$

$f(3) = -2(3)^3 + 9(3)^2 + 3$

$f(3) = -54 + 81 + 3$

$f(3) = 30$

iv.



c. $f(x) = 2x^4 - 4x^2 + 2$

i. $f'(x) = 8x^3 - 8x$

$f'(x) = 0$

$0 = 8x^3 - 8x$

$0 = 8x(x^2 - 1)$

$0 = 8x(x - 1)(x + 1)$

$x = -1$ or $x = 0$ or $x = 1$

ii.

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$x > 1$
$8x$	−	−	+	+
$x - 1$	−	−	−	+
$x + 1$	−	+	+	+
Sign of $f'(x)$	$(-)(-)(-) = -$	$(-)(-)(+) = +$	$(+)(-)(+) = -$	$(+)(+)(+) = +$
Behaviour of $f(x)$	decreasing	increasing	decreasing	increasing

iii. From ii., there is a maximum at $x = 0$ and minima at $x = -1$ and $x = 1$

$f(-1) = 2(-1)^4 - 4(-1)^2 + 2$

$f(-1) = 2 - 4 + 2$

$f(-1) = 0$

$f(0) = 2(0)^4 - 4(0)^2 + 2$

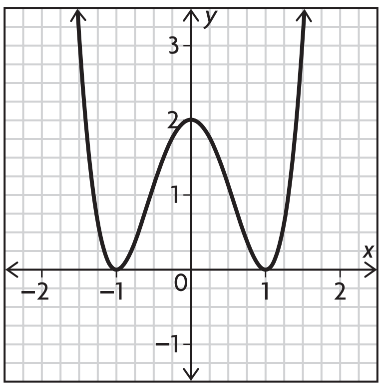
$f(0) = 2$

$f(1) = 2(1)^4 - 4(1)^2 + 2$

$f(1) = 2 - 4 + 2$

$f(1) = 0$

iv.



d. $f(x) = x^5 - 5x$

i. $f'(x) = 5x^4 - 5$

Set $f'(x) = 0$

$0 = 5x^4 - 5$

$0 = 5(x^4 - 1)$

$0 = 5(x^2 - 1)(x^2 + 1)$

$0 = 5(x - 1)(x + 1)(x^2 + 1)$

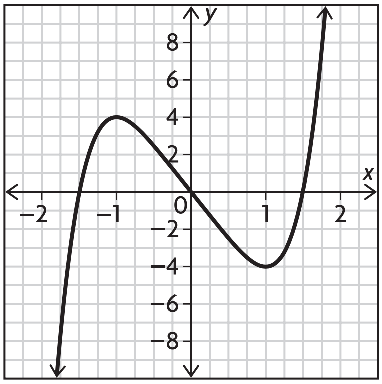
$x = -1$ or $x = 1$

ii.

	$x < -1$	$-1 < x < 1$	$x > 1$
5	+	+	+
$x - 1$	−	−	+
$x + 1$	−	+	+
$x^2 + 1$	+	+	+
Sign of $f'(x)$	$(+)(-)(-)(+) = +$	$(+)(-)(+)(+) = -$	$(+)(+)(+)(+) = +$
Behaviour of $f(x)$	increasing	decreasing	increasing

iii. From ii., there is a maximum at $x = -1$ and a minimum at $x = 1$

$f(-1) = (-1)^5 - 5(-1)$
$f(-1) = -1 + 5$
$f(-1) = 4$
$f(1) = (1)^5 - 5(1)$
$f(1) = -4$



16. a. vertical asymptote: $x = -\frac{1}{2}$, horizontal asymptote $y = \frac{1}{2}$; as x approaches $\frac{1}{2}$ from the left, graph approaches infinity; as x approaches $\frac{1}{2}$ from the right, graph approaches negative infinity.
b. vertical asymptote: $x = -2$, horizontal asymptote: $y = 1$; as x approaches -2 from the left, graph approaches infinity; as x approaches -2 from the right, graph decreases to $(-0.25, -1.28)$ and then approaches to infinity.
c. vertical asymptote: $x = -3$, horizontal asymptote: $y = -1$; as x approaches -3 from the left, graph approaches infinity; as x approaches -3 from the right, graph approaches infinity
d. vertical asymptote: $x = -4$, no horizontal asymptote; as x approaches -4 from the left, graph increases to $(-7.81, -30.23)$ and then decreases to -4; as x approaches -4 from the right, graph decreases to $(-0.19, 0.23)$ then approaches infinity.

17. a. $\lim_{x\to\infty} \frac{3 - 2x}{3x}$

$= \lim_{x\to\infty} \frac{\frac{3}{x} - \frac{2x}{x}}{\frac{3x}{x}}$

$= \frac{0 - 2}{3}$

$= -\frac{2}{3}$

b. $\lim_{x\to\infty} \frac{x^2 - 2x + 5}{6x^2 + 2x - 1}$

$= \lim_{x\to\infty} \frac{\frac{x^2}{x^2} - \frac{2x}{x^2} + \frac{5}{x^2}}{\frac{6x^2}{x^2} + \frac{2x}{x^2} - \frac{1}{x^2}}$

$= \frac{1 - 0 + 0}{6 + 0 - 0}$

$= \frac{1}{6}$

c. $\lim_{x\to\infty} \frac{7 + 2x^2 - 3x^3}{x^3 - 4x^2 + 3x}$

$= \lim_{x\to\infty} \frac{\frac{7}{x^3} + \frac{2x^2}{x^3} - \frac{3x^3}{x^3}}{\frac{x^3}{x^3} - \frac{4x^2}{x^3} + \frac{3x}{x^3}}$

$= \frac{0 + 0 - 3}{1 - 0 + 0}$

$= -3$

d. $\lim_{x\to\infty} \frac{5 + 2x^3}{x^4 - 4x}$

$= \lim_{x\to\infty} \frac{\frac{5}{x^4} - \frac{2x^3}{x^4}}{\frac{x^4}{x^4} - \frac{4x}{x^4}}$

$= \frac{0 - 0}{1 - 0}$

$= 0$

e. $\lim_{x\to\infty} \frac{2x^5 - 1}{3x^4 - x^2 - 2} = \lim_{x\to\infty}\left(\frac{2}{3}x + \frac{\frac{2}{3}x^3 + \frac{4}{3}x - 1}{3x^4 - x^2 - 2}\right)$

$= \lim_{x\to\infty} \frac{2}{3}x + \lim_{x\to\infty} \frac{\frac{\frac{2}{3}x^3}{x^4} + \frac{\frac{4}{3}x}{x^4} - \frac{1}{x^4}}{\frac{3x^4}{x^4} - \frac{x^2}{x^4} - \frac{2}{x^4}}$

$= \infty$

f. $\lim_{x\to\infty} \frac{x^2 + 3x - 18}{(x - 3)^2}$

$= \lim_{x\to\infty} \frac{x^2 + 3x - 18}{x^2 - 6x + 9}$

$= \lim_{x\to\infty} \frac{\frac{x^2}{x^2} + \frac{3x}{x^2} - \frac{18}{x^2}}{\frac{x^2}{x^2} - \frac{6x}{x^2} + \frac{9}{x^2}}$

$= \frac{1 + 0 - 0}{1 - 0 + 0}$

$= 1$

g. $\lim_{x\to\infty} \frac{x^2 - 4x - 5}{x^2 - 1}$

$= \lim_{x\to\infty} \frac{\frac{x^2}{x^2} - \frac{4x}{x^2} - \frac{5}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$

$= \lim_{x\to\infty} \frac{1 - 0 - 0}{1 - 0}$

$= 1$

h. $\lim_{x\to\infty} \left(5x + 4 - \frac{7}{x + 3}\right)$

$= \lim_{x\to\infty} 5x + \lim_{x\to\infty} 4 - \lim_{x\to\infty} \frac{\frac{7}{x}}{\frac{x}{x} + \frac{3}{x}}$

$= \infty$

4.4 Concavity and Points of Inflection, pp. 205–206

1. a. A: negative; B: negative; C: positive; D: positive

b. A: negative; B: negative; C: positive; D: negative

2. a. $y = x^3 - 6x^2 - 15x + 10$

$\frac{dy}{dx} = 3x^2 - 12x - 15$

For critical values, we solve $\frac{dy}{dx} = 0$:

$3x^2 - 12x - 15 = 0$

$x^2 - 4x - 5 = 0$

$(x - 5)(x + 1) = 0$

$x = 5$ or $x = -1$

The critical points are $(5, -105)$ and $(-1, 20)$.

Now, $\frac{d^2y}{dx^2} = 6x - 12$.

At $x = 5$, $\frac{d^2y}{dx^2} = 18 > 0$. There is a local minimum at this point.

At $x = -1$, $\frac{d^2y}{dx^2} = -18 < 0$. There is a local maximum at this point.

The local minimum is $(5, -105)$ and the local maximum is $(-1, 20)$

b. $y = \frac{25}{x^2 + 48}$

$\frac{dy}{dx} = -\frac{50x}{(x^2 + 48)^2}$

For critical values, solve $\frac{dy}{dx} = 0$ or $\frac{dy}{dx}$ does not exist.

Since $x^2 + 48 > 0$ for all x, the only critical point is $\left(0, \frac{25}{48}\right)$

$\frac{d^2y}{dx^2} = -50(x^2 + 48)^{-2} + 100x(x^2 + 48)^{-3}(2x)$

$= -\frac{50}{(x^2 + 48)^2} + \frac{200x^2}{(x^2 + 48)^3}$

At $x = 0$, $\frac{d^2y}{dx^2} = -\frac{50}{48^2} < 0$. The point $\left(0, \frac{25}{48}\right)$ is a local maximum.

c. $s = t + t^{-1}$

$\frac{ds}{dt} = 1 - \frac{1}{t^2}, t \neq 0$

For critical values, we solve $\frac{ds}{dt} = 0$:

$1 - \frac{1}{t^2} = 0$

$t^2 = 1$

$t = \pm 1.$

The critical points are $(-1, -2)$ and $(1, 2)$

$\frac{d^2s}{dt^2} = \frac{2}{t^3}$

At $t = -1$, $\frac{d^2s}{dt} = -2 < 0$. The point $(-1, -2)$ is a local maximum. At $t = 1$, $\frac{d^2s}{dt^2} = 2 > 0$. The point $(1, 2)$ is a local minimum.

d. $y = (x - 3)^3 + 8$

$\frac{dy}{dx} = 3(x - 3)^2$

$x = 3$ is a critical value.

The critical point is $(3, 8)$

$\frac{d^2y}{dx^2} = 6(x - 3)$

At $x = 3$, $\frac{d^2y}{dx^2} = 0$.

The point $(3, 8)$ is neither a relative (local) maximum or minimum.

3. a. For possible point(s) of inflection, solve $\frac{d^2y}{dx^2} = 0$:

$6x - 8 = 0$

$x = \frac{4}{3}.$

Interval	$x < \frac{4}{3}$	$x = \frac{4}{3}$	$x > \frac{4}{3}$
$f''(x)$	<0	$=0$	>0
Graph of $f(x)$	Concave Down	Point of Inflection	Concave Up

The point $\left(\frac{4}{3}, -14\frac{20}{27}\right)$ is point of inflection.

b. For possible point(s) of inflection, solve

$$\frac{d^2y}{dx^2} = 0:$$

$200x^2 - 50x^2 - 2400 = 0$

$150x^2 = 2400.$

Since $x^2 + 48 > 0$:

$x = \pm 4.$

Interval	$x < -4$	$x = -4$	$-4 < x < 4$	$x = 4$	$x > 4$
$f''(x)$	>0	$=0$	<0	$=0$	>0
Graph of $f(x)$	Concave Up	Point of Inflection	Concave Down	Point of Inflection	Concave Up

$\left(-4, \frac{25}{64}\right)$ and $\left(4, \frac{25}{64}\right)$ are points of inflection.

c. $\frac{d^2s}{dt^2} = \frac{3}{t^2}$

Interval	$t < 0$	$t = 0$	$t > 0$
$f''(t)$	<0	Undefined	>0
Graph of $f(t)$	Concave Down	Undefined	Concave Up

The graph does not have any points of inflection.

d. For possible points of inflection, solve

$$\frac{d^2y}{dx^2} = 0:$$

$6(x - 3) = 0$

$x = 3.$

Interval	$x < 3$	$x = 3$	$x > 3$
$f''(x)$	<0	$=0$	>0
Graph of $f(x)$	Concave Down	Point of Inflection	Concave Up

(3, 8) is a point of inflection.

4. a. $f(x) = 2x^3 - 10x + 3$ at $x = 2$

$f'(x) = 6x^2 - 10$

$f''(x) = 12x$

$f''(2) = 24 > 0$

The curve lies above the tangent at $(2, -1)$.

b. $g(x) = x^2 - \frac{1}{x}$ at $x = -1$

$$g'(x) = 2x + \frac{1}{x^2}$$

$$g''(x) = 2 - \frac{2}{x^3}$$

$g''(-1) = 2 + 2 = 4 > 0$

The curve lies above the tangent line at $(-1, 2)$.

c. $p(w) = \frac{w}{\sqrt{w^2 + 1}}$ at $w = 3$

$$p(w) = w(w^2 + 1)^{\frac{1}{2}}$$

$$\frac{dp}{dw} = (w^2 + 1)^{\frac{1}{2}} + w\left(-\frac{1}{2}\right)(w^2 + 1)^{\frac{3}{2}}(2w)$$

$$= (w^2 + 1)^{\frac{1}{2}} - w^2(w^2 + 1)^{\frac{3}{2}}$$

$$\frac{d^2p}{dw^2} = -\frac{1}{2}(w^2 + 1)^{\frac{3}{2}}(2w) - 2w(w^2 + 1)^{\frac{3}{2}} + w^2\left(\frac{3}{2}\right)(w^2 + 1)^{\frac{5}{2}}(2w)$$

At $w = 3$, $\frac{d^2p}{dw^2} = -\frac{3}{10\sqrt{10}} - \frac{6}{10\sqrt{10}} + \frac{81}{100\sqrt{10}}$

$$= -\frac{9}{100\sqrt{10}} < 0.$$

The curve is below the tangent line at $\left(3, \frac{3}{\sqrt{10}}\right)$.

d. The first derivative is

$$s'(t) = \frac{(t - 4)(2) - (2t)(1)}{(t - 4)^2}$$

$$= \frac{-8}{(t - 4)^2}$$

The second derivative is

$$s''(t) = \frac{(t - 4)^2(0) - (-8)2(t - 4)^1}{(t - 4)^4}$$

$$= \frac{16}{(t - 4)^3}$$

So $s''(-2) = \frac{16}{(-2 - 4)^3}$

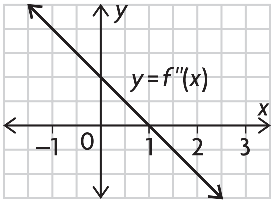
$$= -\frac{16}{216} = -\frac{2}{27}$$

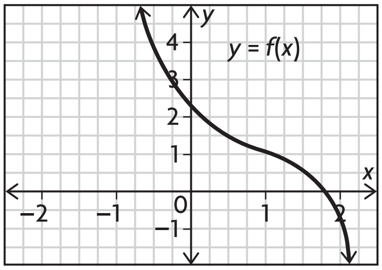
Since the second derivative is negative at this point, the function lies below the tangent there.

5. For the graph on the left: **i.** $f''(x) > 0$ for $x < 1$ Thus, the graph of $f(x)$ is concave up on $x < 1$. $f''(x) \leq 0$ for $x > 1$. The graph of $f(x)$ is concave down on $x > 1$.

ii. There is a point of inflection at $x = 1$.

iii.



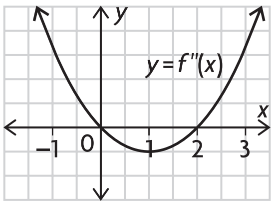


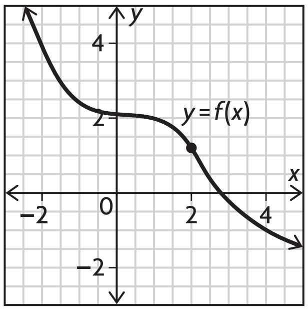
For the graph on the right: **i.** $f''(x) > 0$ for $x < 0$ or $x > 2$

The graph of $f(x)$ is concave up on $x < 0$ or $x > 2$.
The graph of $f(x)$ is concave down on $0 < x < 2$.

ii. There are points of inflection at $x = 0$ and $x = 2$.

iii.





6. For any function $y = f(x)$, find the critical points, i.e., the values of x such that $f'(x) = 0$ or $f'(x)$ does not exist. Evaluate $f''(x)$ for each critical value.
If the value of the second derivative at a critical point is positive, the point is a local minimum. If the value of the second derivative at a critical point is negative, the point is a local maximum.

7. Step 4: Use the first derivative test or the second derivative test to determine the type of critical points that may be present.

8. a. $f(x) = x^4 + 4x^3$

i. $f'(x) = 4x^3 + 12x^2$
$f''(x) = 12x^2 + 24x$

For possible points of inflection, solve $f''(x) = 0$:

$$12x^2 + 24x = 0$$
$$12x(x + 2) = 0$$
$$x = 0 \text{ or } x = -2.$$

Interval	$x < -2$	$x = -2$	$-2 < x < 0$	$x = 0$	$x > 0$
$f''(x)$	>0	$=0$	<0	$=0$	>0
Graph of $f(x)$	Concave Up	Point of Inflection	Concave Down	Point of Inflection	Concave Up

The points of inflection are $(-2, -16)$ and $(0, 0)$.

ii. If $x = 0$, $y = 0$.
For critical points, we solve $f'(x) = 0$:

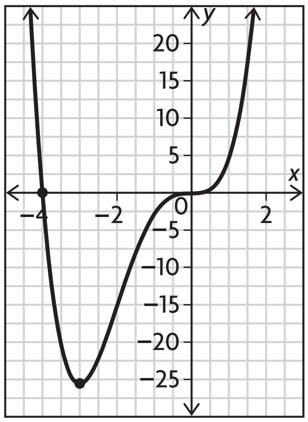
$$4x^3 + 12x^2 = 0$$
$$4x^2(x + 3) = 0$$
$$x = 0 \text{ and } x = -3.$$

Interval	$x < -3$	$x = -3$	$-3 < x < 0$	$x = 0$	$x > 0$
$f'(x)$	<0	$=0$	>0	$=0$	>0
Graph of $f(x)$	Decreasing	Local Min	Increasing		Increasing

If $y = 0$, $x^4 + 4x^3 = 0$

$$x^3(x + 4) = 0$$
$$x = 0 \text{ or } x = -4$$

The x-intercepts are 0 and -4.



b. d. $g(w) = \dfrac{4w^2 - 3}{w^3}$

$$= \frac{4}{3} - \frac{3}{w^3},\ w \neq 0$$

i. $g'(w) = -\dfrac{4}{w^2} + \dfrac{9}{w^4}$

$$= \frac{9 - 4w^2}{w^4}$$

$$g''(w) = \frac{8}{w^3} - \frac{36}{w^3}$$

$$= \frac{8w^2 - 36}{w^5}$$

For possible points of inflection, we solve $g''(w) = 0$:

$8w^2 - 36 = 0$, since $w^5 \neq 0$

$$w^2 = \frac{9}{2}$$

$$w = \pm\frac{3}{\sqrt{2}}.$$

Interval	$w < -\frac{3}{\sqrt{2}}$	$w = -\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}} < w < 0$	$0 < w < \frac{3}{\sqrt{2}}$	$w = \frac{3}{\sqrt{2}}$	$w > \frac{3}{\sqrt{2}}$
$g'(w)$	<0	$=0$	>0	<0	0	>0
Graph of $g(w)$	Concave Down	Point of Inflection	Concave Up	Concave Down	Point of Inflection	Concave Up

The points of inflection are $\left(-\frac{3}{\sqrt{2}}, -\frac{8\sqrt{2}}{9}\right)$ and $\left(\frac{3}{\sqrt{2}}, -\frac{8\sqrt{2}}{9}\right)$.

ii. There is no y-intercept.

The x-intercept is $\pm\frac{3}{\sqrt{2}}$.

For critical values, we solve $g'(w) = 0$:

$9 - 4w^2 = 0$ since $w^4 \neq 0$

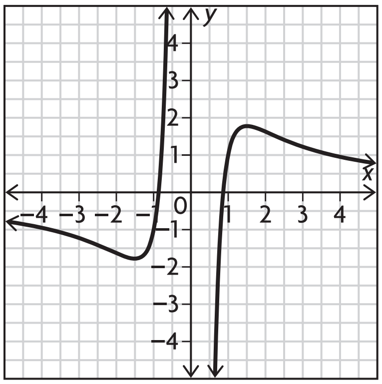
$$w = \pm\frac{3}{2}.$$

Interval	$w < -\frac{3}{2}$	$w = -\frac{3}{2}$	$-\frac{3}{2} < w < 0$	$0 < w < \frac{3}{2}$	$w = \frac{3}{2}$	$w > \frac{3}{2}$
$g'(w)$	<0	$=0$	>0	>0	0	<0
Graph of $g(w)$	Decreasing Down	Local Min	Increasing	Increasing	Local Max	Decreasing

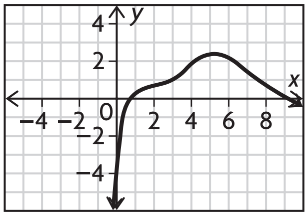
$$\lim_{w\to 0^-} \frac{4w^2 - 3}{w^3} = \infty, \quad \lim_{w\to 0^+} \frac{4w^2 - 3}{w^3} = -\infty$$

$$\lim_{w\to -\infty} \left(\frac{4}{w} - \frac{3}{w^3}\right) = 0, \quad \lim_{w\to \infty} \left(\frac{4}{w} - \frac{3}{w^3}\right) = 0$$

Thus, $y = 0$ is a horizontal asymptote and $x = 0$ is a vertical asymptote.



9. The graph is increasing when $x < 2$ and when $2 < x < 5$.
The graph is decreasing when $x > 5$.
The graph has a local maximum at $x = 5$.
The graph has a horizontal tangent line at $x = 2$.
The graph is concave down when $x < 2$ and when $4 < x < 7$.
The graph is concave up when $2 < x < 4$ and when $x > 7$.
The graph has points of inflection at $x = 2$, $x = 4$, and $x = 7$.
The y-intercept of the graph is -4.



10. $f(x) = ax^3 + bx^2 + c$

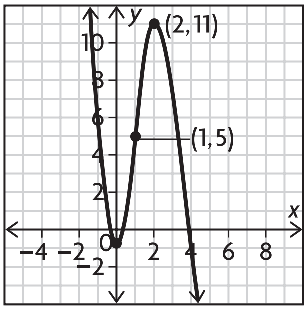
$f'(x) = 3ax^2 + 2bx$

$f''(x) = 6ax + 2b$

Since (2, 11) is a relative extremum,
$f(2) = 12a + 4b = 0$.
Since (1, 5) is an inflection point,
$f''(1) = 6a + 2b = 0$.
Since the points are on the graph, $a + b + c = 5$ and
$8a + 4b + c = 11$

$$7a + 3b = 6$$
$$9a + 3b = 0$$
$$2a = -6$$
$$a = -3$$
$$b = 9$$
$$c = -1.$$

Thus, $f(x) = -3x^3 + 9x^2 - 1$.



11. $f(x) = (x + 1)^{\frac{1}{2}} + bx^{-1}$

$$f'(x) = \frac{1}{2}(x + 1)^{\frac{1}{2}} - bx^{-2}$$

$$f''(x) = -\frac{1}{4}(x + 1)^{\frac{3}{2}} + 2bx^{-3}$$

Since the graph of $y = f(x)$ has a point of inflection at $x = 3$:

$$-\frac{1}{4}(4)^{\frac{3}{2}} + \frac{2b}{27} = 0$$

$$-\frac{1}{32} + \frac{2b}{27} = 0$$

$$b = \frac{27}{64}.$$

12. $f(x) = ax^4 + bx^3$

$f'(x) = 4ax^3 + 3bx^2$

$f''(x) = 12ax^2 + 6bx$

For possible points of inflection, we solve $f''(x) = 0$:

$12ax^2 + 6bx = 0$

$6x(2ax + b) = 0$

$x = 0$ or $x = -\frac{b}{2a}$.

The graph of $y = f''(x)$ is a parabola with x-intercepts 0 and $-\frac{b}{2a}$.

We know the values of $f''(x)$ have opposite signs when passing through a root. Thus at $x = 0$ and at $x = -\frac{b}{2a}$, the concavity changes as the graph goes through these points. Thus, $f(x)$ has points of inflection at $x = 0$ and $x = -\frac{b}{2a}$. To find the x-intercepts, we solve $f(x) = 0$

$$x^3(ax + b) = 0$$

$$x = 0 \text{ or } x = -\frac{b}{a}.$$

The point midway between the x-intercepts has x-coordinate $-\frac{b}{2a}$.

The points of inflection are (0, 0) and $\left(-\frac{b}{2a}, -\frac{b}{16a^3}\right)$.

13. a. $y = \frac{x^3 - 2x^2 + 4x}{x^2 - 4} = x - 2 + \frac{8x - 8}{x^2 - 4}$ (by division of polynomials). The graph has discontinuities at $x = \pm 2$.

$$\left.\begin{aligned}\lim_{x \to -2^-}\left(x - 2 + \frac{8x - 8}{x^2 - 4}\right) = -\infty \\ \lim_{x \to -2^+}\left(x - 2 + \frac{8x - 8}{x^2 - 4}\right) = -\infty\end{aligned}\right\}$$

$$\left.\begin{aligned}\lim_{x \to -2^-}\left(x - 2 + \frac{8x - 8}{x^2 - 4}\right) = -\infty \\ \lim_{x \to -2^+}\left(x - 2 + \frac{8x - 8}{x^2 - 4}\right) = -\infty\end{aligned}\right\}$$

When $x = 0$, $y = 0$.

Also, $y = \frac{x(x^2 - 2x + 4)}{x^2 - 4} = \frac{x[(x - 1)^2 + 3]}{x^2 - 4}$.

Since $(x - 1)^2 + 3 > 0$, the only x-intercept is $x = 0$.

Since $\lim_{x \to \infty} \frac{8x - 8}{x^2 - 4} = 0$, the curve approaches the value $x - 2$ as $x \to \infty$. This suggests that the line $y = x - 2$ is an oblique asymptote. It is verified by the limit $\lim_{x \to \infty}[x - 2 - f(x)] = 0$. Similarly, the curve approaches $y = x - 2$ as $x \to -\infty$.

$$\frac{dy}{dx} = 1 + \frac{8(x^2 - 4) - 8(x - 1)(2x)}{(x^2 - 4)^2}$$

$$= 1 - \frac{8(x^2 - 2x + 4)}{(x^2 - 4)^2}$$

We solve $\frac{dy}{dx} = 0$ to find critical values:

$8x^2 - 16x + 32 = x^4 - 8x^2 + 16$

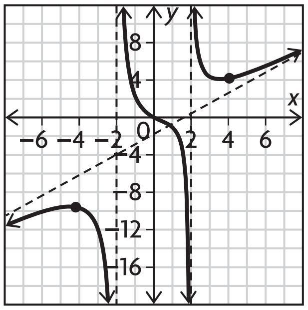
$x^4 - 16x^2 - 16 = 0$

$x^2 = 8 + 4\sqrt{5}$ $\left(8 - 4\sqrt{5}\right.$ is inadmissible$\left.\right)$

$x \doteq \pm 4.12.$

$\lim_{x \to \infty} y = \infty$ and $\lim_{x \to -\infty} y = -\infty$

Interval	$x < -4.12$	$x = -4.12$	$-4.12 < x < 2$	$-2 < x < 2$	$2 < x$ -4.12	$x = 4.12$	$x > 4.12$
$\frac{dy}{dx}$	>0	$=0$	<0	<0	<0	0	>0
Graph of y	Increasing	Local Max	Decreasing	Decreasing	Decreasing	Local Min	Increasing



b. Answers may vary. For example, there is a section of the graph that lies between the two sections of the graph that approach the asymptote.

14. For the various values of n, $f(x) = (x - c)^n$ has the following properties:

n	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$f'(x)$	1	$2(x - c)$	$3(x - c)^2$	$4(x - c)^3$
$f''(x)$	0	2	$6(x - c)$	$12(x - c)^2$
Infl. Pt.	None	None	$x = c$	$x = c$

It appears that the graph of f has an inflection point at $x = c$ when $n \geq 3$.

4.5 An Algorithm for Curve Sketching, pp. 212–213

1. A cubic polynomial that has a local minimum must also have a local maximum. If the local minimum is to the left of the local maximum, then $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$. If the local minimum is to the right of the local maximum, then $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

2. Since each local maximum and minimum of a function corresponds to a zero of its derivative, the number of zeroes of the derivative is the maximum number of local extreme values that the function can have. For a polynomial of degree n, the derivative has degree $n - 1$, so it has at most $n - 1$ zeroes, and thus at most $n - 1$ local extremes. A polynomial of degree three has at most 2 local extremes. A polynomial of degree four has at most 3 local extremes.

3. a. This function is discontinuous when

$$x^2 + 4x + 3 = 0$$
$$(x + 3)(x + 1) = 0$$

$x = -3$ or $x = -1$. Since the numerator is non-zero at both of these points, they are both equations of vertical asymptotes.

b. This function is discontinuous when

$x^2 - 6x + 12$

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(12)}}{2(1)}$$
$$= \frac{6 \pm \sqrt{-12}}{2}$$

This equation has no real solutions, so the function has no vertical asymptotes.

c. This function is discontinuous when

$$x^2 - 6x + 9 = 0$$
$$(x - 3)^2 = 0$$

$x = 3$. Since the numerator is non-zero at this point, it is the equation of a vertical asymptote.

4. a. $y = x^3 - 9x^2 + 15x + 30$

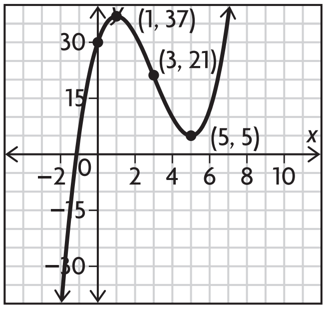
We know the general shape of a cubic polynomial with leading coefficient positive. The local extrema will help refine the graph.

$$\frac{dy}{dx} = 3x^2 - 18x + 15$$

Set $\frac{dy}{dx} = 0$ to find the critical values:

$$3x^2 - 18x + 15 = 0$$
$$x^2 - 6x + 5 = 0$$
$$(x - 1)(x - 5) = 0$$
$$x = 1 \text{ or } x = 5.$$

The local extrema are (1, 37) and (5, 5).



b. $f(x) = 4x^3 + 18x^2 + 3$

The graph is that of a cubic polynomial with leading coefficient negative. The local extrema will help refine the graph.

$$\frac{dy}{dx} = 12x^2 + 36x$$

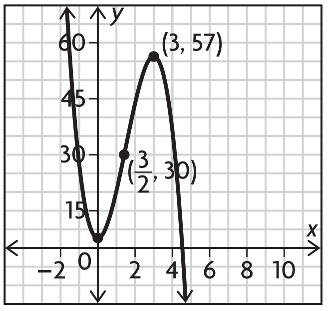
To find the critical values, we solve $\frac{dy}{dx} = 0$:

$$-12x(x - 3) = 0$$
$$x = 0 \text{ or } x = 3.$$

The local extrema are (0, 3) and (3, 57).

$$\frac{d^2y}{dx^2} = -24x + 36$$

The point of inflection is $\left(\frac{3}{2}, 30\right)$.

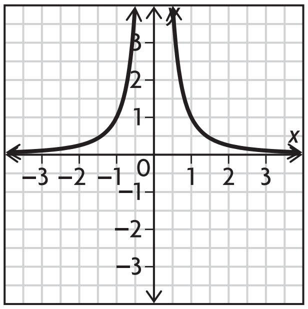


c. $y = 3 + \frac{1}{(x + 2)^2}$

We observe that $y = 3 + \frac{1}{(x + 2)^2}$ is just a translation of $y = \frac{1}{x^2}$.

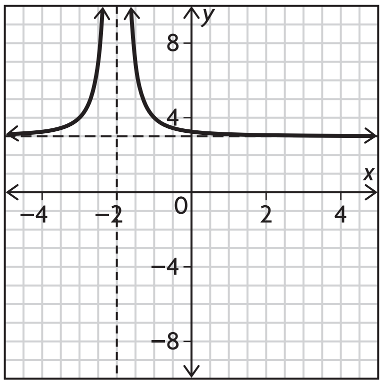
The graph of $y = \frac{1}{x^2}$ is

The reference point (0, 0) for $y = \frac{1}{x^2}$ becomes the point $(-2, 3)$ for $y = 3 + \frac{1}{(x + 2)^2}$. The vertical asymptote is $x = -2$, and the horizontal asymptote is $y = 3$.



$\frac{dy}{dx} = -\frac{2}{(x + 2)^3}$, hence there are no critical points.

$\frac{d^2y}{dx^2} = \frac{6}{(x + 2)^4} > 0$, hence the graph is always concave up.



d. $f(x) = x^4 - 4x^3 - 8x^2 + 48x$

We know the general shape of a fourth degree polynomial with leading coefficient positive. The local extrema will help refine the graph.

$f'(x) = 4x^3 - 12x^2 - 16x + 48$

For critical values, we solve $f'(x) = 0$

$x^3 - 3x^2 - 4x + 12 = 0.$

Since $f'(2) = 0$, $x - 2$ is a factor of $f'(x)$.

The equation factors are

$(x - 2)(x - 3)(x + 2) = 0.$

The critical values are $x = -2, 2, 3$.

$f''(x) = 12x^2 - 24x - 16$

Since $f''(-2) = 80 > 0$, $(-2, -80)$ is a local minimum.

Since $f''(2) = -16 < 0$, $(2, 48)$ is a local maximum.

Since $f''(3) = 20 > 0$, $(3, 45)$ is a local minimum.

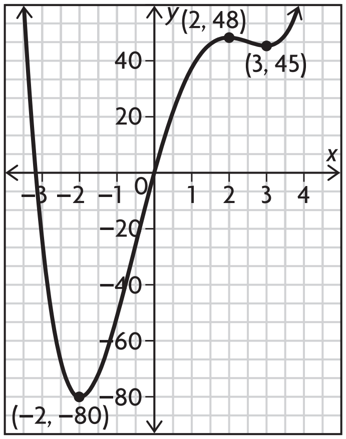
The graph has x-intercepts 0 and -3.2

The points of inflection can be found by solving $f''(x) = 0$:

$3x^2 - 6x - 4 = 0$

$x = \frac{6 \pm \sqrt{84}}{6}$

$x \doteq -\frac{1}{2}$ or $\frac{5}{2}$.



e. $y = \frac{2x}{x^2 - 25}$

There are discontinuities at $x = -5$ and $x = 5$.

$\lim_{x \to 5^-} \left(\frac{2x}{x^2 - 25}\right) = -\infty$ and $\lim_{\to -5^+} \left(\frac{2x}{x^2 - 25}\right) = \infty$

$\lim_{x \to 5^-} \left(\frac{2x}{x^2 - 25}\right) = -\infty$ and $\lim_{\to -5^+} \left(\frac{2x}{x^2 - 25}\right) = \infty$

$x = -5$ and $x = 5$ are vertical asymptotes.

$\frac{dy}{dx} = \frac{2(x^2 - 25) - 2x(2x)}{(x^2 - 25)^2} = -\frac{2x^2 + 50}{(x^2 - 25)^2} < 0$ for all x in the domain. The graph is decreasing throughout the domain.

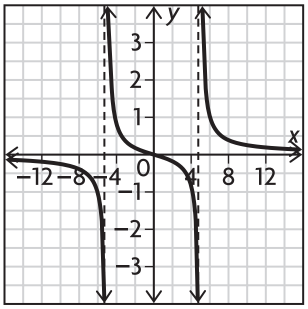
$$\left.\begin{aligned} \lim_{x \to \infty}\left(\frac{2x}{x^2 - 25}\right) &= \lim_{x \to \infty}\left(\frac{\frac{2}{x}}{1 - \frac{25}{x^2}}\right) \\ &= 0 \\ \lim_{x \to -\infty}\left(\frac{\frac{2}{x}}{1 - \frac{25}{x^2}}\right) &= 0 \end{aligned}\right\} \text{$y = 0$ is a horizontal asymptote.}$$

$$\frac{d^2y}{dx^2} = -\frac{4x(x^2-25)^2 - (2x^2+50)(2)(x^2-25)(2x)}{(x^2-25)^4}$$

$$= \frac{4x^3 + 300x}{(x^2-25)^3} = \frac{4x(x^2+75)}{(x^2-25)^3}$$

There is a possible point of inflection at $x = 0$.

Interval	$x < -5$	$-5 < x < 0$	$x = 0$	$0 < x < 5$	$x > 5$
$\frac{d^2y}{dx^2}$	<0	>0	$=0$	<0	>0
Graph of y	Concave Down	Point of Up	Concave Inflection	Point of Down	Concave Up



f. This function is discontinuous when

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0$ or $x = 4$. The numerator is non-zero at these points, so the function has vertical asymptotes at both of them. The behaviour of the function near these asymptotes is:

x-values	1	x	$x - 4$	$f(x)$	$\lim_{x\to c} f(x)$
$x \to 0^-$	> 0	< 0	< 0	> 0	$+\infty$
$x \to 0^+$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 4^-$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 4^+$	> 0	> 0	> 0	> 0	$+\infty$

To check for a horizontal asymptote:

$$\lim_{x\to\infty} \frac{1}{x^2 - 4x} = \lim_{x\to\infty} \frac{1}{x^2\left(1 - \frac{4}{x}\right)}$$

$$= \frac{\lim_{x\to\infty}(1)}{\lim_{x\to\infty}\left(x^2\left(1 - \frac{4}{x}\right)\right)}$$

$$= \frac{\lim_{x\to\infty}(1)}{\lim_{x\to\infty}(x^2) \times \lim_{x\to\infty}\left(1 - \frac{4}{x}\right)}$$

$$= \lim_{x\to\infty}\frac{1}{x^2} \times \frac{1}{1+0}$$

$$= 0$$

Similarly, $\lim_{x\to\infty} \frac{1}{x^2 - 4x} = 0$, so $y = 0$ is a horizontal asymptote of the function.

Since $y = 0$ and $x = 0$ are both asymptotes of the function, it has no x- or y- intercepts.

The derivative is

$$f'(x) = \frac{(x^2 - 4x) - (1)(2x - 4)}{(x^2 - 4x)^2}$$

$$= \frac{4 - 2x}{(x^2 - 4x)^2}$$, and the second derivative is

$$f''(x) = \frac{(x^2 - 4x)^2(-2) - (4 - 2x)(2(x^2 - 4x)(2x - 4))}{(x^2 - 4x)^4}$$

$$= \frac{-2x^2 + 8x + 8x^2 - 32x + 32}{(x^2 - 4x)^3}$$

$$= \frac{6x^2 - 24x + 32}{(x^2 - 4x)^3}$$

Letting $f'(x) = 0$ shows that $x = 2$ is a critical point of the function. The inflection points can be found by letting $f''(x) = 0$, so

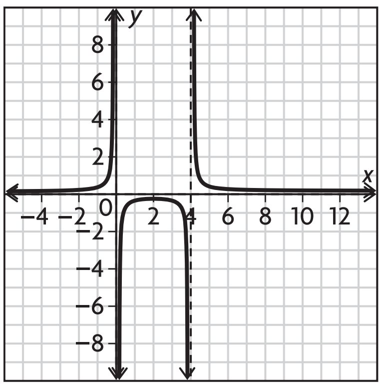
$2(3x^2 - 12x + 16) = 0$

$$x = \frac{12 \pm \sqrt{(-12)^2 - 4(3)(16)}}{2(3)}$$

$$= \frac{12 \pm \sqrt{-48}}{6}$$

This equation has no real solutions, so the graph of f has no inflection points.

x	$x < 0$	$0 < x < 2$	$x = 0$	$2 < x < 4$	$x > 4$
$f'(x)$	+	+	0	−	−
Graph	Inc.	Inc.	Local Max	Dec.	Dec.
$f''(x)$	+	−	−	−	+
Concavity	Up	Down	Down	Down	Up



g. $y = \dfrac{6x^2 - 2}{x^3}$

$= \dfrac{6}{x} - \dfrac{2}{x^3}$

There is a discontinuity at $x = 0$.

$\lim_{x \to 0^-} \dfrac{6x^2 - 2}{x^3} = \infty$ and $\lim_{x \to 0^+} \dfrac{6x^2 - 2}{x^3} = -\infty$

The y-axis is a vertical asymptote. There is no y-intercept. The x-intercept is a vertical asymptote. There is no y-intercept. The x-intercept is $\pm\frac{1}{\sqrt{3}}$.

$\dfrac{dy}{dx} = -\dfrac{6}{x^2} + \dfrac{6}{x^4} = \dfrac{-6x^2 + 6}{x^4}$

$\dfrac{dy}{dx} = 0$ when $6x^2 = 6$

$x = \pm 1$

Interval	$x < -1$	$x = -1$	$-1 < x < 0$	$0 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	<0	$=0$	>0	>0	$=0$	<0
Graph of $y = f(x)$	Decreasing	Local Min	Increasing	Increasing	Local Max	Decreasing

There is a local minimum at $(-1, -4)$ and a local maximum at $(1, 4)$.

$\dfrac{d^2y}{dx^2} = \dfrac{12}{x^3} = \dfrac{24}{x^3} = \dfrac{12x^2 - 24}{x^3}$

For possible points of inflection, we solve $\dfrac{d^2y}{dx^2} = 0$ $(x^3 \neq 0)$:

$12x^2 = 24$

$x = \pm\sqrt{2}.$

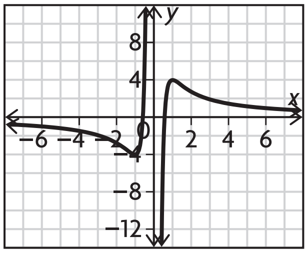
Interval	$x < -\sqrt{2}$	$x < -\sqrt{2}$	$-\sqrt{2} < x < 0$	$0 < x < \sqrt{2}$	$x = \sqrt{2}$	$x > \sqrt{2}$
$\frac{d^2y}{dx^2}$	<0	$=0$	>0	<0	$=0$	>0
Graph of $y = f(x)$	Concave Down	Point of Inflection	Concave Up	Concave Down	Point of Inflection	Convave Up

There are points of inflection at $\left(-\sqrt{2}, -\frac{5}{\sqrt{2}}\right)$ and $\left(\sqrt{2}, \frac{5}{\sqrt{2}}\right)$.

$\lim_{x \to \infty} \dfrac{6x^2 - 2}{x^3} = \lim_{x \to \infty} \dfrac{\frac{6}{x} - \frac{2}{x^3}}{1} = 0$

$\lim_{x = -\infty} \dfrac{\frac{6}{x} - \frac{2}{x^3}}{x} = 0$

The x-axis is a horizontal asymptote.



h. $y = \dfrac{x + 3}{x^2 - 4}$

There are discontinuities at $x = -2$ and at $x = 2$.

$\lim_{x \to -2^-} \left(\dfrac{x + 3}{x^2 - 4}\right) = \infty$ and $\lim_{x \to 2^-} \left(\dfrac{x + 3}{x^2 - 4}\right) = -\infty$

$\lim_{x \to 2^-} \left(\dfrac{x + 3}{x^2 - 4}\right) = -\infty$ and $\lim_{x \to 2^-} \left(\dfrac{x + 3}{x^2 - 4}\right) = \infty$

There are vertical asymptotes at $x = -2$ and $x = 2$.

When $x = 0$, $y = -\frac{3}{4}$. The x-intercept is -3.

$\dfrac{dy}{dx} = \dfrac{(1)(x^2 - 4) - (x + 3)(2x)}{(x^2 - 4)^2}$

$= \dfrac{-x^2 - 6x - 4}{(x^2 - 4)^2}$

For critical values, we solve $\dfrac{dy}{dx} = 0$:

$x^2 + 6x + 4 = 0$

$x = -\dfrac{6 \pm \sqrt{36 - 16}}{2}$

$= -3 \pm \sqrt{5}$

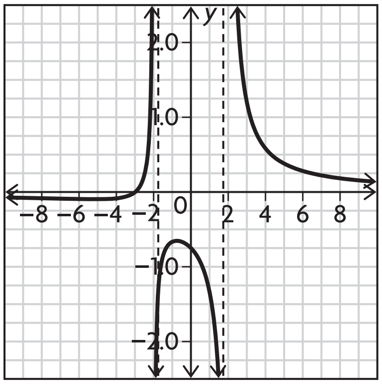
$\doteq -5.2$ or -0.8.

Interval	$x < -5.2$	$x = -5.2$	$-5.2 < x < -2$	$-2 < x < -0.8$	$x = -0.8$	$-0.8 < x < 2$	$x > 2$
$\frac{dy}{dx}$	< 0	$= 0$	> 0	> 0	$= 0$	< 0	< 0
Graph of y	Decreasing	Local Min	Increasing	Increasing	Local Max	Decreasing	Decreasing

$\lim_{x \to \infty} y = \lim_{x \to \infty} \left(\dfrac{\frac{1}{x} + \frac{3}{x^2}}{1 - \frac{4}{x^2}}\right) = 0$

$\lim_{x \to \infty} \left(\dfrac{\frac{1}{x} + \frac{3}{x^2}}{1 - \frac{4}{x^2}}\right) = 0$

The x-axis is a horizontal asymptote.



i. $y = \dfrac{x^2 - 3x + 6}{x - 1}$

$= x - 2 + \dfrac{4}{x - 1}$

$$\begin{array}{r} x - 2 \\ x - 1 \overline{)\, x^2 - 3x + 6} \\ \underline{x^2 - x} \\ -2x + 6 \\ \underline{-2x + 2} \\ 4 \end{array}$$

There is a discontinuity at $x = 1$.

$\lim\limits_{x \to 1^-} \left(\dfrac{x^2 - 3x + 6}{x - 1} \right) = -\infty$

$\lim\limits_{x \to 1^+} \left(\dfrac{x^2 - 3x + 6}{x - 1} \right) = \infty$

Thus, $x = 1$ is a vertical asymptote.

The y-intercept is -6.

There are no x-intercepts ($x^2 - 3x + 6 > 0$ for all x in the domain).

$\dfrac{dy}{dx} = 1 - \dfrac{4}{(x - 1)^2}$

For critical values, we solve $\dfrac{dy}{dx} = 0$:

$1 - \dfrac{4}{(x - 1)^2} = 0$

$(x - 1)^2 = 4$

$x - 1 = \pm 2$

$x = -1$ or $x = 3$.

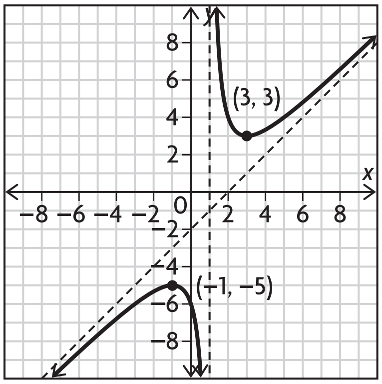
Interval	$x < -1$	$x = -1$	$-1 < x < 1$	$1 < x < 3$	$x = 3$	$x > 3$
$\frac{dy}{dx}$	>0	$=0$	<0	<0	$=0$	>0
Graph of y	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing

$\dfrac{d^2y}{d^2x} = \dfrac{8}{(x - 1)^3}$

For $x < 1$, $\dfrac{d^2y}{dx^2} < 0$ and y is always concave down.

For, $x > 1$, $\dfrac{d^2y}{dx^2} > 0$ and y is always concave up.

The line $y = x - 2$ is an oblique asymptote.



j. This function is continuous everywhere, so it has no vertical asymptotes. It also has no horizontal asymptote, because

$\lim\limits_{x \to \infty} (x - 4)^{\frac{2}{3}} = \infty$ and $\lim\limits_{x \to -\infty} (x - 4)^{\frac{2}{3}} = \infty$.

The x-intercept of the function is found by letting $f(x) = 0$, which gives

$(x - 4)^{\frac{2}{3}} = 0$

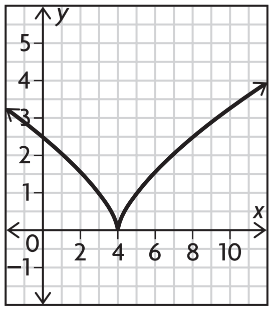
$x = 4$

The y-intercept is found by letting $x = 0$, which gives $f(0) = (0 - 4)^{\frac{2}{3}} \doteq 2.5$.

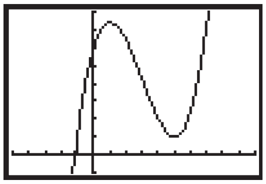
The derivative of the function is

$f'(x) = \left(\frac{2}{3}\right)(x - 4)^{-\frac{1}{3}}$ and the second derivative is $f''(x) = \left(-\frac{2}{9}\right)(x - 4)^{-\frac{4}{3}}$. Neither of these derivatives has a zero, but each is undefined for $x = 4$, so it is a critical value and a possible point of inflection.

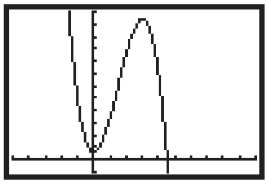
x	$x < 4$	$x = 4$	$x > 4$
$f'(x)$	$-$	Undefined	$+$
Graph	Dec.	Local Min	Inc.
$f''(x)$	$-$	Undefined	$-$
Concavity	Down	Undefined	Down



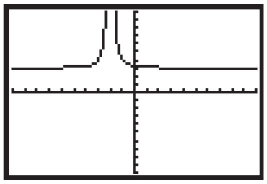
5. a.



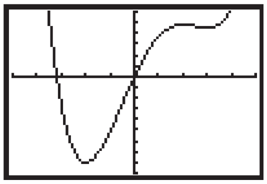
b.



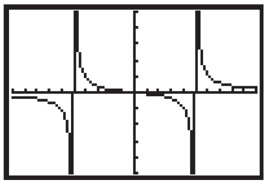
c.



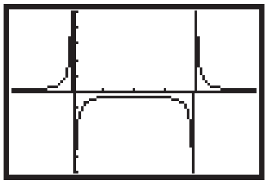
d.



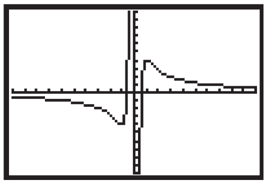
e.



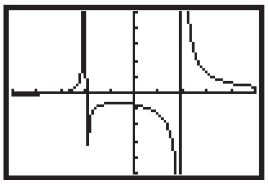
f.



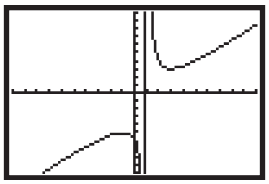
g.



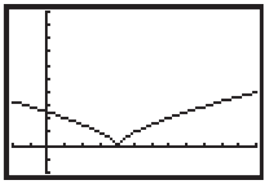
h.



i.



j.



6. $y = ax^3 + bx^2 + cx + d$

Since (0, 0) is on the curve $d = 0$:

$\frac{dy}{dx} = 3ax^2 + 2bx + c$

At $x = 2$, $\frac{dy}{dx} = 0$.

Thus, $12a + 4b + c = 0$.

Since (2, 4) is on the curve, $8a + 4b + 2c = 4$

or $4a + 2b + c = 2$.

$$\frac{d^2y}{dx^2} = 6ax + 2b$$

Since (0, 0) is a point of inflection, $\frac{d^2y}{dx^2} = 0$ when $x = 0$.

Thus, $2b = 0$

$b = 0$.

Solving for a and c:

$12a + c = 0$

$4a + c = 2$

$8a = -2$

$a = -\frac{1}{4}$

$c = 3$.

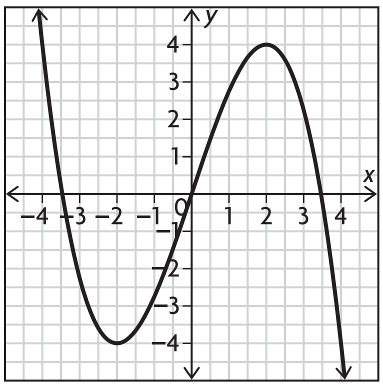
The cubic polynomial is $y = -\frac{1}{4}x^3 + 3x$.

The y-intercept is 0. The x-intercepts are found by setting $y = 0$:

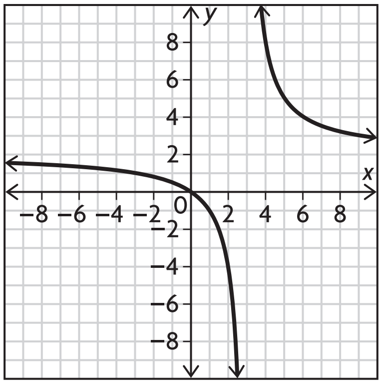
$-\frac{1}{4}x(x^2 - 12) = 0$

$x = 0, \quad \text{or} \quad x = \pm 2\sqrt{3}$.

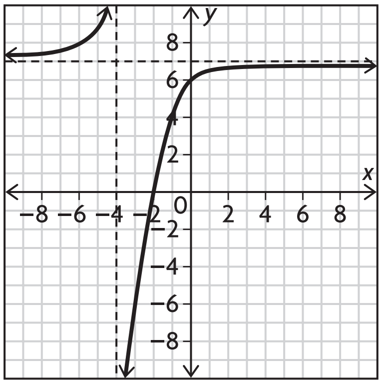
Let $y = f(x)$. Since $f(-x) = \frac{1}{4}x^3 - 3x = -f(x)$, $f(x)$ is an odd function. The graph of $y = f(x)$ is symmetric when reflected in the origin.



7. a. Answers may vary. For example:



b. Answers may vary. For example:



8. $f(x) = \dfrac{k - x}{k^2 + x^2}$

There are no discontinuities.

The y-intercept is $\dfrac{1}{k}$ and the x-intercept is k.

$$f'(x) = \frac{(-1)(k^2 + x^2) - (k - x)(2x)}{(k^2 + x^2)^2}$$
$$= \frac{x^2 - 2kx - k^2}{(k^2 + x^2)^2}$$

For critical points, we solve $f'(x) = 0$:

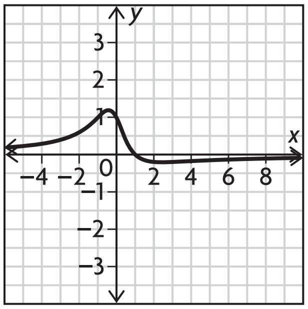
$$x^2 - 2kx - k^2 = 0$$
$$x^2 - 2kx - k^2 = 2k^2$$
$$(x - k)^2 = 2k^2$$
$$x - k = \pm\sqrt{2}k$$
$$x = \left(1 + \sqrt{2}\right)k \text{ or } x = \left(1 - \sqrt{2}\right)k.$$

Interval	$x < -0.41k$	$x = 0.41k$	$-0.41k < x < 2.41k$	$x = 2.41k$	$x > 2.41k$
$f(x)$	>0	<0	<0	$=0$	>0
Graph of $f(x)$	Increasing	Local Max	Decreasing	Local Min	Increasing

$$\lim_{x\to\infty}\left(\frac{k - x}{k^2 + x^2}\right) = \lim_{x\to\infty}\left(\frac{\frac{k}{x^2} - \frac{1}{x}}{\frac{k^2}{x^2} + 1}\right) = 0$$

$$\lim_{x\to\infty}\left(\frac{\frac{k}{x^2} - \frac{1}{x}}{\frac{k^2}{x^2} + 1}\right) = 0$$

Hence, the x-axis is a horizontal asymptote.



9. $g(x) = x^{\frac{1}{3}}(x + 3)^{\frac{2}{3}}$

There are no discontinuities.

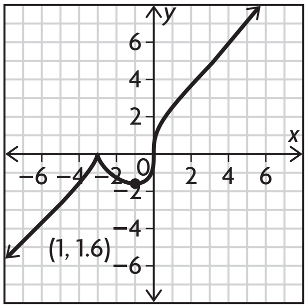
$$g'(x) = \frac{1}{3}x^{-\frac{2}{3}}(x + 3)^{\frac{2}{3}} + x^{\frac{1}{3}}\left(\frac{2}{3}\right)(x + 3)^{-\frac{1}{3}}(1)$$
$$= \frac{x + 3 + 2x}{3x^{\frac{2}{3}}(x + 3)^{\frac{1}{3}}} = \frac{3(x + 1)}{3x^{\frac{2}{3}}(x + 3)^{\frac{1}{3}}}$$
$$= \frac{x + 1}{3x^{\frac{2}{3}}(x + 3)^{\frac{1}{3}}}$$

$g'(x) = 0$ when $x = -1$.
$g'(x)$ doesn't exist when $x = 0$ or $x = -3$.

Interval	$x < -3$	$x = -3$	$-3 < x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$x < 0$
$g^2(x)$	>0	Does not Exist	<0	$=0$	>0	Does not Exist	>0
Graph of $g(x)$	Increasing	Local Max	Decreasing	Local Min	Increasing		Increasing

There is a local maximum at $(-3, 0)$ and a local minimum at $(-1, -1.6)$. The second derivative is algebraically complicated to find.

Interval	$x < -3$	$x = -3$	$-3 < x < 0$	$x = 0$	$x > 0$
$g''(x)$	>0	Does Not Exist	>0	Does Not Exist	>0
Graph $g''(x)$	Concave Down	Cusp	Concave Up	Point of Inflection	Concave Down



10. a. $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}$

$$= \frac{x}{|x|\sqrt{1 + \frac{1}{x^2}}}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x\sqrt{1 + \frac{1}{x^2}}}, \text{ since } x > 0$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1 + \frac{1}{x^2}}}$$

$$= 1$$

$y = 1$ is a horizontal asymptote to the right-hand branch of the graph.

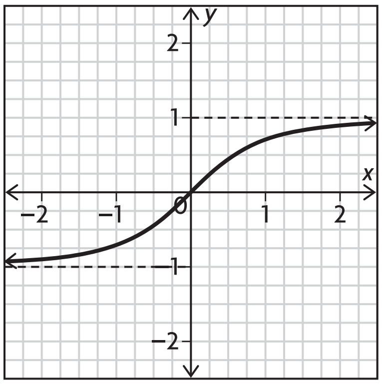
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{-x\sqrt{1 + \frac{1}{x^2}}}, \text{ since } |x| = -x$$

for $x < 0$

$$= \lim_{x\to-\infty} \frac{x}{-\sqrt{1 + \frac{1}{x^2}}}$$

$$= -1$$

$y = -1$ is a horizontal asymptote to the left-hand branch of the graph.



b. $g(t) = \sqrt{t^2 + 4t} - \sqrt{t^2 + t}$

$$= \frac{(\sqrt{t^2 + 4t} - \sqrt{t^2 + t})(\sqrt{t^2 + 4t} + \sqrt{t^2 + t})}{\sqrt{t^2 + 4t} + \sqrt{t^2 + t}}$$

$$= \frac{3t}{\sqrt{t^2 + 4t} + \sqrt{t^2 + t}}$$

$$= \frac{3t}{|t|\sqrt{1 + \frac{4}{t}} + |t|\sqrt{1 + \frac{1}{t}}}$$

$\lim_{x\to\infty} g(t) = {}^{3} = \frac{3}{2}$, since $|t| = t$ for $t > 0$

$\lim_{x\to-\infty} g(t) = \frac{3}{-1-1} = -\frac{3}{2}$, since $|t| = -t$ for $t < 0$

$y = \frac{3}{2}$ and $y = -\frac{3}{2}$ are horizontal asymptotes.

11. $y = ax^3 + bx^2 + cx + d$

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

$$\frac{d^2y}{dx^2} = 6ax + 2b = 6a\left(x + \frac{b}{3a}\right)$$

For possible points of inflection, we solve $\frac{d^2y}{dx^2} = 0$:

$x = -\dfrac{b}{3a}$.

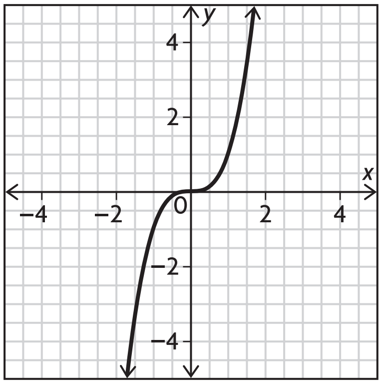
The sign of $\frac{d^2y}{dx^2}$ changes as x goes from values less than $-\frac{b}{3a}$ to values greater than $-\frac{b}{3a}$. Thus, there is a point of inflection at $x = -\frac{b}{3a}$.

At $x = \dfrac{b}{3a}$, $\dfrac{dy}{dx} = 3a\left(-\dfrac{b}{3a}\right)^2 + 2b\left(-\dfrac{b}{3a}\right) + c$

$$= c - \frac{b^2}{3a}.$$

Review Exercise, pp. 216–219

1. a. i. $x < 1$
ii. $x > 1$
iii. (1, 20)
b. i. $x < -3$, $-3 < x < 1$, $x > 6.5$
ii. $1 < x < 3$, $3 < x < 6.5$
iii. $(1, -1)$, $(6.5, -1)$
2. No. A counter example is sufficient to justify the conclusion. The function $f(x) = x^3$ is always increasing yet the graph is concave down for $x < 0$ and concave up for $x > 0$.



3. a. $f(x) = -2x^3 + 9x^2 + 20$
$f'(x) = -6x^2 + 18x$
For critical values, we solve:
$f'(x) = 0$
$-6x(x - 3) = 0$
$x = 0$ or $x = 3$.
$f''(x) = -12x + 18$
Since $f''(0) = 18 > 0$, (0, 20) is a local minimum point. The tangent to the graph of $f(x)$ is horizontal at (0, 20). Since $f''(3) = -18 < 0$, (3, 47) is a local maximum point. The tangent to the graph of $f(x)$ is horizontal at (3, 47).
b. $f(x) = x^4 - 8x^3 + 18x^2 + 6$
$f(x) = 4x^3 - 24x^2 + 36x$
$f(x) = 4x(x^2 - 6x + 9)$
$f(x) = 4x(x - 3)^2$
Let $f(x) = 0$:
$4x(x - 3)^2 = 0$
$x = 0$ or $x = 3$
The critical points are $(0, 6)$ and $(3, 33)$.

x	$x < 0$	0	$0 < x < 3$	3	$x > 3$
$\frac{dy}{dx}$	−	0	+	0	+
Graph	Dec	Local Min	Inc		Inc

Local minimum at $(0, 6)$
$(3, 33)$ is neither a local maximum nor a local minimum.
The tangent is parallel to the x-axis at both points because the derivative is defined at both points.
c. $h(x) = \dfrac{x - 3}{x^2 + 7}$

$$h(x) = \frac{(1)(x^2 + 7) - (x - 3)(2x)}{(x^2 + 7)^2}$$
$$= \frac{7 + 6x - x^2}{(x^2 + 7)^2}$$
$$= \frac{(7 - x)(1 + x)}{(x^2 + 7)^2}$$

Since $x^2 + 7 > 0$ for all x, the only critical values occur when $h'(x) = 0$. The critical values are $x = 7$ and $x = -1$.

Interval	$x < -1$	$x = -1$	$-1 < x < 7$	$x = 7$	$x > 7$
$h'(x)$	< 0	$= 0$	> 0	$= 0$	< 0
Graph of $h(t)$	Decreas-ing	Local Min	Increas-ing	Local Max	Decreas-ing

There is a local minimum at $(-1, -\frac{1}{2})$ and a local maximum at $(7, \frac{1}{14})$. At both points, the tangents are parallel to the x-axis.
d) $g(x) = (x - 1)^{\frac{1}{3}}$

$$g'(x) = \frac{1}{3}(x - 1)^{\frac{-2}{3}}$$

Let $g'(x) = 0$:

$$\frac{1}{3}(x - 1)^{\frac{-2}{3}} = 0$$

There are no solutions, but $g'(x)$ is undefined for $x = 1$, so the point $(1, 0)$ is a critical point.

x	$x < 1$	1	$x > 1$
$f'(x)$	+	Undefined	+
Graph	Inc.		Inc.

$(1, 0)$ is neither a local maximum nor a local minimum.
The tangent is not parallel to the x-axis because it is not defined for $x = 1$.
4. a. $a < x < b$, $x > e$
b. $b < x < c$
c. $x < a$, $d < x < e$
d. $c < x < d$
5. a. $y = \dfrac{2x}{x - 3}$
There is a discontinuity at $x = 3$.

$$\lim_{x \to 3^-}\left(\frac{2x}{x - 3}\right) = -\infty \text{ and } \lim_{x \to 3^+}\left(\frac{2x}{x - 3}\right) = \infty$$

Therefore, $x = 3$ is a vertical asymptote.
b. $g(x) = \dfrac{x - 5}{x + 5}$
There is a discontinuity at $x = -5$.

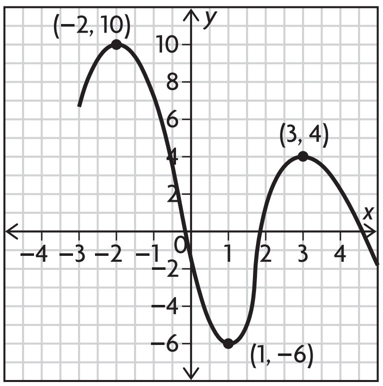
$$\lim_{x \to -5^-}\left(\frac{x - 5}{x + 5}\right) = \infty \text{ and } \lim_{x \to -5^+}\left(\frac{x - 5}{x + 5}\right) = -\infty$$

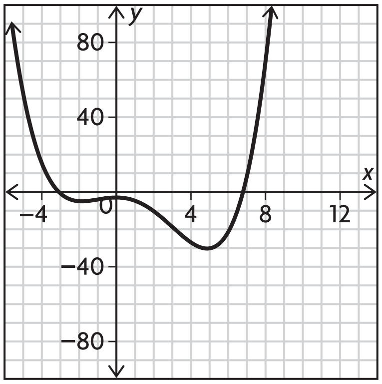
Therefore, $x = -5$ is a vertical asymptote.
c. $f(x) = \dfrac{x^2 - 2x - 15}{x + 3}$

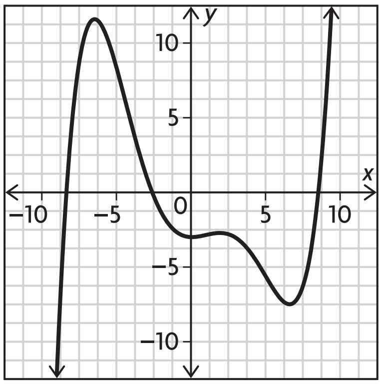
$$= \frac{(x + 3)(x - 5)}{x + 3}$$
$$= x - 5, x \neq -3$$
There is a discontinuity at $x = -3$.
$\lim_{x \to -3^+} f(x) = -8$ and $\lim_{x \to -3^-} f(x) = -8$
There is a hole in the graph of $y = f(x)$ at $(-3, -8)$.

d. $g(x) = \frac{5}{x^2 - x - 20}$
$$g(x) = \frac{5}{(x - 5)(x + 4)}$$
To find vertical asymptotes, set the denominator equal to 0:
$(x - 5)(x + 4) = 0$
$x = -4$ or $x = 5$
Vertical asymptotes at $x = -4$ and $x = 5$
$$\lim_{x \to -4^-} \frac{5}{(x - 5)(x + 4)} = \infty$$
$$\lim_{x \to -4^+} \frac{5}{(x - 5)(x + 4)} = -\infty$$
$$\lim_{x \to 5^-} \frac{5}{(x - 5)(x + 4)} = -\infty$$
$$\lim_{x \to 5^+} \frac{5}{(x - 5)(x + 4)} = \infty$$

6. $y = x^3 + 5$
$y' = 3x^2$
$y'' = 6x$
Let $y'' = 0$
$6x = 0$
$x = 0$
The point of inflection is $(0, 5)$
Since the derivative is 0 at $x = 0$, the tangent line is parallel to the x-axis at that point. Because the derivative is always positive, the function is always increasing and therefore must cross the tangent line instead of just touching it.

7.


8. a. i. Concave up: $-1 < x < 3$
Concave down: $x < -1, 3 < x$
ii. Points of inflection at $x = -1$ and $x = 3$
iii.


b. i. Concave up: $-4.5 < x < 1, 5 < x$
Concave down: $x < -4.5, 1 < x < 5$
ii. Points of inflection at $x = -4.5$, $x = 1$, and $x = 5$
iii.


9. a. $g(x) = \frac{ax + b}{(x - 1)(x - 4)}$
$$= \frac{ax + b}{x^2 - 5x + 4}$$
$$g'(x) = \frac{a(x^2 - 5x + 4) - (ax + b)(2x - 5)}{(x^2 - 5x + 4)^2}$$
Since the tangent at $(2, -1)$ has slope 0, $g'(2) = 0$.
Hence, $\frac{-2a + 2a + b}{4} = 0$ and $b = 0$.
Since $(2, -1)$ is on the graph of $g(x)$:
$$-1 = \frac{2a + b}{-2}$$
$2a + 0 = 2$
$a = 1$.
Therefore $g(x) = \frac{x}{(x - 1)(x - 4)}$.

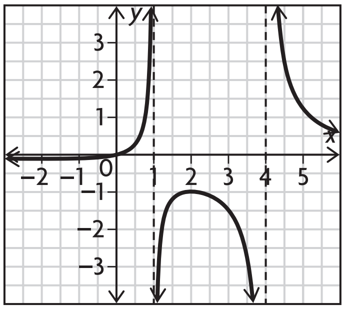
b. There are discontinuities at $x = 1$ and $x = 4$.
$\lim_{x \to 1^-} g(x) = \infty$ and $\lim_{x \to 1^+} g(x) = -\infty$
$\lim_{x \to 4^-} g(x) = -\infty$ and $\lim_{x \to 4^+} g(x) = \infty$
$x = 1$ and $x = 4$ are vertical asymptotes.
The y-intercept is 0.

$$g'(x) = \frac{4 - x^2}{(x^2 - 5x + 4)^2}$$

$g'(x) = 0$ when $x = \pm 2$.

Interval	$x < -2$	$x = -2$	$-2 < x < 1$	$1 < x < 2$	$x = 2$	$2 < x < 4$	$x > 4$
$g'(x)$	< 0	0	> 0	> 0	0	< 0	< 0
Graph of $g(x)$	Decreasing	Local Min	Increasing	Increasing	Local Max	Decreasing	Decreasing

There is a local minimum at $\left(-2, -\frac{1}{9}\right)$ and a local maximum at $(2, -1)$.



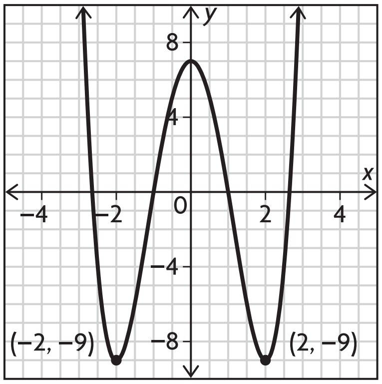
10. a. $y = x^4 - 8x^2 + 7$
This is a fourth degree polynomial and is continuous for all x. The y-intercept is 7.

$$\frac{dy}{dx} = 4x^3 - 16x$$
$$= 4x(x - 2)(x + 2)$$

The critical values are $x = 0$, -2 and 2.

Interval	$x < -2$	$x = -2$	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
$\frac{dy}{dx}$	< 0	$= 0$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of y	Decreasing	Local Min	Increasing	Local Max	Decreasing	Local Min	Increasing

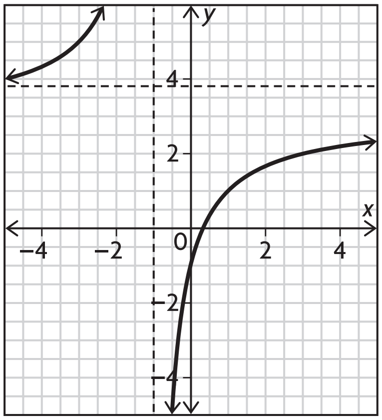
There are local minima at $(-2, -9)$ and at $(2, -9)$, and a local maximum at $(0, 7)$.



b. $f(x) = \dfrac{3x - 1}{x + 1}$

$$= 3 - \frac{4}{x + 1}$$

From experience, we know the graph of $y = -\frac{1}{x}$ is



The graph of the given function is just a transformation of the graph of $y = -\frac{1}{x}$. The vertical asymptote is $x = -1$ and the horizontal asymptote is $y = 3$. The y-intercept is -1 and there is an x-intercept at $\frac{1}{3}$.

c. $g(x) = \dfrac{x^2 + 1}{4x^2 - 9}$

$$= \frac{x^2 + 1}{(2x - 3)(2x + 3)}$$

The function is discontinuous at $x = -\frac{3}{2}$ and at $x = \frac{3}{2}$.

$\lim_{x \to -\frac{3}{2}^-} g(x) = \infty$

$\lim_{x \to -\frac{3}{2}^+} g(x) = -\infty$

$\lim_{x \to \frac{3}{2}^-} g(x) = -\infty$

$\lim_{x \to -\frac{3}{2}^+} g(x) = \infty$

Hence, $x = -\frac{3}{2}$ and $x = \frac{3}{2}$ are vertical asymptotes.
The y-intercept is $-\frac{1}{9}$.

$$g'(x) = \frac{2x(4x^2 - 9) - (x^2 + 1)(8x)}{(4x^2 - 9)^2} = \frac{-26x}{(4x^2 - 9)^2}$$

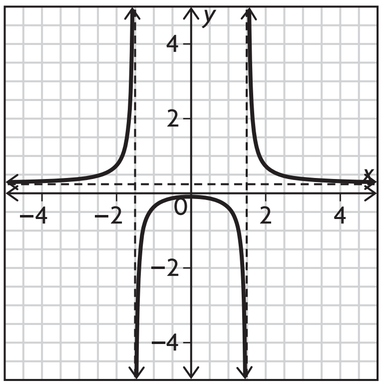
$g'(x) = 0$ when $x = 0$.

Interval	$x < -\frac{3}{2}$	$-\frac{3}{2} < x < 0$	$x = 0$	$0 < x < \frac{3}{2}$	$x > \frac{3}{2}$
$g'(x)$	> 0	> 0	$= 0$	< 0	< 0
Graph $g(x)$	Increasing	Increasing	Local Max	Decreasing	Decreasing

There is a local maximum at $\left(0, -\frac{1}{9}\right)$.

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{1 + \frac{1}{x^2}}{4 - \frac{9}{x^2}} = \frac{1}{4} \text{ and } \lim_{x\to\infty} g(x) = \frac{1}{4}$$

Hence, $y = \frac{1}{4}$ is a horizontal asymptote.



d) $y = x(x - 4)^3$
This is a polynomial function, so there are no discontinuities and no asymptotes. The domain is $\{x \in \mathbf{R}\}$.
x-intercepts at $x = 0$ and $x = 4$
y-intercepts at $y = 0$

$y' = (x - 4)^3 + 3x(x - 4)^2$

$y' = (x - 4)^2(x - 4 + 3x)$

$y' = 4(x - 4)^2(x - 1)$

Let $y' = 0$:

$4(x - 4)^2(x - 1) = 0$

$x = 4$ or $x = 1$

The critical numbers are $(1, -27)$ and $(4, 0)$.

x	$x < 1$	1	$1 < x < 4$	4	$x > 4$
$\frac{dy}{dx}$	−	0	+	0	+
Graph	Dec	Local Min	Inc		Inc

Local minimum at $(1, -27)$
$(4, 0)$ is not a local extremum

$y'' = 4(2(x - 4)(x - 1) + (x - 4)^2)$

$y'' = 4\left(2(x - 4)\left(x - 1 + \frac{x - 4}{2}\right)\right)$

$y'' = 8(x - 4)\left(\frac{3}{2}x - 3\right)$

Let $y'' = 0$:

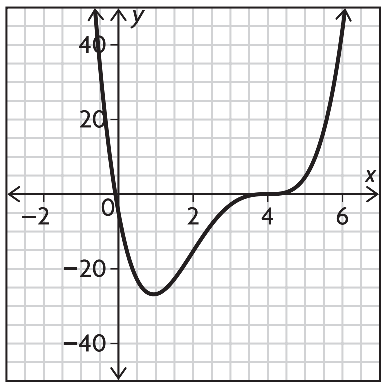
$8(x - 4)\left(\frac{3}{2}x - 3\right) = 0$

$x = 4$ or $x = 2$

The points of inflection are $(2, -16)$ and $(4, 0)$.

x	$x < 2$	2	$2 < x < 4$	4	$x > 4$
$\frac{dy}{dx}$	+	0	−	0	+
Graph	c. up	point of inflection	c. down	point of inflection	c. up

The graph has a local minimum at $(1, -27)$ and points of inflection at $(2, -16)$ and $(4, 0)$, with x-intercepts of 0 and 4 and a y-intercept of 0.



e. $h(x) = \dfrac{x}{x^2 - 4x + 4}$

$= \dfrac{x}{(x - 2)^2} = x(x - 2)^{-2}$

There is a discontinuity at $x = 2$

$\lim_{x\to 2^-} h(x) = \infty = \lim_{x\to 2^+} h(x)$

Thus, $x = 2$ is a vertical asymptote. The y-intercept is 0.

$$h'(x) = (x-2)^{-2} + x(-2)(x-2)^{-3}(1)$$
$$= \frac{x - 2 - 2x}{(x-2)^3}$$
$$= \frac{-2 - x}{(x-2)^3}$$

$h'(x) = 0$ when $x = -2$.

Interval	$x < -2$	$x = -2$	$-2 < x < 2$	$x > 2$
$h'(x)$	< 0	$= 0$	> 0	< 0
Graph of $h(x)$	Decreasing	Local Min	Increasing	Decreasing

There is a local minimum at $\left(-2, -\frac{1}{8}\right)$.

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} = \frac{\frac{1}{x}}{1 - \frac{4}{x} + \frac{4}{x^2}} = 0$$

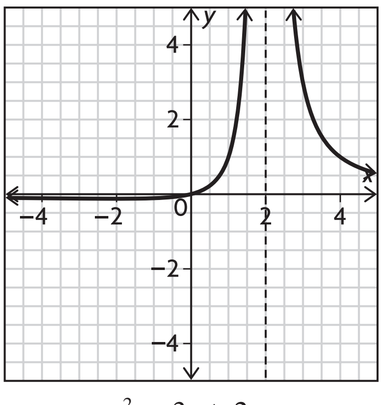
Similarly, $\lim_{x\to\infty} h(x) = 0$

The x-axis is a horizontal asymptote.

$$h''(x) = -2(x-2)^{-3} - 2(x-2)^{-3} + 6x(x-2)^{-4}$$
$$= -4(x-2)^{-3} + 6x(x-2)^{-4}$$
$$= \frac{2x + 8}{(x-2)^4}$$

$h''(x) = 0$ when $x = -4$

The second derivative changes signs on opposite sides $x = -4$, Hence $(-4, -\frac{1}{9})$ is a point of inflection.



f. $f(t) = \dfrac{t^2 - 3t + 2}{t - 3}$
$$= t + \frac{2}{t-3}$$

Thus, $f(t) = t$ is an oblique asymptote. There is a discontinuity at $t = 3$.

$\lim_{t\to 3^-} f(t) = -\infty$ and $\lim_{t\to 3^+} f(t) = \infty$

Therefore, $x = 3$ is a vertical asymptote.

The y-intercept is $-\frac{2}{3}$.

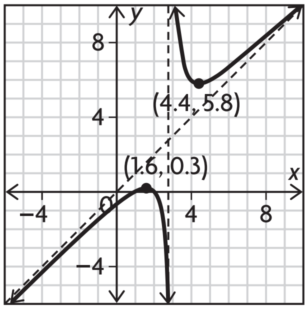
The x-intercepts are $t = 1$ and $t = 2$.

$$f'(t) = 1 - \frac{2}{(t-3)^2}$$

$f'(t) = 0$ when $1 - \dfrac{2}{(t-3)^2} = 0$
$$(t-3)^2 = 2$$
$$t - 3 = \pm\sqrt{2}$$
$$t = 3 \pm \sqrt{2}.$$

Interval	$t < 3 - \sqrt{2}$	$t = 3 - \sqrt{2}$	$3 - \sqrt{2} < t < 3$	$3 < t < 3 + \sqrt{2}$	$t = 3 + \sqrt{2}$	$t > 3 + \sqrt{2}$
$f'(t)$	> 0	$= 0$	< 0	< 0	$= 0$	> 0
Graph of $f(t)$	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing

(1.6, 0.2) is a local maximum and (4.4, 5.8) is a local minimum.



11. a. $f(x) = \dfrac{2x + 4}{x^2 - k^2}$

$$f'(x) = \frac{2(x^2 - k^2) - (2x+4)(2x)}{(x^2 - k^2)^2}$$
$$= -\frac{2x^2 + 8x + 2k^2}{(x^2 - k^2)^2}$$

For critical values, $f'(x) = 0$ and $x \neq \pm k$:

$x^2 + 4x + k^2 = 0$

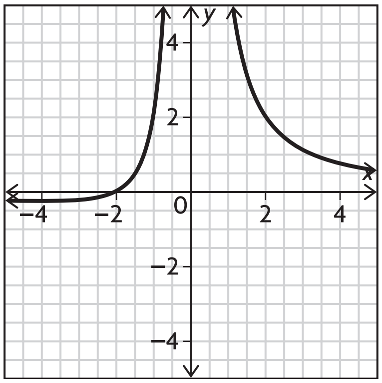
$x = \dfrac{-4 \pm \sqrt{16 - 4k^2}}{2}.$

For real roots, $16 - 4k^2 \geq 0$
$$-2 \leq k \leq 2.$$

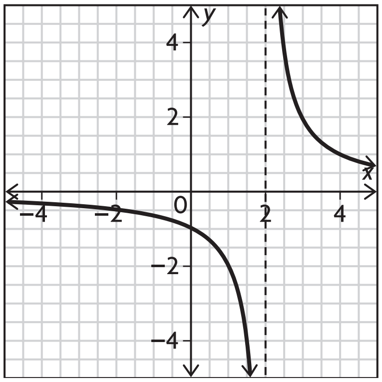
The conditions for critical points to exist are $-2 \leq k \leq 2$ and $x \neq \pm k$.

b. There are three different graphs that results for values of k chosen.

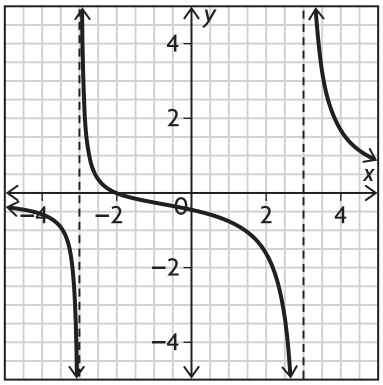
$k = 0$



$k = 2$



For all other values of k, the graph will be similar to that of $1(i)$ in Exercise 9.5.



12. a. $f(x) = \dfrac{2x^2 - 7x + 5}{2x - 1}$

$$f(x) = x - 3 + \frac{2}{2x - 1}$$

The equation of the oblique asymptote is $y = x - 3$.

$$\begin{array}{r}
x - 3 \\
2x - 1\overline{)2x^2 - 7x + 5} \\
\underline{2x^2 - x} \quad\quad \\
-6x + 5 \\
\underline{-6x + 3} \\
2
\end{array}$$

$$\lim_{x\to\infty}[y - f(x)] = \lim_{x\to\infty}\left[x - 3 - \left(x - 3 + \frac{2}{2x - 1}\right)\right]$$
$$= \lim_{x\to\infty}\left[-\frac{2}{2x - 1}\right] = 0$$

b. $f(x) = \dfrac{4x^3 - x^2 - 15x - 50}{x^2 - 3x}$

$$f(x) = 4x + 11 + \frac{18x - 50}{x^2 - 3x}$$

$$\begin{array}{r}
4x + 11 \\
x^2 - 3x\overline{)4x^3 - x^2 - 15x - 50} \\
\underline{4x^3 - 12x^2} \quad\quad\quad\quad \\
11x^2 - 15x \quad\quad \\
\underline{11x^2 - 33x} \quad\quad \\
18x - 50
\end{array}$$

$$\lim_{x\to\infty}[y - f(x)]$$
$$= \lim_{x\to\infty}\left[4x + 11 - \left(4x + 11 + \frac{18x - 50}{x^2 - 3x}\right)\right]$$
$$= \lim_{x\to\infty}\left[\frac{\frac{18}{x} - \frac{50}{x^2}}{1 - \frac{3}{x}}\right]$$
$$= 0$$

13. $g(x) = (x^2 - 4)^2$

$g(x) = (x^2 - 4)(x^2 - 4)$

$g'(x) = 2x(x^2 - 4) + 2x(x^2 - 4)$

$g'(x) = 4x(x^2 - 4)$

$g'(x) = 4x(x - 2)(x + 2)$

Set $g'(x) = 0$

$0 = 4x(x - 2)(x + 2)$

$x = -2$ or $x = 0$ or $x = 2$

	$x < -2$	$-2 < x < 0$	$0 < x < 2$	$x > 2$
$4x$	$-$	$-$	$+$	$+$
$x - 2$	$-$	$-$	$-$	$+$
$x + 2$	$-$	$+$	$+$	$+$
Sign of $g'(x)$	$(-)(-)(-) = -$	$(-)(-)(+) = +$	$(+)(-)(+) = -$	$(+)(+)(+) = +$
Behaviour of $g(x)$	decreasing	increasing	decreasing	increasing

14. $f(x) = x^3 + \dfrac{3}{2}x^2 - 7x + 5, -4 \le x \le 3$

$f'(x) = 3x^2 + 3x - 7$

Set $f'(x) = 0$

$0 = 3x^2 + 3x - 7$

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(3)(-7)}}{2(3)}$$

$$x = \frac{-3 \pm \sqrt{93}}{6}$$
$x \doteq -2.107$ or $x \doteq 1.107$
$f'(x) = 3x^2 + 3x - 7$
$f'(x) = 6x + 3$
When $x = -2.107$,
$f'(-2.107) = 6(-2.107) + 3$
$f'(-2.107) = -9.642$
Since $f''(-2.107) < 0$, a local maximum occurs when $x = -2.107$.
when $x = 1.107$,
$f''(1.107) = 6(1.107) + 3$
$f''(1.107) = 9.642$
Since $f''(1.107) > 0$, a local minimum occurs when $x = (1.107)$.
when $x = -4$,
$$f(-4) = (-4)^3 + \frac{3}{2}(-4)^2 - 7(-4) + 5$$
$f(-4) = -64 + 24 + 28 + 5$
$f(-4) = -7$
when $x = -2.107$,
$$f(-2.107) = (-2.107)^3 + \frac{3}{2}(-2.107)^2 - 7(-2.107) + 5$$
$$f(-2.107) \doteq -9.353\,919 + 6.659\,173\,5 + 14.749 + 5$$
when $x = 1.107$,
$$f(1.107) = (1.107)^3 + \frac{3}{2}(1.107)^2 - 7(1.107) + 5$$
$f(1.107) \doteq 1.356\,572 + 1.838\,173\,5 - 7.749 + 5$
$f(1.107) \doteq 0.446$
when $x = 3$,
$$f(3) = (3)^3 + \frac{3}{2}(3)^2 - 7(3) + 5$$
$f(3) = 27 + 13.5 - 21 + 5$
$f(3) = 24.5$
Local Maximum: $(-2.107, 17.054)$
Local Minimum: $(1.107, 0.446)$
Absolute Maximum: $(3, 24.5)$
Absolute Minimum: $(-4, -7)$

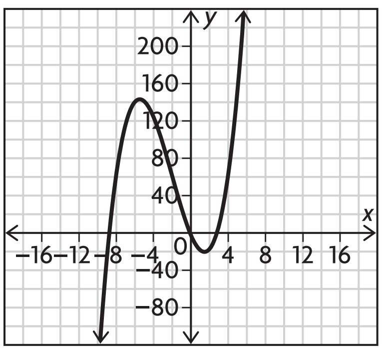
15. $f(x) = 4x^3 + 6x^2 - 24x - 2$
Evaluate $y = 4(0)^3 + 6(0)^2 - 24(0) - 2$
$y = -2$
$f(x) = 4x^3 + 6x^2 - 24x - 2$
$f'(x) = 12x^2 + 12x - 24$
Set $f'(x) = 0$
$0 = 12x^2 + 12x - 24$
$0 = 12(x^2 + x - 2)$
$0 = 12(x - 1)(x + 2)$
$x = -2$ or $x = 1$

	$x < -2$	$-2 < x < 1$	$x > 1$
$12(x - 1)$	$-$	$-$	$+$
$x + 2$	$-$	$+$	$+$
Sign of $f'(x)$	$(-)(-) = +$	$(-)(+) = -$	$(+)(+) = +$
Behaviour of $f(x)$	increasing	decreasing	increasing
	maximum at $x = -2$	minimum at $x = 1$	

when $x = -2$,
$f(-2) = 4(-2)^3 + 6(-2)^2 - 24(-2) - 2$
$f(-2) = -32 + 24 + 48 - 2$
$f(-2) = 38$
when $x = 1$,
$f(1) = 4(1)^3 + 6(1)^2 - 24(1) - 2$
$f(1) = 4 + 6 - 24 - 2$
$f(1) = -16$
Maximum: $(-2, 38)$ Minimum: $(1, -16)$
$f'(x) = 12x^2 + 12x - 24$
$f''(x) = 24x + 12$
Set $f''(x) = 0$
$0 = 24x + 12$
$x = -0.5$

	$x < -0.5$	$x > -0.5$
$f''(x) = 24x + 12$	$-$	$+$
$f(x)$	concave down	concave up
	point of inflection at $x = -0.5$	

when $x = -0.5$,
$f(-0.5) = 4(-0.5)^3 + 6(-0.5)^2 - 24(-0.5) - 2$
$f(-0.5) = -0.5 + 1.5 + 12 - 2$
$f(-0.5) = 11$
Point of inflection: $(-0.5, 11)$



16. a. $p(x)$: oblique asymptote, because the highest degree of x in the numerator is exactly one degree higher than the highest degree of x in the denominator.
$q(x)$: vertical asymptotes at $x = -1$ and $x = 3$; horizontal asymptote at $y = 0$
$r(x)$: vertical asymptotes at $x = -1$ and $x = 1$; horizontal asymptote at $y = 1$
$s(x)$: vertical asymptote at $y = 2$.

b. $r(x) = \dfrac{x^2 - 2x - 8}{x^2 - 1}$
$= \dfrac{(x-4)(x+2)}{(x-1)(x+1)}$

The domain is $\{x \mid x \neq -1, 1, x \in \mathbf{R}\}$.
x-intercepts: $-2, 4$; y-intercept: 8
r has vertical asymptotes at $x = -1$ and $x = 1$.
$r(-1.001) = -2496.75$, so as $x \to -1^-$, $r(x) \to -\infty$
$r(-0.999) = 2503.25$, so as $x \to -1^+$, $r(x) \to \infty$
$r(0.999) = 4502.25$, so as $x \to 1^-$, $r(x) \to \infty$
$r(1.001) = -4497.75$, so as $x \to 1^+$, $r(x) \to -\infty$

$$\lim_{x \to -\infty} \frac{x^2 - 2x - 8}{x^2 - 1} = \lim_{x \to -\infty} \frac{\frac{x^2}{x^2} - \frac{2x}{x^2} - \frac{8}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$$
$$= \lim_{x \to -\infty} \frac{1 - \frac{2}{x} - \frac{8}{x^2}}{1 - \frac{1}{x^2}}$$
$$= \frac{1 - 0 - 0}{1 - 0}$$
$$= 1$$

So, $y = 1$ is a horizontal asymptote on the left.

$$\lim_{x \to \infty} \frac{x^2 - 2x - 8}{x^2 - 1} = \lim_{x \to \infty} \frac{\frac{x^2}{x^2} - \frac{2x}{x^2} - \frac{8}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$$
$$= \lim_{x \to \infty} \frac{1 - \frac{2}{x} - \frac{8}{x^2}}{1 - \frac{1}{x^2}}$$
$$= \frac{1 - 0 - 0}{1 - 0}$$
$$= 1$$

So, $y = 1$ is a horizontal asymptote on the right.

$$r'(x) = \frac{(x^2 - 1)(2x - 2) - (x^2 - 2x - 8)(2x)}{(x^2 - 1)^2}$$
$$= \frac{2x^3 - 2x^2 - 2x + 2 - (2x^3 - 4x^2 - 16x)}{(x^2 - 1)^2}$$
$$= \frac{2x^2 + 14x + 2}{(x^2 - 1)^2}$$
$$= \frac{2(x^2 + 7x + 1)}{(x^2 - 1)^2}$$

r' is defined for all values of x in the domain of r. $r'(x) = 0$ for $x \doteq -0.15$ and $x \doteq -6.85$. $r'(1)$ and $r'(-1)$ do not exist.

	$x < -6.85$	$x = -6.85$	$-6.85 < x < -1$
$x^2 + 7x + 1$	+	0	−
$r'(x)$	+	0	−
	$x = -1$	$-1 < x < -0.15$	$x = -0.15$
$x^2 + 7x + 1$	−	−	0
$r'(x)$	undefined	−	0
	$-0.15 < x < 1$	$x = 1$	$x > 1$
$x^2 + 7x + 1$	+	+	+
$r'(x)$	+	undefined	+

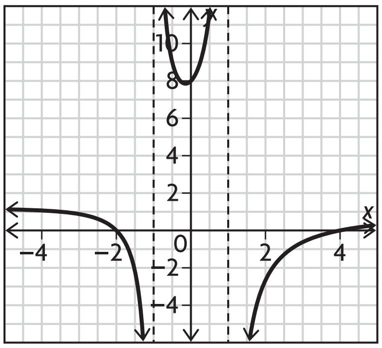
r is increasing when $x < -6.85$, $-0.15 < x < 1$, and $x > 1$. r is decreasing when $-6.85 < x < -1$ and $-1 < x < -0.15$. r has a maximum turning point at $x = -6.85$ and a minimum turning point at $x = -0.15$.

$$r''(x) = \frac{(x^2 - 1)^2(4x + 14)}{(x^2 - 1)^4} - \frac{(2x^2 + 14x + 2)[2(x^2 - 1)(2x)]}{(x^2 - 1)^4}$$
$$= \frac{(x^2 - 1)(4x + 14) - 4x(2x^2 + 14x + 2)}{(x^2 - 1)^3}$$
$$= \frac{4x^3 + 14x^2 - 4x - 14 - 8x^3 - 56x^2 - 8x}{(x^2 - 1)^3}$$
$$= \frac{-4x^3 - 42x^2 - 12x - 14}{(x^2 - 1)^3}$$
$$= \frac{-2(2x^3 + 21x^2 + 6x + 7)}{(x^2 - 1)^3}$$

r'' is defined for all values of x in the domain of r. $r''(x) = 0$ for $x \doteq -10.24$. This is a possible point of inflection. $r''(1)$ and $r''(-1)$ do not exist.

	$x < -10.24$	$x = 10.24$
$-2(2x^3 + 21x^2 + 6x + 7)$	+	0
$(x^2 - 1)^3$	+	+
$r''(x)$	+	0
	$-10.24 < x < -1$	$x = -1$
$-2(2x^3 + 21x^2 + 6x + 7)$	−	−
$(x^2 - 1)^3$	+	0
$r''(x)$	−	undefined
	$-1 < x < 1$	$x = 1$
$-2(2x^3 + 21x^2 + 6x + 7)$	−	−
$(x^2 - 1)^3$	−	0
$r''(x)$	+	undefined
	$x > 1$	
$-2(2x^3 + 21x^2 + 6x + 7)$	−	
$(x^2 - 1)^3$	+	
$r''(x)$	−	

The graph is concave up for $x < -10.24$ and $-1 < x < 1$. The graph is concave down for $-10.24 < x < -1$ and $x > 1$. The graph changes concavity at $x = -10.24$. This is a point of inflection with coordinates $(-10.24, 1.13)$.
$r(-6.85) = 1.15$ and $r(-0.15) = 7.85$. The graph has a local maximum point at $(-6.85, 1.15)$ and a local minimum point at $(-0.15) = 7.85$.



17. The domain is $\{x | x \neq 0, x \in \mathbf{R}\}$: x-intercept: -2, y-intercept: 8; f has a vertical asymptote at $x = 0$. $f(-0.001) = -7999.99$, so $f(x) \to -\infty$ as $x \to 0^-$. $f(0.001) = 8000.00$, so $f(x) \to \infty$ as $x \to 0^+$. There are no horizontal asymptotes.

$$f'(x) = \frac{x(3x^2) - (x^3 + 8)(1)}{x^2} = \frac{3x^3 - x^3 - 8}{x^2} = \frac{2x^3 - 8}{x^2}$$

$f'(x)$ is defined for all values of x in the domain of f. $f'(x) = 0$ when $x = 1.59$. $f'(0)$ does not exist.

	$x < 0$	$x = 0$	$0 < x < 1.59$
$2x^3 - 8$	−	−	−
x^2	+	0	+
$f'(x)$	−	undefined	−
	$x = 1.59$	$x > 1.59$	
$2x^3 - 8$	0	+	
x^2	+	+	
$f'(x)$	0	+	

f is increasing for $x > 1.59$ and decreasing for $x < 0$ and $0 < x < 1.59$. f has a minimum turning point at $x = 1.59$.

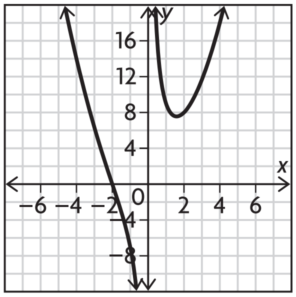
$$f''(x) = \frac{x^2(6x^2) - (2x^3 - 8)(2x)}{x^4} = \frac{x(6x^2) - (2x^3 - 8)2}{x^3} = \frac{6x^3 - 4x^3 + 16}{x^3} = \frac{2x^3 + 16}{x^3}$$

f'' is defined for all values of x in the domain of f. $f''(x) = 0$ when $x = -2$. This is a possible point of inflection. $f(0)$ does not exist.

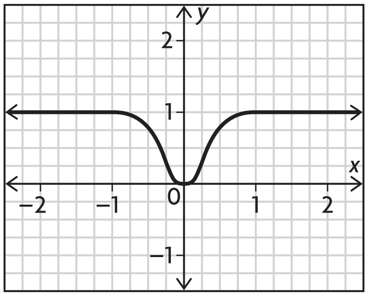
	$x < -2$	$x = -2$	$-2 < x < 0$
$2x^3 + 16$	−	0	+
x^3	−	−	−
$f''(x)$	+	0	−
	$x = 0$	$x > 0$	
$2x^3 + 16$	+	+	
x^3	0	+	
$f''(x)$	undefined	+	

f is concave up when $x < -2$ and $x > 0$. f is concave down when $-2 < x < 0$. The graph changes

concavity where $x = -2$. This is point of inflection with coordinates $(-2, 0)$.
$f(1.59) \doteq 7.56$. The graph has a local minimum at $(1.59, 7.56)$.



18. If $f(x)$ is increasing, then $f'(x) > 0$. From the graph of f', $f'(x) > 0$ for $x > 0$. If $f(x)$ is decreasing, then $f'(x) < 0$. From the graph of f', $f'(x) < 0$ for $x < 0$. At a stationary point, $f'(x) = 0$. From the graph, the zero for $f'(x)$ occurs at $x = 0$. At $x = 0$. $f'(x)$ changes from negative to positive, so f has a local minimum point there.
If the graph of f is concave up, then f'' is positive. From the slope of f', the graph of f is concave up for $-0.6 < x < 0.6$. If the graph of f is concave down, then f'' is negative. From the slope of f', the graph of f is concave down for $x < -0.6$ and $x > 0.6$. Graphs will vary slightly.



19. $f'(x) = \dfrac{(x-1)^2(5) - 5x(2)(x-1)(1)}{(x-1)^4}$

$$= \frac{5(x-1) - 10x}{(x-1)^3}$$

$$= \frac{-5x-5}{(x-1)^3}$$

$$= \frac{-5(x+1)}{(x-1)^3}$$

$$f''(x) = \frac{(x-1)^3(-5)}{(x-1)^6} - \frac{(-5x-5)(3)(x-1)^2(1)}{(x-1)^6}$$

$$= \frac{(x-1)(-5) - 3(-5x-5)}{(x-1)^4}$$

$$= \frac{10x-20}{(x-1)^4}$$

$$= \frac{10(x-2)}{(x-1)^4}$$

The domain is $\{x | x \neq 1, x \in \mathbf{R}\}$. The x- and y-intercepts are both 0. f has a vertical asymptote at $x = 1$.
$f(0.999) = 4\,995\,000$ so as $x \to 1^-$, $f(x) \to \infty$
$f(1.001) = 5\,005\,000$ so as $x \to 1^+$, $f(x) \to \infty$

$\lim\limits_{x\to-\infty} \dfrac{5x}{x^2 - 2x + 1} = 0$ $y = 0$ is a horizontal asymptote on the right.

$\lim\limits_{x\to-\infty} \dfrac{5x}{x^2 - 2x + 1} = 0$ $y = 0$ is a horizontal asymptote on the left.

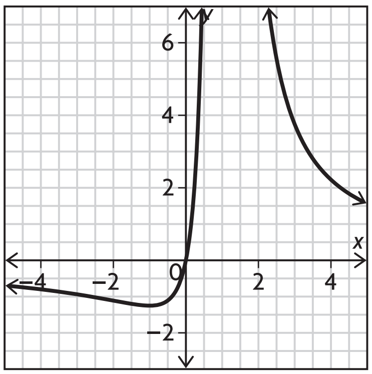
$f'(x)$ is defined for all values of x in the domain of f. $f'(x) = 0$ when $x = -1$. $f(1)$ does not exist.

	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$-5(x+1)$	+	0	−	−	−
$(x-1)^3$	−	−	−	0	+
$f'(x)$	−	0	+	undefined	−

f is decreasing when $x < -1$ and $x > 1$. f is increasing when $-1 < x < 1$. f has a minimum turning point at $x = -1$.
$f''(x)$ is defined for all values of x in the domain of f. $f''(x) = 0$ when $x = -3$. This is a possible point of inflection.
$f(1)$ does not exist.

	$x < -2$	$x = -2$	$-2 < x < 1$	$x = 1$	$x > 1$
$x + 2$	−	0	+	+	+
$f''(x)$	−	0	+	undefined	+

The graph is concave down for $x < -2$ and concave up when $-2 < x < 1$ and $x > 1$. It changes concavity at $x = -2$. f has an inflection point at $x = -2$ with coordinates $(-2, -1.11)$.
$f(-1) = -1.25$. f has a local minimum at $(-1, -1.25)$.



20. a. Graph A is f, graph C is f', and graph B is f''. We know this because when you take the derivative, the degree of the denominator increases by one. Graph A has a squared term in the denominator, graph C has a cubic term in the denominator, and graph B has a term to the power of four in the denominator.

b. Graph F is f, graph E is f' and graph D is f''. We know this because the degree of the denominator increases by one degree when the derivative is taken.

Chapter 4 Test, p. 220

1. a. $x < -9$ or $-6 < x < -3$ or $0 < x < 4$ or $x > 8$

b. $-9 < x < -6$ or $-3 < x < 0$ or $4 < x < 8$

c. $(-9, 1)$, $(-6, -2)$, $(0, 1)$, $(8, -2)$

d. $x = -3$, $x = 4$

e. $f''(x) > 0$

f. $-3 < x < 0$ or $4 < x < 8$

g. $(-8, 0)$, $(10, -3)$

2. a. $g(x) = 2x^4 - 8x^3 - x^2 + 6x$

$g'(x) = 8x^3 - 24x^2 - 2x + 6$

To find the critical points, we solve $g'(x) = 0$:

$8x^3 - 24x^2 - 2x + 6 = 0$

$4x^3 - 12x^2 - x + 3 = 0$

Since $g'(3) = 0$, $(x - 3)$ is a factor.

$(x - 3)(4x^2 - 1) = 0$

$x = 3$ or $x = -\frac{1}{2}$ or $x = \frac{1}{2}$.

Note: We could also group to get

$4x^2(x - 3) - (x - 3) = 0$.

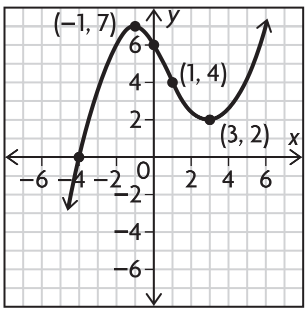
b. $g''(x) = 24x^2 - 48x - 2$

Since $g''\left(-\frac{1}{2}\right) = 28 > 0$, $\left(-\frac{1}{2}, -\frac{17}{8}\right)$ is a local maximum.

Since $g''\left(-\frac{1}{2}\right) = -20 < 0$, $\left(\frac{1}{2}, \frac{15}{8}\right)$ is a local maximum.

Since $g''(3) = 70 > 0$, $(3, -45)$ is a local minimum.

3.



4. $g(x) = \dfrac{x^2 + 7x + 10}{(x - 3)(x + 2)}$

The function $g(x)$ is not defined at $x = -2$ or $x = 3$. At $x = -2$, the value of the numerator is 0. Thus, there is a discontinuity at $x = -2$, but $x = -2$ is not a vertical asymptote.

At $x = 3$, the value of the numerator is 40. $x = 3$ is a vertical asymptote.

$$g(x) = \frac{(x + 2)(x + 5)}{(x - 3)(x + 2)} = \frac{x + 5}{x - 3}, x \neq -2$$

$$\lim_{x \to -2^-} g(x) = \lim_{x \to -2^-} \left(\frac{x + 5}{x - 3}\right) = -\frac{3}{5}$$

$$\lim_{x \to -2^+} g(x) = \lim_{x \to -2^+} \left(\frac{x + 5}{x - 3}\right) = -\frac{3}{5}$$

There is a hole in the graph of $g(x)$ at $\left(-2, -\frac{3}{5}\right)$.

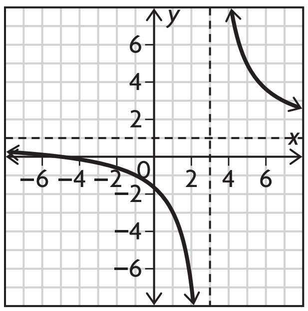
$$\lim_{x \to 3^-} g(x) = \lim_{x \to 3^-} \left(\frac{x + 5}{x - 3}\right) = -\infty$$

$$\lim_{x \to 3^+} g(x) = \lim_{x \to 3^+} \left(\frac{x + 5}{x - 3}\right) = \infty$$

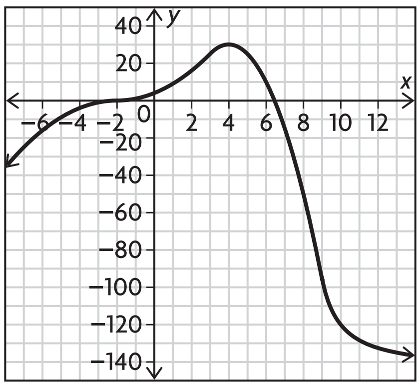
There is a vertical asymptote at $x = 3$.

Also, $\lim_{x \to \infty} g(x) = \lim_{x \to -\infty} g(x) = 1$.

Thus, $y = 1$ is a horizontal asymptote.



5.



6. $f(x) = \dfrac{2x + 10}{x^2 - 9}$

$= \dfrac{2x + 10}{(x - 3)(x + 3)}$

There are discontinuities at $x = -3$ and at $x = 3$.

$\left.\begin{array}{l}\lim\limits_{x \to 3^-} f(x) = \infty \\ \lim\limits_{x \to 3^+} f(x) = -\infty\end{array}\right\} x = -3$ is a vertical asymptote.

$\left.\begin{array}{l}\lim\limits_{x \to 3^-} f(x) = -\infty \\ \lim\limits_{x \to 3^+} f(x) = \infty\end{array}\right\} x = 3$ is a vertical asymptote.

The y-intercept is $-\frac{10}{9}$ and $x = -5$ is an x-intercept.

$f'(x) = \dfrac{2(x^2 - 9) - (2x + 10)(2)}{(x^2 - 9)^2}$

$= \dfrac{-2x^2 - 20x - 18}{(x^2 - 9)^2}$

For critical values, we solve $f'(x) = 0$:

$x^2 + 10x + 9 = 0$

$(x + 1)(x + 9) = 0$

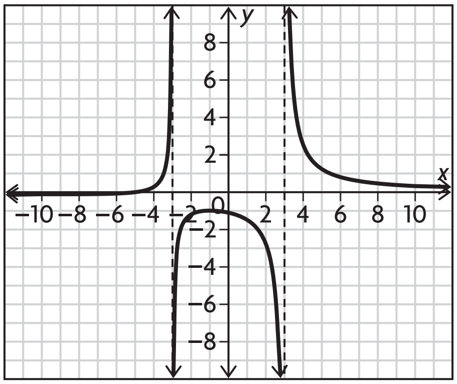
$x = -1$ or $x = -9$.

$\left(-9, -\frac{1}{9}\right)$ is a local minimum and $(-1, -1)$ is a local maximum.

$\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \dfrac{\frac{2}{x} + \frac{10}{x^2}}{1 - \frac{9}{x^2}} = 0$ and

$\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \left(\dfrac{\frac{2}{x} + \frac{10}{x^2}}{1 - \frac{9}{x^2}}\right) = 0$

$y = 0$ is a horizontal asymptote.



7. $f(x) = x^3 + bx^2 + c$

$f'(x) = 3x^2 + 2bx$

Since $f'(-2) = 0$, $12 - 4b = 0$

$b = 3$.

Also, $f(-2) = 6$.

Thus, $-8 + 12 + c = 6$

$c = 2$.

$f'(x) = 3x^2 + 6x$

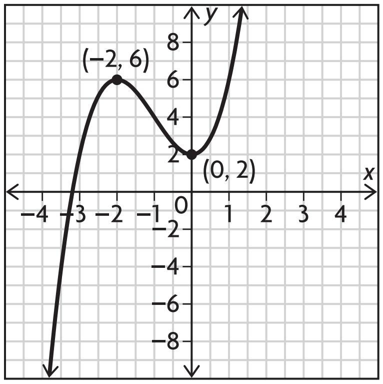
$= 3x(x + 2)$

The critical points are $(-2, 6)$ and $(0, 2)$.

$f''(x) = 6x + 6$

Since $f''(-2) = -6 < 0$, $(-2, 6)$ is a local maximum.

Since $f'(0) = 6 > 0$, $(0, 2)$ is a local minimum.



Chapter 4
Curve Sketching

Review of Prerequisite Skills, pp. 162–163

1. a. $2y^2 + y - 3 = 0$
$(2y + 3)(y - 1) = 0$
$y = -\frac{3}{2}$ or $y = 1$

b. $x^2 - 5x + 3 = 17$
$x^2 - 5x - 14 = 0$
$(x - 7)(x + 2) = 0$
$x = 7$ or $x = -2$

c. $4x^2 + 20x + 25 = 0$
$(2x + 5)(2x + 5) = 0$
$x = -\frac{5}{2}$

d. $y^3 + 4y^2 + y - 6 = 0$
$y = 1$ is a zero, so $y - 1$ is a factor. After synthetic division, the polynomial factors to $(y - 1)(y^2 + 5y + 6)$.
So $(y - 1)(y + 3)(y + 2) = 0$.
$y = 1$ or $y = -3$ or $y = -2$

2. a. $3x + 9 < 2$
$3x < -7$
$x < -\frac{7}{3}$

b. $5(3 - x) \geq 3x - 1$
$15 - 5x \geq 3x - 1$
$16 \geq 8x$
$8x \leq 16$
$x \leq 2$

c. $t^2 - 2t < 3$
$t^2 - 2t - 3 < 0$
$(t - 3)(t + 1) < 0$
Consider $t = 3$ and $t = -1$.

t values	$t < -1$	$-1 < t < 3$	$t > 3$
$(t + 1)$	−	+	+
$(t - 3)$	−	−	+
$(t - 3)(t + 1)$	+	−	+

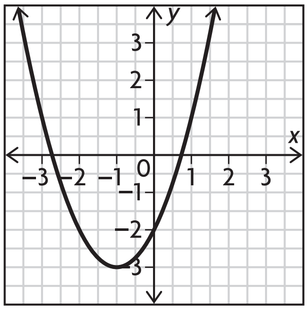
The solution is $-1 < t < 3$.

d. $x^2 + 3x - 4 > 0$
$(x + 4)(x - 1) > 0$
Consider $x = -4$ and $x = 1$.

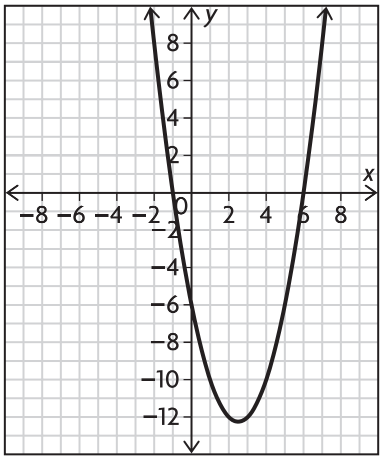
x values	$x < -4$	$-4 < x < 1$	$x > 1$
$(x + 4)$	−	+	+
$(x - 1)$	−	−	+
$(x + 4)(x - 1)$	+	−	+

The solution is $x < -4$ or $x > 1$.

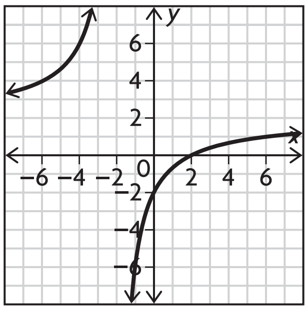
3. a.



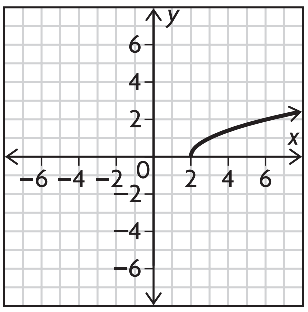
b.



c.



d.



4. a. $\lim_{x \to 2^-} (x^2 - 4) = 2^2 - 4$
$= 0$

b. $\lim_{x \to 2} \frac{x^2 + 3x - 10}{x - 2}$
$= \lim_{x \to 2} \frac{(x + 5)(x - 2)}{x - 2}$
$= \lim_{x \to 2} (x + 5)$
$= 7$

c. $\lim_{x \to 3} \frac{x^3 - 27}{x - 3}$
$= \lim_{x \to 3} \frac{(x - 3)(x^2 + 3x + 9)}{x - 3}$
$= \lim_{x \to 3} (x^2 + 3x + 9)$
$= 3^2 + 3 \times 3 + 9$
$= 27$

d. $\lim_{x \to 4^+} \sqrt{2x + 1}$
$= \sqrt{2 \times 4 + 1}$
$= 3$

5. a. $f(x) = \frac{1}{4}x^4 + 2x^3 - \frac{1}{x}$
$= \frac{1}{4}x^4 + 2x^3 - x^{-1}$
$f'(x) = x^3 + 6x^2 + x^{-2}$

b. $f(x) = \frac{x + 1}{x^2 - 3}$
$f'(x) = \frac{(x^2 - 3)(1) - (x + 1)(2x)}{(x^2 - 3)^2}$
$= \frac{x^2 - 3 - 2x^2 - 2x}{(x^2 - 3)^2}$
$= \frac{-x^2 - 2x - 3}{(x^2 - 3)^2}$
$= -\frac{x^2 + 2x + 3}{(x^2 - 3)^2}$

c. $f(x) = (3x^2 - 6x)^2$
$f'(x) = 2(3x^2 - 6x)(6x - 6)$

d. $f(t) = \frac{2t}{\sqrt{t - 4}}$
$f'(t) = \frac{2\sqrt{t - 4} - \frac{2t}{2\sqrt{t - 4}}}{t - 4}$
$f'(t) = \frac{\frac{4(t - 4)}{2\sqrt{t - 4}} - \frac{2t}{2\sqrt{t - 4}}}{t - 4}$
$f'(t) = \frac{4(t - 4) - 2t}{2(t - 4)^{\frac{3}{2}}}$
$= \frac{2t - 16}{2(t - 4)^{\frac{3}{2}}}$
$= \frac{t - 8}{(t - 4)^{\frac{3}{2}}}$

6. a.

$$
\begin{array}{r}
x - 8 \\
x + 3 \overline{\smash{)}\, x^2 - 5x + 4} \\
\underline{x^2 + 3x} \\
-8x + \ \ 4 \\
\underline{-8x - 24} \\
28
\end{array}
$$

$(x^2 - 5x - 4) \div (x + 3) = x - 8 + \frac{28}{x + 3}$

b.

$$
\begin{array}{r}
x + 7 \\
x - 1 \overline{\smash{)}\, x^2 + 6x - 9} \\
\underline{x^2 - x} \\
7x - 9 \\
\underline{7x - 7} \\
-2
\end{array}
$$

$(x^2 - 6x - 9) \div (x - 1) = x + 7 - \frac{2}{x - 1}$

7. $f(x) = x^3 + 0.5x^2 - 2x + 3$
$f'(x) = 3x^2 + x - 2$
Let $f'(x) = 0$:
$3x^2 + x - 2 = 0$
$(3x - 2)(x + 1) = 0$
$x = \frac{2}{3}$ or $x = -1$

The points are $\left(\frac{2}{3}, 2.19\right)$ and $(-1, 4.5)$.

8. a. If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$.

b. If $f(x) = k$, where k is a constant, then $f'(x) = 0$.

c. If $k(x) = f(x)g(x)$, then
$k'(x) = f'(x)g(x) - f(x)g'(x)$

d. If $h(x) = \frac{f(x)}{g(x)}$, then $h'(x)$
$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$, $g(x) \neq 0$.

e. If f and g are functions that have derivatives, then the composite function $h(x) = f(g(x))$ has a derivative given by $h'(x) = f'(g(x))g'(x)$.
f. If u is a function of x, and n is a positive integer, then $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$

9. a. $\lim_{x\to\infty} 2x^2 - 3x + 4 = \infty$

$\lim_{x\to-\infty} 2x^2 - 3x + 4 = \infty$

b. $\lim_{x\to\infty} 2x^3 + 4x - 1 = \infty$

$\lim_{x\to-\infty} 2x^3 + 4x - 1 = -\infty$

c. $\lim_{x\to\infty} -5x^4 + 2x^3 - 6x^2 + 7x - 1 = -\infty$

$\lim_{x\to-\infty} -5x^4 + 2x^3 - 6x^2 + 7x - 1 = -\infty$

10. a. $\frac{1}{f(x)} = \frac{1}{2x}$
Let $2x = 0$
$x = 0$, so the graph has a vertical asymptote at $x = 0$.
b. $\frac{1}{f(x)} = \frac{1}{-x + 3}$
Let $-x + 3 = 0$
$x = 3$, so the graph has a vertical asymptote at $x = 3$.
c. $\frac{1}{f(x)} = \frac{1}{(x + 4)^2 + 1}$
Let $(x + 4)^2 + 1 = 0$
There is no solution, so the graph has no vertical asymptotes.
d. $\frac{1}{f(x)} = \frac{1}{(x + 3)^2}$
Let $(x + 3)^2 = 0$
$x = -3$, so the graph has a vertical asymptote at $x = -3$.

11. a. $\lim_{x\to\infty} \frac{5}{x + 1} = 0$, so the horizontal asymptote is $y = 0$.
b. $\lim_{x\to\infty} \frac{4x}{x - 2} = 4$, so the horizontal asymptote is $y = 4$.
c. $\lim_{x\to\infty} \frac{3x - 5}{6x - 3} = \frac{1}{2}$, so the horizontal asymptote is $y = \frac{1}{2}$.
d. $\lim_{x\to\infty} \frac{10x - 4}{5x} = 2$, so the horizontal asymptote is $y = 2$.

12. a. i. $y = \frac{5}{x + 1}$
To find the x-intercept, let $y = 0$.
$\frac{5}{x + 1} = 0$
There is no solution, so there is no x-intercept.
The y-intercept is $y = \frac{5}{0 + 1} = 5$.
ii. $y = \frac{4x}{x - 2}$
To find the x-intercept, let $y = 0$.
$\frac{4x}{x - 2} = 0$
$x = 0$
The y-intercept is $y = \frac{0}{0 - 2} = 0$.
iii. $y = \frac{3x - 5}{6x - 3}$
To find the x-intercept, let $y = 0$:
$\frac{3x - 5}{6x - 3} = 0$
Therefore, $3x - 5 = 0$
$x = \frac{5}{3}$
The y-intercept is $y = \frac{0 - 5}{0 - 3} = \frac{5}{3}$.
iv. $y = \frac{10x - 4}{5x}$
To find the x-intercept, let $y = 0$.
$\frac{10x - 4}{5x} = 0$
Therefore, $10x - 4 = 0$
$x = \frac{2}{5}$
The y-intercept is $y = \frac{0 - 4}{0}$, which is undefined, so there is no y-intercept.
b. i. $y = \frac{5}{x + 1}$
Domain: $\{x \in \mathbf{R} | x \neq -1\}$
Range: $\{y \in \mathbf{R} | y \neq 0\}$
ii. $y = \frac{4x}{x - 2}$
Domain: $\{x \in \mathbf{R} | x \neq 2\}$
Range: $\{y \in \mathbf{R} | y \neq 4\}$
iii. $y = \frac{3x - 5}{6x - 3}$
Domain: $\left\{x \in \mathbf{R} | x \neq \frac{1}{2}\right\}$
Range: $\left\{y \in \mathbf{R} | y \neq \frac{1}{2}\right\}$
iv. $y = \frac{10x - 4}{5x}$
Domain: $\{x \in \mathbf{R} | x \neq 0\}$
Range: $\{y \in \mathbf{R} | y \neq 2\}$

4.1 Increasing and Decreasing Functions, pp. 169–171

1. a. $f(x) = x^3 + 6x^2 + 1$

$f'(x) = 3x^2 + 12x$

Let $f'(x) = 0$: $3x(x + 4) = 0$

$x = 0$ or $x = -4$

The points are $(0, 1)$ and $(-4, 33)$.

b. $f(x) = \sqrt{x^2 + 4}$

$= (x^2 + 4)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(x^2 + 4)^{-\frac{1}{2}}(2x)$

$= \frac{x}{\sqrt{x^2 + 4}}$

Let $f'(x) = 0$:

$\frac{x}{\sqrt{x^2 + 4}} = 0$

So $x = 0$.

The point is $(0, 2)$.

c. $f(x) = (2x - 1)^2(x^2 - 9)$

$f'(x) = 2(2x - 1)(2)(x^2 - 9) + 2x(2x - 1)^2$

Let $f'(x) = 0$:

$2(2x - 1)(2(x^2 - 9) + x(2x - 1)) = 0$

$2(2x - 1)(4x^2 - x - 18) = 0$

$2(2x - 1)(4x - 9)(x + 2) = 0$

$x = \frac{1}{2}$ or $x = \frac{9}{4}$ or $x = -2$.

This points are $\left(\frac{1}{2}, 0\right)$, $(2.25, -48.2)$ and $(-2, -125)$.

d. $f(x) = \frac{5x}{x^2 + 1}$

$f'(x) = \frac{5(x^2 + 1) - 5x(2x)}{(x^2 + 1)^2} = \frac{5(1 - x^2)}{(x^2 + 1)^2}$

Let $f'(x) = 0$:

$\frac{5(1 - x^2)}{(x^2 + 1)^2} = 0$

Therefore, $5(1 - x^2) = 0$

$(1 - x)(1 + x) = 0$

$x = \pm 1$

The points are $\left(1, \frac{5}{2}\right)$ and $\left(-1, -\frac{5}{2}\right)$.

2. A function is increasing when $f'(x) > 0$ and is decreasing when $f'(x) < 0$.

3. a. i. $x < -1, x > 2$

ii. $-1 < x < 2$

iii. $(-1, 4), (2, -1)$

b. i. $-1 < x < 1$

ii. $x < -1, x > 1$

iii. $(-1, 2), (2, 4)$

c. i. $x < -2$

ii. $-2 < x < 2, 2 < x$

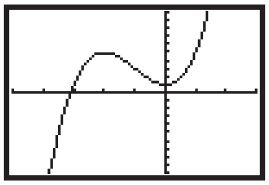
iii. none

d. i. $-1 < x < 2, 3 < x$

ii. $x < -1, 2 < x < 3$

iii. $(2, 3)$

4.



a. $f(x) = x^3 + 3x^2 + 1$

$f'(x) = 3x^2 + 6x$

Let $f'(x) = 0$

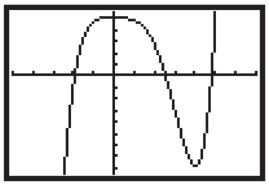
$3x^2 + 6x = 0$

$3x(x + 2) = 0$

$x = 0$ or $x = -2$

x	$x < -2$	-2	$-2 < x < 0$	0	$x > 0$
$f'(x)$	+	0	−	0	+
Graph	Increasing		Decreasing		Increasing

b.



$f(x) = x^5 - 5x^4 + 100$

$f'(x) = 5x^4 - 20x^3$

Let $f'(x) = 0$:

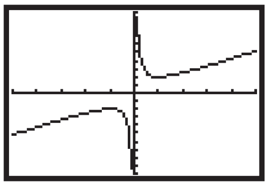
$5x^4 - 20x^3 = 0$

$5x^3(x - 4) = 0$

$x = 0$ or $x = 4$.

x	$x < 0$	0	$0 < x < 4$	4	$x > 4$
$f'(x)$	+	0	−	0	+
Graph	Increasing		Decreasing		Increasing

c.



$$f(x) = x + \frac{1}{x}$$

$$f'(x) = 1 - \frac{1}{x^2}$$

Let $f'(x) = 0$

$$1 - \frac{1}{x^2} = 0$$

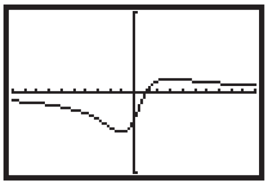
$$x^2 - 1 = 0$$

$$x = -1 \text{ or } x = 1$$

Also note that $f(x)$ is undefined for $x = 0$.

x	$x < -1$	-1	$-1 < x < 0$	0	$0 < x < 1$	1	$x > 1$
$f'(x)$	+	0	−	undefined	−	0	+
Graph	Increasing		Decreasing		Decreasing		Increasing

d.



$$f(x) = \frac{x - 1}{x^2 + 3}$$

$$f'(x) = \frac{x^2 + 3 - 2x(x - 1)}{(x^2 + 3)^2}$$

Let $f'(x) = 0$, therefore, $-x^2 + 2x + 3 = 0$.

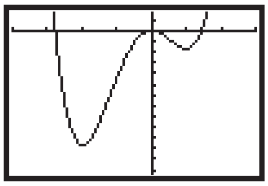
Or $x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3$ or $x = -1$

x	$x < -1$	-1	$-1 < x < 3$	3	$x > 3$
$f'(x)$	−	0	+	0	−
Graph	Decreasing		Increasing		Decreasing

e.



$y = 3x^4 + 4x^3 - 12x^2$

$y' = 12x^3 + 12x^2 - 24x$

Intervals of increasing:

$12x^3 + 12x^2 - 24x > 0$

$x(x^2 + x - 2) > 0$

$x(x - 1)(x + 2) > 0$

Intervals of decreasing:

$12x^3 + 12x^2 - 24x < 0$

$x(x^2 + x - 2) < 0$

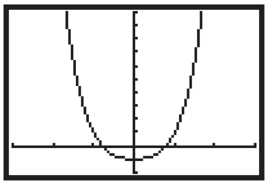
$x(x - 1)(x + 2) < 0$

	$x < -2$	$-2 < x < 0$	$0 < x < 1$	$x < 1$
x	−	−	+	+
$x - 1$	+	−	−	+
$x + 2$	−	+	+	+
y'	+	+	−	+

Intervals of increasing: $-2 < x < 0, x > 1$

Intervals of decreasing: $x < -2, 0 < x < 1$

f.



$y = x^4 + x^2 - 1$

$y' = 4x^3 + 2x$

Interval of increasing:

$4x^3 + 2x > 0$

$x(2x^2 + 1) > 0$

Interval of decreasing:

$4x^3 + 2x < 0$

$x(2x^2 + 1) < 0$

But $2x^2 + 1$ is always positive.

Interval of increasing: $x > 0$

Interval of decreasing: $x < 0$

5. $f'(x) = (x - 1)(x + 2)(x + 3)$

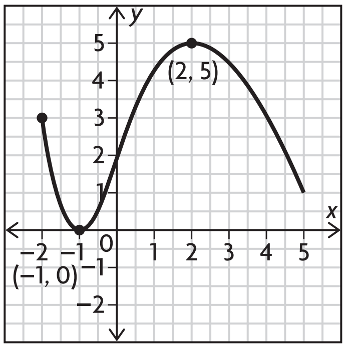
Let $f'(x) = 0$:

Then $(x - 1)(x + 2)(x + 3) = 0$

$x = 1$ or $x = -2$ or $x = -3$.

x	$x < -3$	-3	$-3 < x < -3$	-2	$-2 < x < 1$	1	$x > 1$
$f'(x)$	−	0	+	0	−	0	+
Graph	Decreasing		Increasing		Decreasing		Increasing

6.



7. $f(x) = x^3 + ax^2 + bx + c$

$f'(x) = 3x^2 + 2ax + b$

Since $f(x)$ increases to $(-3, 18)$ and then decreases, $f'(3) = 0$.

Therefore, $27 - 6a + b = 0$ or $6a - b = 27$. (1)

Since $f(x)$ decreases to the point $(1, -14)$ and then increases $f'(1) = 0$.

Therefore, $3 + 2ab + b = 0$ or $2a + b = -3$. (2)

Add (1) to (2) $8a = 24$ and $a = 3$.

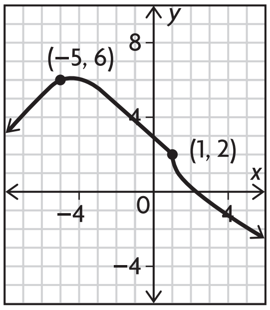
When $a = 3$, $b = 6 + b = -3$ or $b = -9$.

Since $(1, -14)$ is on the curve and $a = 3$, $b = -9$, then $-14 = 1 + 3 - 9 + c$

$c = -9$.

The function is $f(x) = x^3 + 3x^2 - 9x - 9$.

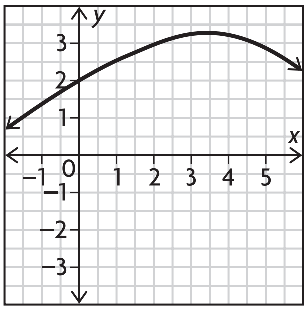
8.



9. a. i. $x < 4$

ii. $x > 4$

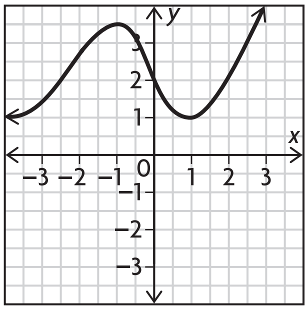
iii. $x = 4$



b. i. $x < -1, x > 1$

ii. $-1 < x < 1$

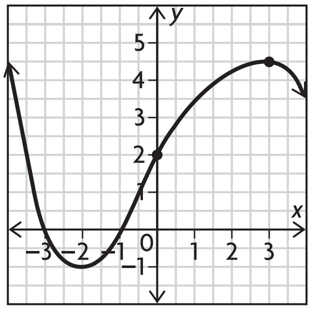
iii. $x = -1, x = 1$



c. i. $-2 < x < 3$

ii. $x < -2, x > 3$

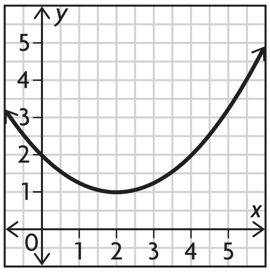
iii. $x = -2, x = 3$



d. i. $x > 2$

ii. $x < 2$

iii. $x = 2$



10. $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

Let $f'(x) = 0$, then $x = \frac{-b}{2a}$.

If $x < \frac{-b}{2a}$, $f'(x) < 0$, therefore the function is decreasing.

If $x > \frac{-b}{2a}$, $f'(x) > 0$, therefore the function is increasing.

11. $f(x) = x^4 - 32x + 4$

$f'(x) = 4x^3 - 32$

Let $f'(x) = 0$:

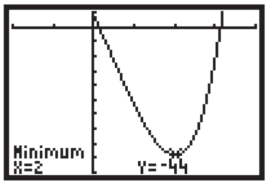
$4x^3 - 32 = 0$

$4x^3 = 32$

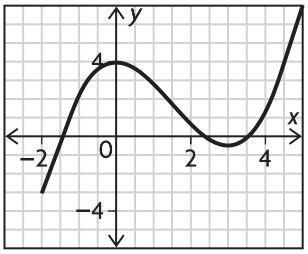
$x^3 = 8$
$x = 2$

x	$x < 2$	2	$x > 2$
$f(x)$	−	0	+
Graph	Dec.	Local Min	Inc

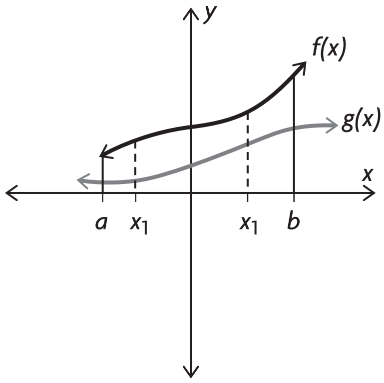
Therefore the function is decreasing for $x < 2$ and increasing for $x > 2$. The function has a local minimum at the point $(2, -44)$.



12.



13. Let $y = f(x)$ and $u = g(x)$.
Let x_1 and x_2 be any two values in the interval $a \le x \le b$ so that $x_1 < x_2$.
Since $x_1 < x_2$, both functions are increasing:
$f(x_2) > f(x_1) \quad (1)$
$g(x_2) > g(x_1) \quad (2)$
$yu = f(x) \cdot g(x)$.
$(1) \times (2)$ results in $f(x_2) \cdot g(x_2) > f(x_1)g(x_1)$.
The function yu or $f(x) \cdot g(x)$ is strictly increasing.



14. Let x_1, x_2 be in the interval $a \le x \le b$, such that $x_1 < x_2$. Therefore, $f(x_2) > f(x_1)$, and $g(x_2) > g(x_1)$. In this case, $f(x_1), f(x_2), g(x_1)$, and $g(x_2) < 0$. Multiplying an inequality by a negative will reverse its sign.

Therefore, $f(x_2) \cdot g(x_2) < f(x_1) \cdot g(x_1)$.
But LS > 0 and RS > 0.
Therefore, the function fg is strictly decreasing.

4.2 Critical Points, Relative Maxima, and Relative Minima, pp. 178–180

1. Finding the critical points means determining the points on the graph of the function for which the derivative of the function at the x-coordinate is 0.

2. a. Take the derivative of the function. Set the derivative equal to 0. Solve for x. Evaluate the original function for the values of x. The (x, y) pairs are the critical points.

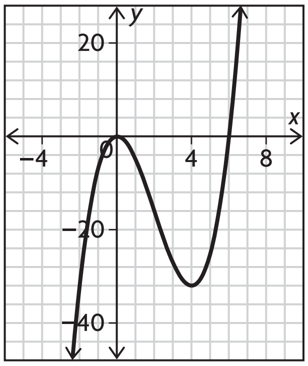
b. $y = x^3 - 6x^2$

$$\frac{dy}{dx} = 3x^2 - 12x$$
$$= 3x(x - 4)$$

Let $\frac{dy}{dx} = 0$.

$3x(x - 4) = 0$
$x = 0, 4$

The critical points are $(0, 0)$ and $(4, -32)$.



3. a. $y = x^4 - 8x^2$

$$\frac{dy}{dx} = 4x^3 - 16x = 4x(x^2 - 4)$$
$$= 4x(x + 2)(x - 2)$$

Let $\frac{dy}{dx} = 0$

$4x(x + 2)(x - 2) = 0$
$x = 0, \pm 2$.

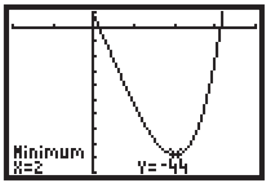
The critical points are $(0, 0)$, $(-2, 16)$, and $(2, -16)$.

x	$x < -2$	-2	$-2 < x < 0$	0	$0 < x < 2$	2	$x < 2$
$\frac{dy}{dx}$	−	0	+	0	−	0	+
Graph	Dec.	Local Min	Inc.	Local Max	Dec.	Local Min	Inc.

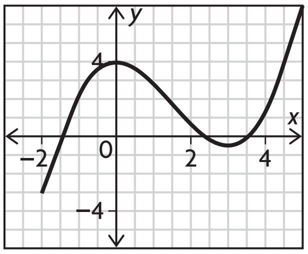
$$x^3 = 8$$
$$x = 2$$

x	$x < 2$	2	$x > 2$
$f(x)$	$-$	0	$+$
Graph	Dec.	Local Min	Inc

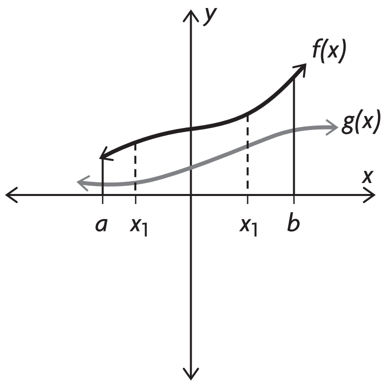
Therefore the function is decreasing for $x < 2$ and increasing for $x > 2$. The function has a local minimum at the point $(2, -44)$.



12.



13. Let $y = f(x)$ and $u = g(x)$.
Let x_1 and x_2 be any two values in the interval $a \le x \le b$ so that $x_1 < x_2$.
Since $x_1 < x_2$, both functions are increasing:
$f(x_2) > f(x_1)$ (1)
$g(x_2) > g(x_1)$ (2)
$yu = f(x) \cdot g(x)$.
(1) $\times$ (2) results in $f(x_2) \cdot g(x_2) > f(x_1)g(x_1)$.
The function yu or $f(x) \cdot g(x)$ is strictly increasing.



14. Let x_1, x_2 be in the interval $a \le x \le b$, such that $x_1 < x_2$. Therefore, $f(x_2) > f(x_1)$, and $g(x_2) > g(x_1)$. In this case, $f(x_1)$, $f(x_2)$, $g(x_1)$, and $g(x_2) < 0$. Multiplying an inequality by a negative will reverse its sign.

Therefore, $f(x_2) \cdot g(x_2) < f(x_1) \cdot g(x_1)$.
But LS > 0 and RS > 0.
Therefore, the function fg is strictly decreasing.

4.2 Critical Points, Relative Maxima, and Relative Minima, pp. 178–180

1. Finding the critical points means determining the points on the graph of the function for which the derivative of the function at the x-coordinate is 0.

2. a. Take the derivative of the function. Set the derivative equal to 0. Solve for x. Evaluate the original function for the values of x. The (x, y) pairs are the critical points.

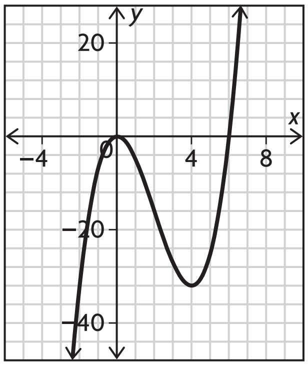
b. $y = x^3 - 6x^2$

$$\frac{dy}{dx} = 3x^2 - 12x$$
$$= 3x(x - 4)$$

Let $\frac{dy}{dx} = 0$.

$$3x(x - 4) = 0$$
$$x = 0, 4$$

The critical points are $(0, 0)$ and $(4, -32)$.



3. a. $y = x^4 - 8x^2$

$$\frac{dy}{dx} = 4x^3 - 16x = 4x(x^2 - 4)$$
$$= 4x(x + 2)(x - 2)$$

Let $\frac{dy}{dx} = 0$

$$4x(x + 2)(x - 2) = 0$$
$$x = 0, \pm 2.$$

The critical points are $(0, 0)$, $(-2, 16)$, and $(2, -16)$.

x	$x < -2$	-2	$-2 < x < 0$	0	$0 < x < 2$	2	$x < 2$
$\frac{dy}{dx}$	$-$	0	$+$	0	$-$	0	$+$
Graph	Dec.	Local Min	Inc.	Local Max	Dec.	Local Min	Inc.

Local minima at $(-2, -16)$ and $(2, -16)$
Local maximum at $(0, 0)$

b. $f(x) = \frac{2x}{x^2 + 9}$

$$f'(x) = \frac{2(x^2 + 9) - 2x(2x)}{(x^2 + 9)^2}$$
$$= \frac{18 - 2x^2}{(x^2 + 9)^2}$$

Let $f'(x) = 0$
Therefore, $18 - 2x^2 = 0$
$x^2 = 9$
$x = \pm 3.$

x	$x < -3$	-3	$-3 < x < 3$	3	$x > 3$
$f'(x)$	$-$	0	$+$	0	$-$
Graph	Decreasing	Local Min	Increasing	Local Max	Decreasing

Local minimum at $(-3, -0.3)$ and local maximum at $(3, 0.3)$.

c. $y = x^3 + 3x^2 + 1$

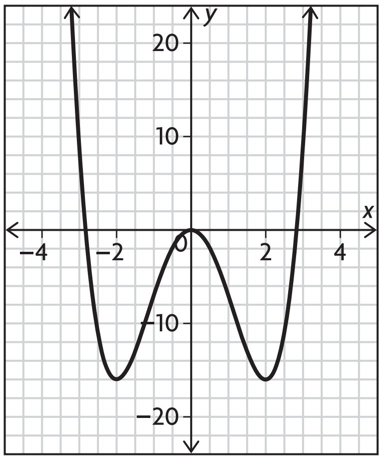
$$\frac{dy}{dx} = 3x^2 + 6x = 3x(x + 2)$$

Let $\frac{dy}{dx} = 0$
$3x(x + 2) = 0$
$x = 0, -2$
The critical points are $(0, 1)$ and $(-2, 5)$.

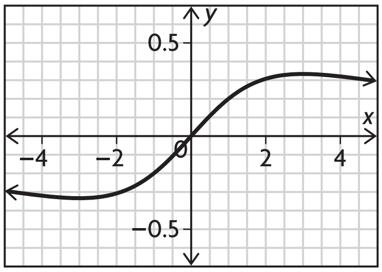
x	$x < -2$	-2	$-2 < x < 0$	0	$x < 0$
$\frac{dy}{dx}$	$+$	0	$-$	0	$+$
Graph	Inc.	Local Min		Local Max	Inc.

Local maximum at $(-2, 5)$
Local minimum at $(0, 1)$

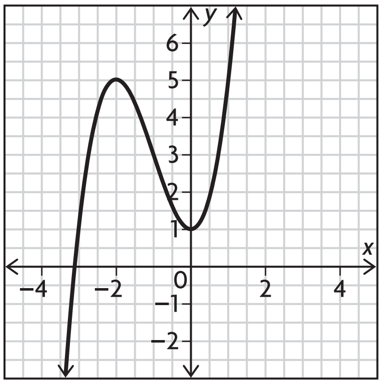
4. a. $y = x^4 - 8x^2$
To find the x-intercepts, let $y = 0$.
$x^4 - 8x^2 = 0$
$x^2(x^2 - 8) = 0$
$x = 0, \pm\sqrt{8}$
To find the y-intercepts, let $x = 0$.
$y = 0$



b. $f(x) = \frac{2x}{x^2 + 9}$
To find the x-intercepts, let $y = 0$.
$\frac{2x}{x^2 + 9} = 0$
Therefore, $2x = 0$
$x = 0$
To find the y-intercepts, let $x = 0$.
$y = \frac{0}{9} = 0$



c. $y = x^3 + 3x^2 + 1$
To find the x-intercepts, let $y = 0$.
$0 = x^3 + 3x^2 + 1$
The x-intercept cannot be easily obtained algebraically. Since the function has a local maximum when $x = -2$, it must have an x-intercept prior to this x-value. Since $f(-3) = 1$ and $f(-4) = -15$, an estimate for the x-intercept is about -3.1.
To find the y-intercepts, let $x = 0$.
$y = 1$



5. a. $h(x) = -6x^3 + 18x^2 + 3$
$h'(x) = -18x^2 + 36x$
Let $h'(x) = 0$:
$-18x^2 + 36x = 0$
$18x(2 - x) = 0$
$x = 0$ or $x = 2$
The critical points are (0, 3) and (2, 27).
Local minimum at (0, 3)
Local maximum at (2, 27)
Since the derivative is 0 at both points, the tangent is parallel to the horizontal axis for both.

b. $g(t) = t^5 + t^3$
$g'(t) = 5t^4 + 3t^2$
Let $g'(t) = 0$:
$5t^4 + 3t^2 = 0$
$t^2(5t^2 + 3) = 0$
$t = 0$

x	$x < 0$	0	$0 < x < 2$	0	$x > 2$
$\frac{dy}{dx}$	+	0	−	0	+
Graph	Inc.	Local Min	Dec.	Local Max	Inc.

The critical point is (0, 0).

t	$t < 0$	0	$t > 0$
$g'(x)$	+	0	+
Graph	Inc.	Local Min	Inc.

(0, 0) is neither a maximum nor a minimum
Since the derivative at (0, 0) is 0, the tangent is parallel to the horizontal axis there.

c. $y = (x - 5)^{\frac{1}{3}}$
$$\frac{dy}{dx} = \frac{1}{3}(x - 5)^{\frac{-2}{3}}$$
$$= \frac{1}{3(x - 5)^{\frac{2}{3}}}$$
$$\frac{dy}{dx} \neq 0$$

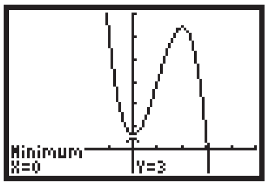
The critical point is at $(5, 0)$, but is neither a maximum or minimum. The tangent is not parallel to the x-axis.

d. $f(x) = (x^2 - 1)^{\frac{1}{3}}$
$$f'(x) = \frac{1}{3}(x^2 - 1)^{-\frac{2}{3}}(2x)$$
Let $f'(x) = 0$:
$$\frac{1}{3}(x^2 - 1)^{-\frac{2}{3}}(2x) = 0$$
$$x = 0$$
There is a critical point at $(0, -1)$. Since the derivative is undefined for $x = \pm 1$, (1, 0) and $(-1, 0)$ are also critical points.

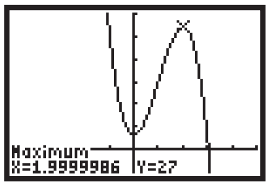
x	$x < -1$	-1	$-1 < x < 0$	0	$0 < x < 1$	0	$x < 1$
$\frac{dy}{dx}$	−	DNE	−	0	+	DNE	+
Graph	Dec.		Dec.	Local Min	Inc.		Inc.

Local minimum at $(0, -1)$
The tangent is parallel to the horizontal axis at $(0, -1)$ because the derivative is 0 there. Since the derivative is undefined at $(-1, 0)$ and (1, 0), the tangent is not parallel to the horizontal axis at either point.

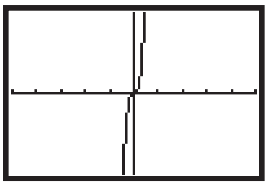
6. a.



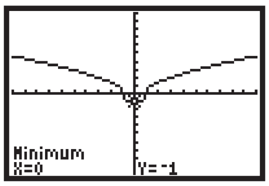
b.



c.



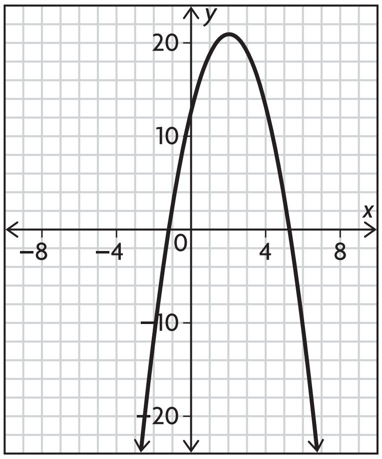
d.



7. a. $f(x) = -2x^2 + 8x + 13$
$f'(x) = -4x + 8$
Let $f'(x) = 0$:
$-4x + 8 = 0$
$x = 2$

The critical point is (2, 21).
Local maximum at (2, 21)

x	$x < 2$	2	$x > 2$
$f'(x)$	+	0	−
Graph	Inc.	Local Max.	Dec.



b. $f(x) = \frac{1}{3}x^3 - 9x + 2$

$f'(x) = x^2 - 9$

Let $f'(x) = 0$:

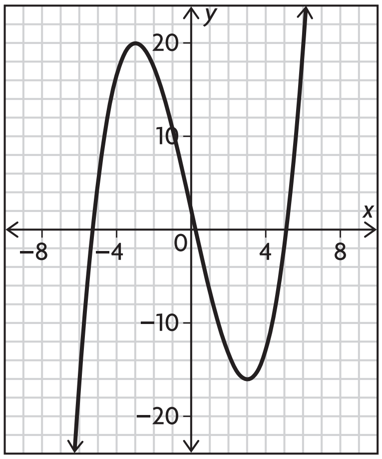
$x^2 - 9 = 0$

$x^2 = 9$

$x = \pm 3$

The critical points are $(-3, 20)$ and $(3, -16)$
Local maximum at $(-3, 20)$
Local minimum at $(3, -16)$

x	$x < -3$	-3	$-3 < x < 3$	3	$x > 3$
$\frac{dy}{dx}$	+	0	−	0	+
Graph	Inc.	Local Max	Dec.	Local Min	Inc.



c. $f(x) = 2x^3 + 9x^2 + 12x$

$f'(x) = 6x^2 + 18x + 12$

Let $f'(x) = 0$:

$6x^2 + 18x + 12 = 0$

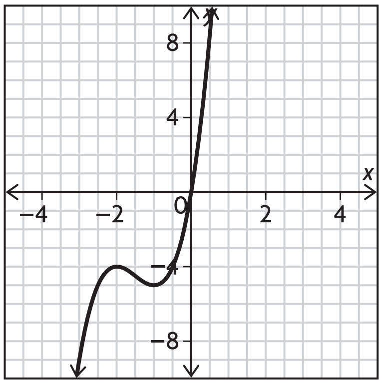
$6(x + 2)(x + 1) = 0$

$x = -2$ or $x = -1$

The critical points are $(-2, -4)$ and $(-1, -5)$.

x	$x < -2$	-2	$-2 < x < -1$	-1	$x > -1$
$\frac{dy}{dx}$	+	0	−	0	+
Graph	Inc.	Local Max	Dec.	Local Min	Inc.

Local maximum at $(-2, -4)$
Local minimum at $(-1, -5)$



d. $f(x) = -3x^3 - 5x$

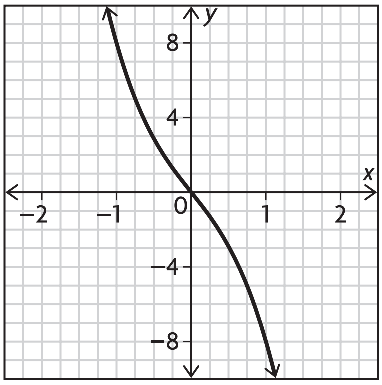
$f'(x) = -9x^2 - 5$

Let $f'(x) = 0$:

$-9x^2 - 5 = 0$

$x^2 = -\frac{5}{9}$

This equation has no solution, so there are no critical points.



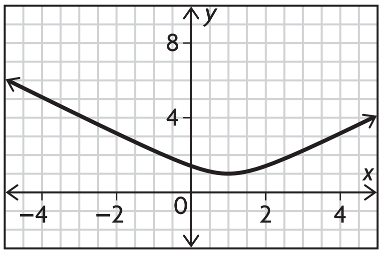
e. $f(x) = \sqrt{x^2 - 2x + 2}$

$f'(x) = \frac{2x - 2}{2\sqrt{x^2 - 2x + 2}} = \frac{x - 1}{\sqrt{x^2 - 2x + 2}}$

Let $f'(x) = 0$:
Therefore, $x - 1 = 0$
$x = 1$
The critical point is (1, 1).
$\sqrt{x^2 - 2x + 2}$ is never undefined or equal to zero, so (1, 1) is the only critical point.

x	$x < 1$	1	$x > 1$
$f'(x)$	−	0	+
Graph	Dec.	Local Min	Inc.

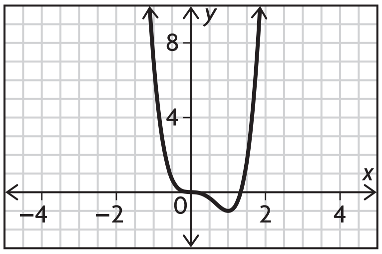
Local minimum at (1, 1)



f. $f(x) = 3x^4 - 4x^3$
$f'(x) = 12x^3 - 12x^2$
Let $f'(x) = 0$:
$12x^3 - 12x^2 = 0$
$12x^2(x - 1) = 0$
$x = 0$ or $x = 1$

x	$x < 0$	0	$0 < x < 1$	1	$x > 1$
$\frac{dy}{dx}$	−	0	−	0	+
Graph	Dec.		Dec.	Local Min	Inc.

There are critical points at (0, 0) and (1, −1).
Neither local minimum nor local maximum at (0, 0)
Local minimum at (1, −1)

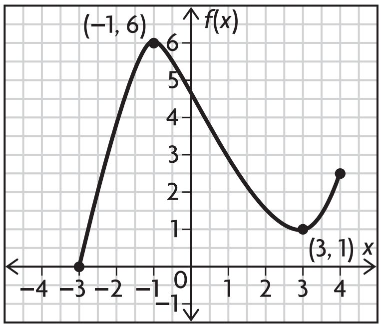


8. $f'(x) = (x + 1)(x - 2)(x + 6)$
Let $f'(x) = 0$:
$(x + 1)(x - 2)(x + 6) = 0$
$x = -6$ or $x = -1$ or $x = 2$
The critical numbers are −6, −1, and 2.

x	$x < -6$	−6	$-6 < x < -1$	−1	$-1 < x < 2$	2	$x < 2$
$\frac{dy}{dx}$	−	0	+	0	−	0	+
Graph	Dec.	Local Min	Inc.	Local Max	Dec.	Local Min	Inc.

Local minima at $x = -6$ and $x = 2$
Local maximum at $x = -1$

9.



10. $y = ax^2 + bx + c$
$\frac{dy}{dx} = 2ax + b$
Since a relative maximum occurs at $x = 3$, then $2ax + b = 0$ at $x = 3$. Or, $6a + b = 0$. Also, at $(0, 1)$, $1 = 0 + 0 + c$ or $c = 1$. Therefore, $y = ax^2 + bx + 1$. Since $(3, 12)$ lies on the curve,
$12 = 9a + 3b + 1$
$9a + 3b = 11$
$6a + b = 0.$
Since $b = -6a$,
Then $9a - 18a = 11$
or $a = -\frac{11}{9}$
$b = \frac{22}{3}.$
The equation is $y = -\frac{11}{9}x^2 + \frac{22}{3}x + 1$.

11. $f(x) = x^2 + px + q$
$f'(x) = 2x + p$
In order for 1 to be an extremum, $f'(1)$ must equal 0.
$2(1) + p = 0$
$p = -2$
To find q, substitute the known values for p and x into the original equation and set it equal to 5.

x	$x < 1$	1	$x > 1$
$f'(x)$	−	0	+
Graph	Dec.	Local Min	Inc.

$(1)^2 + (1)(-2) + q = 5$

$q = 6$

This extremum is a minimum value.

12. a. $f(x) = x^3 - kx$

$f'(x) = 3x^2 - k$

In order for f to have no critical numbers, $f'(x) = 0$ must have no solutions. Therefore, $3x^2 = k$ must have no solutions, so $k < 0$.

b. $f(x) = x^3 - kx$

$f'(x) = 3x^2 - k$

In order for f to have one critical numbers, $f'(x) = 0$ must have exactly one solution. Therefore, $3x^2 = k$ must have one solution, which occurs when $k = 0$.

c. $f(x) = x^3 - kx$

$f'(x) = 3x^2 - k$

In order for f to have two critical numbers, $f'(x) = 0$ must have two solutions. Therefore, $3x^2 = k$ must have two solutions, which occurs when $k > 0$.

13. $g(x) = ax^3 + bx^2 + cx + d$

$g'(x) = 3ax^2 + 2bx + c$

Since there are local extrema at $x = 0$ and $x = 2$, $0a + 0b + c = 0$ and $12a + 4b + c = 0$

Therefore, $c = 0$ and $12a + 4b = 0$

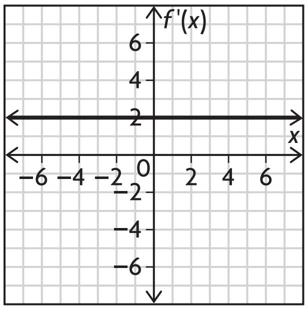
Going back to the original equation, we have the points (2, 4) and (0, 0). Substitute these values of x in the original function to get two more equations: $8a + 4b + 2c + d = 4$ and $d = 0$. We now know that $c = 0$ and $d = 0$. We are left with two equations to find a and b:

$12a + 4b = 0$

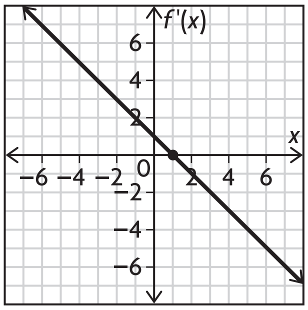
$8a + 4b = 4$

Subtract the second equation from the first to get $4a = -4$. Therefore $a = -1$, and $b = 3$.

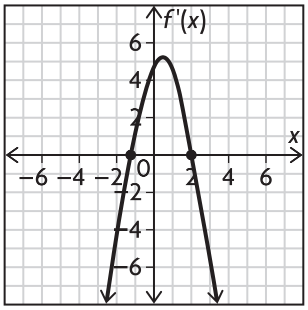
14. a.



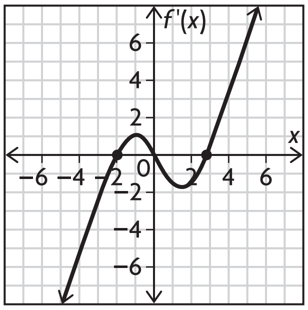
b.



c.



d.



15. $f(x) = 3x^4 + ax^3 + bx^2 + cx + d$

a. $f'(x) = 12x^3 + 3ax^2 + 2bx + c$

At $x = 0$, $f'(0) = 0$, then $f'(0) = 0 + 0 + 0 + c$ or $c = 0$.

At $x = -2$, $f'(-2) = 0$,

$-96 + 12a - 4b = 0.$ (1)

Since $(0, -9)$ lies on the curve,

$-9 = 0 + 0 + 0 + 0 + d$ or $d = -9$.

Since $(-2, -73)$ lies on the curve,

$-73 = 48 - 8a + 4b + 0 - 9$

$-8a + 4b = -112$

or $2a - b = 28$ (2)

Also, from (1): $3a - b = 24$

$2a - b = -28$

$a = -4$

$b = -36.$

The function is $f(x) = 3x^4 - 4x^3 - 36x^2 - 9$.

b. $f'(x) = 12x^3 - 12x^2 - 72x$

Let $f'(x) = 0$:

$$x^3 - x^2 - 6x = 0$$

$$x(x - 3)(x + 2) = 0.$$

Third point occurs at $x = 3$,

$f(3) = -198$.

c.

Local minimum is at $(-2, -73)$ and $(3, -198)$.

x	$x < -2$	-2	$-2 < x < 0$	0	$0 < x < 3$	3	$x > 3$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
Graph	Decreasing	Local Min	Increasing	Local Max	Decreasing	Local Min	Increasing

Local maximum is at $(0, -9)$.

16. a. $y = 4 - 3x^2 - x^4$

$$\frac{dy}{dx} = -6x - 4x^3$$

Let $\frac{dy}{dx} = 0$:

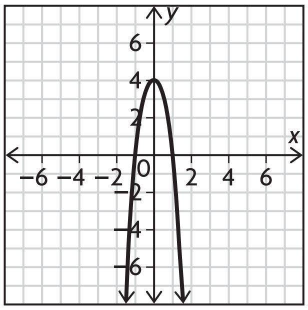
$$-6x - 4x^3 = 0$$

$$-2x(2x^2 + 3) = 0$$

$x = 0$ or $x^2 = -\frac{3}{2}$; inadmissible

x	$x < 0$	0	$x > 0$
$\frac{dy}{dx}$	$+$	0	$-$
Graph	Increasing	Local Max	Decreasing

Local maximum is at $(0, 4)$.



b. $y = 3x^5 - 5x^3 - 30x$

$$\frac{dy}{dx} = 15x^4 - 15x^2 - 30$$

Let $\frac{dy}{dx} = 0$:

$$15x^4 - 15x^2 - 30 = 0$$

$$x^4 - x^2 - 2 = 0$$

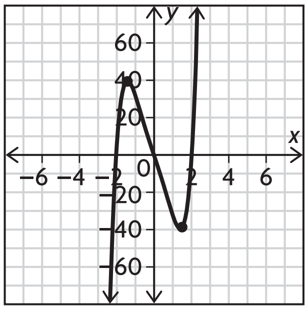
$$(x^2 - 2)(x^2 + 1) = 0$$

$x^2 = 2$ or $x^2 = -1$

$x = \pm\sqrt{2}$; inadmissible

At $x = 100$, $\frac{dy}{dx} > 0$.

Therefore, function is increasing into quadrant one, local minimum is at $(1.41, -39.6)$ and local maximum is at $(-1.41, 39.6)$.



17. $h(x) = \frac{f(x)}{g(x)}$

Since $f(x)$ has local maximum at $x = c$, then $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$.

Since $g(x)$ has a local minimum at $x = c$, then $g'(x) < 0$ for $x < c$ and $g'(x) > 0$ for $x > c$.

$$h(x) = \frac{f(x)}{g(x)}$$

$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

If $x < c$, $f'(x) > 0$ and $g'(x) < 0$, then $h'(x) > 0$.

If $x > c$, $f'(x) < 0$ and $g'(x) > 0$, then $h'(x) < 0$.

Since for $x < c$, $h'(x) > 0$ and for $x > c$, $h'(x) < 0$.

Therefore, $h(x)$ has a local maximum at $x = c$.

4.3 Vertical and Horizontal Asymptotes, pp. 193–195

1. a. vertical asymptotes at $x = -2$ and $x = 2$; horizontal asymptote at $y = 1$

b. vertical asymptote at $x = 0$; horizontal asymptote at $y = 0$

2. $f(x) = \frac{g(x)}{h(x)}$

Conditions for a vertical asymptote: $h(x) = 0$ must have at least one solution s, and $\lim_{x \to \infty} f(x) = \infty$.

Conditions for a horizontal asymptote: $\lim_{x \to \infty} f(x) = k$, where $k \in \mathbf{R}$,

or $\lim_{x \to -\infty} f(x) = k$ where $k \in \mathbf{R}$.

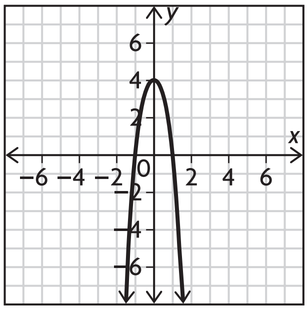
The function is $f(x) = 3x^4 - 4x^3 - 36x^2 - 9$.
b. $f'(x) = 12x^3 - 12x^2 - 72x$
Let $f'(x) = 0$:
$x^3 - x^2 - 6x = 0$
$x(x - 3)(x + 2) = 0$.
Third point occurs at $x = 3$,
$f(3) = -198$.
c.
Local minimum is at $(-2, -73)$ and $(3, -198)$.

x	$x < -2$	-2	$-2 < x < 0$	0	$0 < x < 3$	3	$x > 3$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
Graph	Decreasing	Local Min	Increasing	Local Max	Decreasing	Local Min	Increasing

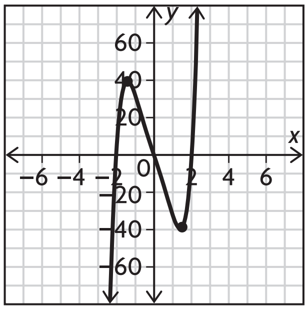
Local maximum is at $(0, -9)$.
16. a. $y = 4 - 3x^2 - x^4$
$$\frac{dy}{dx} = -6x - 4x^3$$
Let $\frac{dy}{dx} = 0$:
$-6x - 4x^3 = 0$
$-2x(2x^2 + 3) = 0$
$x = 0$ or $x^2 = -\frac{3}{2}$; inadmissible

x	$x < 0$	0	$x > 0$
$\frac{dy}{dx}$	$+$	0	$-$
Graph	Increasing	Local Max	Decreasing

Local maximum is at $(0, 4)$.



b. $y = 3x^5 - 5x^3 - 30x$
$$\frac{dy}{dx} = 15x^4 - 15x^2 - 30$$
Let $\frac{dy}{dx} = 0$:
$15x^4 - 15x^2 - 30 = 0$
$x^4 - x^2 - 2 = 0$
$(x^2 - 2)(x^2 + 1) = 0$
$x^2 = 2$ or $x^2 = -1$
$x = \pm\sqrt{2}$; inadmissible
At $x = 100$, $\frac{dy}{dx} > 0$.
Therefore, function is increasing into quadrant one, local minimum is at $(1.41, -39.6)$ and local maximum is at $(-1.41, 39.6)$.



17. $h(x) = \frac{f(x)}{g(x)}$
Since $f(x)$ has local maximum at $x = c$, then $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$.
Since $g(x)$ has a local minimum at $x = c$, then $g'(x) < 0$ for $x < c$ and $g'(x) > 0$ for $x > c$.
$$h(x) = \frac{f(x)}{g(x)}$$
$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$
If $x < c$, $f'(x) > 0$ and $g'(x) < 0$, then $h'(x) > 0$.
If $x > c$, $f'(x) < 0$ and $g'(x) > 0$, then $h'(x) < 0$.
Since for $x < c$, $h'(x) > 0$ and for $x > c$, $h'(x) < 0$.
Therefore, $h(x)$ has a local maximum at $x = c$.

4.3 Vertical and Horizontal Asymptotes, pp. 193–195

1. a. vertical asymptotes at $x = -2$ and $x = 2$; horizontal asymptote at $y = 1$
b. vertical asymptote at $x = 0$; horizontal asymptote at $y = 0$
2. $f(x) = \frac{g(x)}{h(x)}$
Conditions for a vertical asymptote: $h(x) = 0$ must have at least one solution s, and $\lim_{x\to\infty} f(x) = \infty$.
Conditions for a horizontal asymptote: $\lim_{x\to\infty} f(x) = k$, where $k \in \mathbf{R}$,
or $\lim_{x\to-\infty} f(x) = k$ where $k \in \mathbf{R}$.

Condition for an oblique asymptote is that the highest power of $g(x)$ must be one more than the highest power of $h(x)$.

3. a. $\lim_{x\to\infty}\frac{2x+3}{x-1}=\lim_{x\to\infty}\frac{x\left(2+\frac{3}{x}\right)}{x\left(x-\frac{1}{x}\right)}$

$=\lim_{x\to\infty}\frac{1+\frac{3}{2x}}{1-\frac{1}{x}}$

$=\frac{\lim_{x\to\infty}\left(2+\frac{3}{x}\right)}{\lim_{x\to\infty}\left(1-\frac{1}{x}\right)}$

$=\frac{2+0}{1-0}$

$=2$

Similarly, $\lim_{x\to-\infty}\frac{2x+3}{x-1}=2.$

b. $\lim_{x\to\infty}\frac{5x^2-3}{x^2+2}=\lim_{x\to\infty}\frac{x^2\left(5-\frac{3}{x^2}\right)}{x^2\left(1+\frac{2}{x^2}\right)}$

$=\lim_{x\to\infty}\frac{5-\frac{3}{x^2}}{1+\frac{2}{x^2}}$

$=\frac{\lim_{x\to\infty}\left(5-\frac{3}{x^2}\right)}{\lim_{x\to\infty}\left(1+\frac{2}{x^2}\right)}$

$=\frac{5-0}{1+0}$

$=5$

Similarly, $\lim_{x\to-\infty}\frac{5x^2-3}{x^2+2}=5.$

c. $\lim_{x\to\infty}\frac{-5x^2+3x}{2x^2-5}=\lim_{x\to\infty}\frac{x^2\left(-5+\frac{3}{x}\right)}{x^2\left(2-\frac{5}{x^2}\right)}$

$=\lim_{x\to\infty}\frac{-5+\frac{3}{x}}{2-\frac{5}{x^2}}$

$=\frac{\lim_{x\to\infty}\left(-5+\frac{3}{x}\right)}{\lim_{x\to\infty}\left(2-\frac{5}{x^2}\right)}$

$=\frac{-5+0}{2-0}$

$=-\frac{5}{2}$

Similarly, $\lim_{x\to-\infty}\frac{-5x^2+3x}{2x^2-5}=-\frac{5}{2}.$

d. $\lim_{x\to\infty}\frac{2x^5-3x^2+5}{3x^4+5x-4}=\lim_{x\to\infty}\frac{x^5\left(2-\frac{3}{x^3}+\frac{5}{x^5}\right)}{x^4\left(3+\frac{5}{x^3}-\frac{4}{x^4}\right)}$

$=\lim_{x\to\infty}\frac{x\left(2-\frac{3}{x^3}+\frac{5}{x^5}\right)}{3+\frac{5}{x^3}-\frac{4}{x^4}}$

$=\frac{\lim_{x\to\infty}\left(x\left(2-\frac{3}{x^3}+\frac{5}{x^5}\right)\right)}{\lim_{x\to\infty}\left(3+\frac{5}{x^3}-\frac{4}{x^4}\right)}$

$=\frac{\lim_{x\to\infty}(x)\times\lim_{x\to\infty}\left(2-\frac{3}{x^3}+\frac{5}{x^5}\right)}{\lim_{x\to\infty}\left(3+\frac{5}{x^3}-\frac{4}{x^4}\right)}$

$=\lim_{x\to\infty}(x)\times\frac{2-0+0}{3+0-0}$

$=\infty$

Similarly, $\lim_{x\to-\infty}\frac{2x^5-3x^2+5}{3x^4+5x-4}=\lim_{x\to\infty}(x)=-\infty.$

4. a. This function is discontinuous at $x=-5$. The numerator is non-zero at this point, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

x-values	x	$x+5$	y	$\lim_{x\to c} y$
$x\to-5^-$	<0	<0	>0	$+\infty$
$x\to-5^+$	<0	>0	<0	$-\infty$

b. This function is discontinuous at $x=2$. The numerator is non-zero at this point, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

x-values	$x+2$	$x-2$	$f(x)$	$\lim_{x\to c} f(x)$
$x\to2^-$	<0	<0	<0	$-\infty$
$x\to2^+$	>0	>0	>0	$+\infty$

c. This function is discontinuous at $t=3$. The numerator is non-zero at this point, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

t-values	1	$(t-3)^2$	s	$\lim_{t\to c} s$
$x\to3^-$	>0	>0	>0	$+\infty$
$x\to3^+$	>0	>0	>0	$+\infty$

d. This function is discontinuous at $x = 3$. However, the numerator also has value 0 there, since $3^2 - 3 - 6 = 0$, so this function has no vertical asymptotes.

e. The denominator of the function has value 0 when

$(x + 3)(x - 1) = 0$

$x = -3$ or $x = 1$. The numerator is non-zero at these points, so the function has vertical asymptotes there. The behaviour of the function near the asymptotes is:

x-values	6	$x + 3$	$x - 1$	$f(x)$	$\lim_{x\to c} f(x)$
$x \to -3^-$	> 0	< 0	< 0	> 0	$+\infty$
$x \to -3^+$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 1^-$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 1^+$	> 0	> 0	> 0	> 0	$+\infty$

f. This function is discontinuous when

$x^2 - 1 = 0$

$(x + 1)(x - 1) = 0$

$x = -1$ or $x = 1$. The numerator is non-zero at these points, so the function has vertical asymptotes there. The behaviour of the function near the asymptotes is:

x-values	x^2	$x + 1$	$x - 1$	y	$\lim_{x\to c} y$
$x \to -1^-$	> 0	< 0	< 0	> 0	$+\infty$
$x \to -1^+$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 1^-$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 1^+$	> 0	> 0	> 0	> 0	$+\infty$

5. a.
$$\lim_{x\to\infty} \frac{x}{x+4} = \lim_{x\to\infty} \frac{x}{x\left(1 + \frac{4}{x}\right)}$$
$$= \lim_{x\to\infty} \frac{1}{1 + \frac{4}{x}}$$
$$= \frac{\lim_{x\to\infty} (1)}{\lim_{x\to\infty} \left(1 + \frac{4}{x}\right)}$$
$$= \frac{1}{1 + 0}$$
$$= 1$$

Similarly, $\lim_{x\to-\infty} \frac{x}{x+4} = 1$, so $y = 1$ is a horizontal asymptote of the function.

At a point x, the difference between the function $y = \frac{x}{x+4}$ and its asymptote $y = 1$ is
$$\frac{x}{x+4} - 1 = \frac{x - (x + 4)}{x + 4}$$
$$= -\frac{4}{x+4}.$$

When x is large and positive, this difference is negative, which means that the curve approaches the asymptote from below. When x is large and negative, this difference is positive, which means that the curve approaches the asymptote from above.

b.
$$\lim_{x\to\infty} \frac{2x}{x^2 - 1} = \lim_{x\to\infty} \frac{2x}{x^2\left(1 - \frac{1}{x^2}\right)}$$
$$= \lim_{x\to\infty} \frac{2}{x\left(1 - \frac{1}{x^2}\right)}$$
$$= \frac{\lim_{x\to\infty} (2)}{\lim_{x\to\infty} x\left(1 - \frac{1}{x^2}\right)}$$
$$= \frac{\lim_{x\to\infty} (2)}{\lim_{x\to\infty} (x) \times \lim_{x\to\infty} \left(1 - \frac{1}{x^2}\right)}$$
$$= \lim_{x\to\infty} \frac{1}{x} \times \frac{1}{1 - 0}$$
$$= 0$$

Similarly, $\lim_{x\to-\infty} \frac{2x}{x^2 - 1} = 0$, so $y = 0$ is a horizontal asymptote of the function.

At a point x, the difference between the function $f(x) = \frac{2x}{x^2 - 1}$ and its asymptote $y = 0$ is $\frac{2x}{x^2 - 1}$.

When x is large and positive, this difference is positive, which means that the curve approaches the asymptote from above. When x is large and negative, this difference is negative, which means that the curve approaches the asymptote from below.

c.
$$\lim_{x\to\infty} \frac{3t^2 + 4}{t^2 - 1} = \lim_{x\to\infty} \frac{t^2\left(3 + \frac{4}{t^2}\right)}{t^2\left(1 - \frac{1}{t^2}\right)}$$
$$= \lim_{x\to\infty} \frac{3 + \frac{4}{t^2}}{1 - \frac{1}{t^2}}$$
$$= \frac{\lim_{x\to\infty} \left(3 + \frac{4}{t^2}\right)}{\lim_{x\to\infty} \left(1 - \frac{1}{t^2}\right)}$$

$$= \frac{3 + 0}{1 - 0}$$
$$= 3$$

Similarly, $\lim_{x \to -\infty} \frac{3t^2 + 4}{t^2 - 1} = 3$, so $y = 3$ is a horizontal asymptote of the function.

At a point x, the difference between the function $g(t) = \frac{3t^2 + 4}{t^2 - 1}$ and its asymptote $y = 3$ is

$$\frac{3t^2 + 4}{t^2 - 1} - 3 = \frac{3t^2 + 4 - 3(t^2 - 1)}{t^2 - 1}$$
$$= \frac{7}{t^2 - 1}.$$

When x is large and positive, this difference is positive, which means that the curve approaches the asymptote from above. When x is large and negative, this difference is positive, which means that the curve approaches the asymptote from above.

d. $$\lim_{x \to \infty} \frac{3x^2 - 8x - 7}{x - 4} = \lim_{x \to \infty} \frac{x^2\left(3 - \frac{8}{x} - \frac{7}{x^2}\right)}{x\left(1 - \frac{4}{x}\right)}$$
$$= \lim_{x \to \infty} \frac{x\left(3 - \frac{8}{x} - \frac{7}{x^2}\right)}{1 - \frac{4}{x}}$$
$$= \frac{\lim_{x \to \infty} \left(x\left(3 - \frac{8}{x} - \frac{7}{x^2}\right)\right)}{\lim_{x \to \infty} \left(1 - \frac{4}{x}\right)}$$
$$= \frac{\lim_{x \to \infty} (x) \times \lim_{x \to \infty} \left(3 - \frac{8}{x} - \frac{7}{x^2}\right)}{\lim_{x \to \infty} \left(1 - \frac{4}{x}\right)}$$
$$= \lim_{x \to \infty} (x) \times \frac{3 - 0 - 0}{1 - 0}$$
$$= \infty$$

Similarly, $\lim_{x \to \infty} \frac{3x^2 - 8x - 7}{x - 4} = \lim_{x \to \infty} (x) = -\infty$, so this function has no horizontal asymptotes.

6. a. This function is discontinuous at $x = -5$. Since the numerator is not equal to 0 there, the function has a vertical asymptote at this point. The behaviour of the function near the asymptote is:

x-values	$x - 3$	$x + 5$	y	$\lim_{x \to c} y$
$x \to -5^-$	< 0	< 0	> 0	$+\infty$
$x \to -5^+$	< 0	> 0	< 0	$-\infty$

To check for a horizontal asymptote:

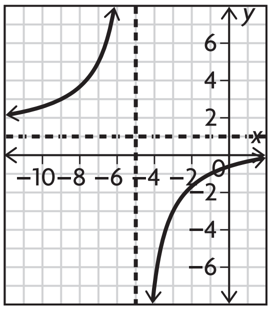
$$\lim_{x \to \infty} \frac{x - 3}{x + 5} = \lim_{x \to \infty} \frac{x\left(1 - \frac{3}{x}\right)}{x\left(1 + \frac{5}{x}\right)}$$
$$= \lim_{x \to \infty} \frac{1 - \frac{3}{x}}{1 + \frac{5}{x}}$$
$$= \frac{\lim_{x \to \infty} \left(1 - \frac{3}{x}\right)}{\lim_{x \to \infty} \left(1 + \frac{5}{x}\right)}$$
$$= \frac{1 - 0}{1 + 0}$$
$$= 1$$

Similarly, $\lim_{x \to -\infty} \frac{x - 3}{x + 5} = 1$, so $y = 1$ is a horizontal asymptote of the function.

At a point x, the difference between the function $y = \frac{x - 3}{x + 5}$ and its asymptote $y = 1$ is

$$\frac{x - 3}{x + 5} - 1 = \frac{x - 3 - (x + 5)}{x + 5} = -\frac{8}{x + 5}.$$

When x is large and positive, this difference is negative, which means that the curve approaches the asymptote from below. When x is large and negative, this difference is positive, which means that the curve approaches the asymptote from above.



b. This function is discontinuous at $x = -2$. Since the numerator is non-zero there, the function has a vertical asymptote at this point. The behaviour of the function near the asymptote is:

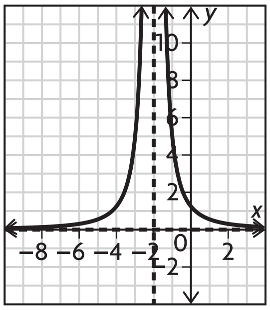
x-values	5	$(x + 2)^2$	$f(x)$	$\lim_{x \to c} f(x)$
$x \to 2^-$	> 0	> 0	> 0	$+\infty$
$x \to 2^+$	> 0	> 0	> 0	$+\infty$

To check for a horizontal asymptote:

$$\lim_{x\to\infty}\frac{5}{(x+2)^2} = \lim_{x\to\infty}\frac{5}{x^2+4x+4}$$
$$= \lim_{x\to\infty}\frac{5}{x^2\left(1+\frac{4}{x}+\frac{4}{x^2}\right)}$$
$$= \frac{\lim_{x\to\infty}(5)}{\lim_{x\to\infty}\left(x^2\left(1+\frac{4}{x}+\frac{4}{x^2}\right)\right)}$$
$$= \frac{\lim_{x\to\infty}(5)}{\lim_{x\to\infty}(x^2)\times\lim_{x\to\infty}\left(1+\frac{4}{x}+\frac{4}{x^2}\right)}$$
$$= \lim_{x\to\infty}\frac{1}{x^2}\times\frac{5}{1+0+0}$$
$$= 0$$

Similarly, $\lim_{x\to-\infty}\frac{5}{(x+2)^2} = 0$, so $y = 0$ is a horizontal asymptote of the function.

At a point x, the difference between the function $f(x) = \frac{5}{(x+2)^2}$ and its asymptote $y = 0$ is $\frac{5}{(x+2)^2}$. When x is large and positive, this difference is positive, which means that the curve approaches the asymptote from above. When x is large and negative, this difference is positive, which means that the curve approaches the asymptote from above.

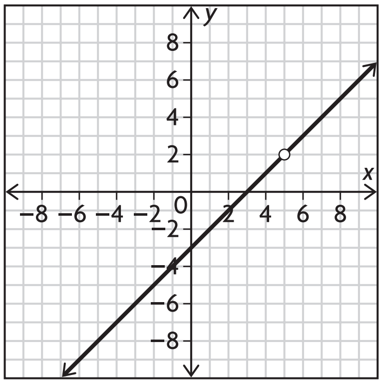


c. This function is discontinuous at $t = 5$. However, the numerator is equal to zero there, since $5^2 - 2(5) - 15 = 0$, so this function has no vertical asymptote.

To check for an oblique asymptote:

$$\begin{array}{r} t-3 \\ t-5\overline{)t^2-2t-15} \\ t^2-5t \\ 0+3t-15 \\ \underline{0+3t-15} \\ 0+0+0 \end{array}$$

So $g(t)$ can be written in the form
$g(t) = t - 3$



d. This function is discontinuous when
$x^2 - 3x = 0$
$x(x - 3) = 0$
$x = 0$ or $x = 3$

The numerator is non-zero at these points, so the function has vertical asymptotes at both of them. The behaviour of the function near these asymptotes is:

x-values	$2 + x$	$3 - 2x$	x	$x - 3$	y	$\lim_{x\to c} y$
$x \to 0^-$	> 0	> 0	< 0	< 0	> 0	$+\infty$
$x \to 0^+$	> 0	> 0	> 0	< 0	< 0	$-\infty$
$x \to 3^-$	> 0	< 0	> 0	< 0	> 0	$+\infty$
$x \to 3^+$	> 0	< 0	> 0	> 0	< 0	$-\infty$

To check for horizontal asymptotes:

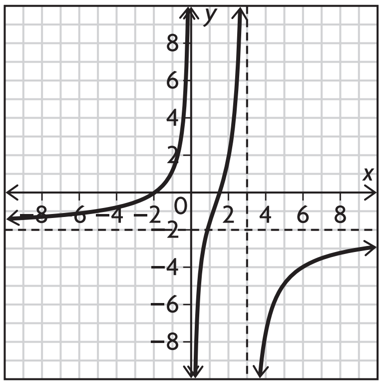
$$\lim_{x\to\infty}\frac{(2+x)(3-2x)}{x^2-3x} = \lim_{x\to\infty}\frac{-2x^2-x+6}{x^2-3x}$$
$$= \lim_{x\to\infty}\frac{x^2\left(-2-\frac{1}{x}+\frac{6}{x^2}\right)}{x^2\left(1-\frac{3}{x}\right)}$$
$$= \lim_{x\to\infty}\frac{-2-\frac{1}{x}+\frac{6}{x^2}}{1-\frac{3}{x}}$$
$$= \frac{\lim_{x\to\infty}\left(-2-\frac{1}{x}+\frac{6}{x^2}\right)}{\lim_{x\to\infty}\left(1-\frac{3}{x}\right)}$$
$$= \frac{-2-0+0}{1-0}$$
$$= -2$$

Similarly, $\lim_{x\to-\infty}\frac{(2+x)(3-2x)}{x^2-3x} = -2$, so $y = -2$ is a horizontal asymptote of the function.

At a point x, the difference between the function $y = \frac{-2x^2 - x + 6}{x^2 - 3x}$ and its asymptote $y = -2$ is

$$\frac{-2x^2 - x + 6}{x^2 - 3x} + 2 = \frac{-2x^2 - x + 6 + 2(x^2 - 3x)}{x^2 - 3x}$$
$$= \frac{-7x + 6}{x^2 - 3x}.$$

When x is large and positive, this difference is negative, which means that the curve approaches the asymptote from below. When x is large and negative, this difference is positive, which means that the curve approaches the asymptote from above.



7. a.

$$\begin{array}{r} 3x - 7 \\ x - 3\overline{)3x^2 - 2x - 17} \\ \underline{3x^2 - 9x} \\ 7x - 17 \\ \underline{7x - 21} \\ 4 \end{array}$$

So $f(x)$ can be written in the form $f(x) = 3x - 7 + \frac{4}{x - 3}$. Since $\lim\limits_{x\to\infty} \frac{4}{x - 3} = 0$ and $\lim\limits_{x\to\infty} \frac{4}{x - 3} = 0$, the line $y = 3x - 7$ is an asymptote to the function $f(x)$.

b.

$$\begin{array}{r} x + 3 \\ 2x + 3\overline{)2x^2 + 9x + 2} \\ \underline{2x^2 + 3x} \\ 6x + 2 \\ \underline{6x + 9} \\ -7 \end{array}$$

So $f(x)$ can be written in the form $f(x) = x + 3 - \frac{7}{2x + 3}$. Since $\lim\limits_{x\to\infty} \frac{7}{2x + 3} = 0$ and $\lim\limits_{x\to-\infty} \frac{7}{2x + 3} = 0$, the line $y = x + 3$ is an asymptote to the function $f(x)$.

c.

$$\begin{array}{r} x - 2 \\ x^2 + 2x\overline{)x^3 + 0x^2 + 0x - 1} \\ \underline{x^3 + 2x^2} \\ -2x^2 + 0x - 1 \\ \underline{-2x^2 - 4x} \\ 4x - 1 \end{array}$$

So $f(x)$ can be written in the form $f(x) = x - 2 + \frac{4x - 1}{x^2 + 2x}$. Since

$$\lim_{x\to\infty} \frac{4x - 1}{x^2 + 2x} = \lim_{x\to\infty} \frac{x\left(4 - \frac{1}{x}\right)}{x^2\left(1 + \frac{2}{x}\right)}$$
$$= \lim_{x\to\infty} \frac{4 - \frac{1}{x}}{x\left(1 + \frac{2}{x}\right)}$$
$$= \frac{\lim\limits_{x\to\infty}\left(4 - \frac{1}{x}\right)}{\lim\limits_{x\to\infty}\left(x\left(1 + \frac{2}{x}\right)\right)}$$
$$= \frac{\lim\limits_{x\to\infty}\left(4 - \frac{1}{x}\right)}{\lim\limits_{x\to\infty}(x) \times \lim\limits_{x\to\infty}\left(1 + \frac{2}{x}\right)}$$
$$= \lim_{x\to\infty}\left(\frac{1}{x}\right) \times \frac{4 - 0}{1 + 0}$$
$$= 0,$$

and similarly $\lim\limits_{x\to-\infty} \frac{4x - 1}{x^2 + 2x} = 0$, the line $y = x - 2$ is an asymptote to the function $f(x)$.

d.

$$\begin{array}{r} x + 3 \\ x^2 - 4x + 3\overline{)x^3 - x^2 - 9x + 15} \\ \underline{x^3 - 4x^2 + 3x} \\ 3x^2 - 12x + 15 \\ \underline{3x^2 - 12x + 9} \\ 6 \end{array}$$

So $f(x)$ can be written in the form $f(x) = x + 3 + \frac{6}{x^2 - 4x + 3}$. Since $\lim\limits_{x\to\infty} \frac{6}{x^2 - 4x + 3}$ and $\lim\limits_{x\to-\infty} \frac{6}{x^2 - 4x + 3} = 0$, the line $y = x + 3$ is an asymptote to the function $f(x)$.

8. a. At a point x, the difference between the function $f(x) = f(x) = 3x - 7 + \frac{4}{x - 3}$ and its oblique asymptote $y = 3x - 7$ is $3x - 7 + \frac{4}{x - 3} - (3x - 7) = \frac{4}{x - 3}$. When x is large and positive, this difference is positive, which means that the curve approaches the asymptote from above. When x is large and negative, this difference is negative, which means that the curve approaches the asymptote from below.

b. At a point x, the difference between the function $f(x) = x + 3 - \frac{7}{2x + 3}$ and its oblique asymptote $y = x + 3$ is $x + 3 - \frac{7}{2x + 3} - (x + 3) = -\frac{7}{2x + 3}$. When x is large and positive, this difference is negative, which means that the curve approaches the asymptote from below. When x is large and negative, this difference is positive, which means that the curve approaches the asymptote from above.

9. a. This function is discontinuous at $x = -5$. The numerator is non-zero at this point, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

x-values	$3x - 1$	$x + 5$	$f(x)$	$\lim_{x\to c} f(x)$
$x \to -5^-$	< 0	< 0	> 0	$+\infty$
$x \to -5^+$	< 0	> 0	< 0	$-\infty$

To check for a horizontal asymptote:

$$\lim_{x\to\infty} \frac{3x - 1}{x + 5} = \lim_{x\to\infty} \frac{x\left(3 - \frac{1}{x}\right)}{x\left(1 + \frac{5}{x}\right)}$$
$$= \lim_{x\to\infty} \frac{3 - \frac{1}{x}}{1 + \frac{5}{x}}$$
$$= \frac{\lim_{x\to\infty}\left(3 - \frac{1}{x}\right)}{\lim_{x\to\infty}\left(1 + \frac{5}{x}\right)}$$
$$= \frac{3 - 0}{1 + 0}$$
$$= 1$$

Similarly, $\lim_{x\to\infty} \frac{3x - 1}{x + 5} = 3$, so $y = 3$ is a horizontal asymptote of the function.

b. This function is discontinuous at $x = 1$. The numerator is non-zero at this point, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

To check for a horizontal asymptote:

x-values	$x^2 + 3x - 2$	$(x - 1)^2$	$g(x)$	$\lim_{x\to c} g(x)$
$x \to 1^-$	> 0	> 0	> 0	$+\infty$
$x \to 1^+$	> 0	> 0	> 0	$+\infty$

$$\lim_{x\to\infty} \frac{x^2 + 3x - 2}{(x - 1)^2} = \lim_{x\to\infty} \frac{x^2 + 3x - 2}{x^2 - 2x + 1}$$
$$= \lim_{x\to\infty} \frac{x^2\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{x^2\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}$$
$$= \lim_{x\to\infty} \frac{1 + \frac{3}{x} - \frac{2}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}}$$
$$= \frac{\lim_{x\to\infty}\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{\lim_{x\to\infty}\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}$$
$$= \frac{1 + 0 - 0}{1 - 0 + 0}$$
$$= 1$$

Similarly, $\lim_{x\to\infty} \frac{x^2 + 3x - 2}{(x - 1)^2} = 1$, so $y = 1$ is a horizontal asymptote of the function.

c. This function is discontinuous when

$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = \pm 2.$$

At $x = 2$ the numerator is 0, since $2^2 + 2 - 6 = 0$, so the function has no vertical asymptote there. At $x = -2$, however, the numerator is non-zero, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

x-values	$x^2 + x - 6$	$x^2 - 4$	$h(x)$	$\lim_{x\to c} h(x)$
$x \to -2^-$	< 0	> 0	< 0	$-\infty$
$x \to -2^+$	< 0	< 0	> 0	$+\infty$

To check for a horizontal asymptote:

$$\lim_{x\to\infty} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x\to\infty} \frac{x^2\left(1 + \frac{1}{x} - \frac{6}{x^2}\right)}{x^2\left(1 - \frac{4}{x^2}\right)}$$
$$= \lim_{x\to\infty} \frac{1 + \frac{1}{x} - \frac{6}{x^2}}{1 - \frac{4}{x^2}}$$
$$= \frac{\lim_{x\to\infty}\left(1 + \frac{1}{x} - \frac{6}{x^2}\right)}{\lim_{x\to\infty}\left(1 - \frac{4}{x^2}\right)}$$

$$= \frac{1 + 0 - 0}{1 - 0}$$
$$= 1$$

Similarly, $\lim\limits_{x \to -\infty} \frac{x^2 + x - 6}{x^2 - 4} = 1$, so $y = 1$ is a horizontal asymptote of the function.

d. This function is discontinuous at $x = 2$. The numerator is non-zero at this point, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

x-values	$5x^2 - 3x + 2$	$x - 2$	$m(x)$	$\lim\limits_{x \to c} m(x)$
$x \to 2^-$	> 0	< 0	< 0	$-\infty$
$x \to 2^+$	> 0	> 0	> 0	$+\infty$

To check for a horizontal asymptote:

$$\lim_{x \to \infty} \frac{x^2 + 3x - 2}{(x - 1)^2} = \lim_{x \to \infty} \frac{x^2 + 3x - 2}{x^2 - 2x + 1}$$
$$= \lim_{x \to \infty} \frac{x^2\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{x^2\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}$$
$$= \lim_{x \to \infty} \frac{1 + \frac{3}{x} - \frac{2}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}}$$
$$= \frac{\lim\limits_{x \to \infty}\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{\lim\limits_{x \to \infty}\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}$$
$$= \frac{1 + 0 - 0}{1 - 0 + 0}$$
$$= 1$$

Similarly, $\lim\limits_{x \to \infty} \frac{x^2 + 3x - 2}{(x - 1)^2} = 1$, so $y = 1$ is a horizontal asymptote of the function.

10. a. $f(x) = \frac{3 - x}{2x + 5}$

Discontinuity is at $x = -2.5$.

$$\lim_{x \to -25^-} \frac{3 - x}{2x + 5} = -\infty$$
$$\lim_{x \to -25^+} \frac{3 - x}{2x + 5} = +\infty$$

Vertical asymptote is at $x = -2.5$.

Horizontal asymptote:

$$\lim_{x \to \infty} \frac{3 - x}{2x + 5} = -\frac{1}{2},$$
$$\lim_{x \to -\infty} \frac{3 - x}{2x + 5} = -\frac{1}{2}.$$

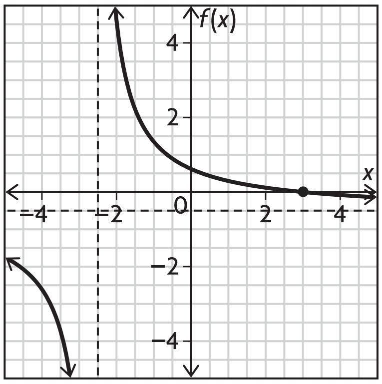
Horizontal asymptote is at $y = -\frac{1}{2}$.

$$f'(x) = \frac{-(2x + 5) - 2(3 - x)}{(2x + 5)^2} = \frac{-11}{(2x + 5)^2}$$

Since $f'(x) \neq 0$, there are no maximum or minimum points.

y-intercept, let $x = 0$, $y = \frac{3}{5} = 0.6$

x-intercept, let $y = 0$, $\frac{3 - x}{2x + 5} = 0$, $x = 3$



b. This function is a polynomial, so it is continuous for every real number. It has no horizontal, vertical, or oblique asymptotes.

The y-intercept can be found by letting $t = 0$, which gives $y = -10$.

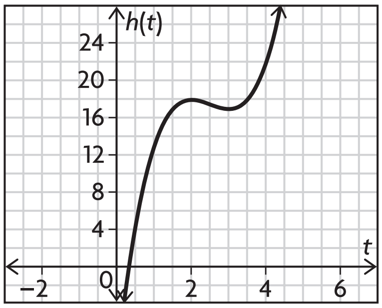
$h'(t) = 6t^2 - 30t + 36$

Set $h'(t) = 0$ and solve for t to determine the critical points.

$$6t^2 - 30t + 36 = 0$$
$$t^2 - 5t + 6 = 0$$
$$(t - 2)(t - 3) = 0$$
$$t = 2 \text{ or } t = 3$$

t	$t < 2$	$t = 2$	$2 < t < 3$	$t = 3$	$t > 3$
$h'(t)$	$+$	0	$-$	0	$+$
Graph	Inc.	Local Max	Dec.	Local Min	Inc.

The x-intercept cannot be easily obtained algebraically. Since the polynomial function has a local maximum when $x = 2$, it must have an x-intercept prior to this x-value. Since $f(0) = -10$ and $f(1) = 13$, an estimate for the x-intercept is about 0.3.



c. This function is discontinuous when
$x^2 + 4 = 0$
$x^2 = -4$
This equation has no real solutions, however, so the function is continuous everywhere.
To check for a horizontal asymptote:

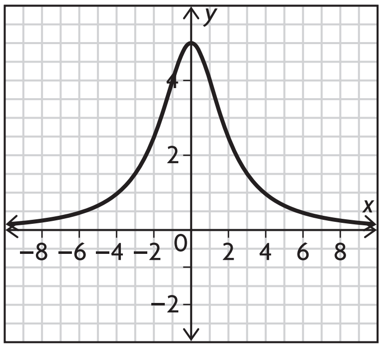
$$\lim_{x\to\infty} \frac{20}{x^2+4} = \lim_{x\to\infty} \frac{20}{x^2\left(1+\frac{4}{x^2}\right)}$$
$$= \frac{\lim_{x\to\infty}(20)}{\lim_{x\to\infty}\left(x^2\left(1+\frac{4}{x^2}\right)\right)}$$
$$= \frac{\lim_{x\to\infty}(20)}{\lim_{x\to\infty}(x^2) \times \lim_{x\to\infty}\left(1+\frac{4}{x^2}\right)}$$
$$= \lim_{x\to\infty}\frac{1}{x^2} \times \frac{20}{1+0}$$
$$= 0$$

Similarly, $\lim_{x\to-\infty} \frac{20}{x^2+4} = 0$, so $y = 0$ is a horizontal asymptote of the function.
The y-intercept of this function can be found by letting $x = 0$, which gives $y = \frac{20}{0^2+4} = 5$. Since the numerator of this function is never 0, it has no x-intercept. The derivative can be found by rewriting the function as $y = 20(x^2+4)^{-1}$, then

$$y' = -20(x^2+4)^{-1}(2x)$$
$$= -\frac{40x}{(x^2+4)^2}$$

Letting $y' = 0$ shows that $x = 0$ is a critical point of the function.

x	$x < 0$	$x = 0$	$x > 0$
y'	+	0	−
Graph	Inc.	Local Max	Dec.

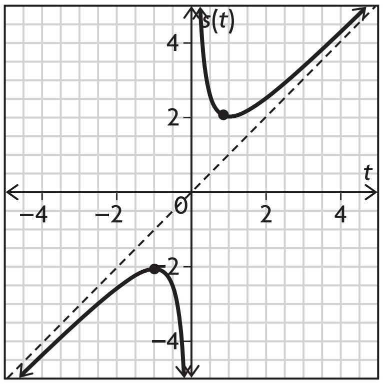


d. $s(t) = t + \frac{1}{t}$
Discontinuity is at $t = 0$.

$$\lim_{t\to0^+}\left(t + \frac{1}{t}\right) = +\infty$$
$$\lim_{t\to0^-}\left(t + \frac{1}{t}\right) = -\infty$$

Oblique asymptote is at $s(t) = t$.
$s'(t) = 1 - \frac{1}{t^2}$
Let $s'(t) = 0$, $t^2 = 1$
$t = \pm1$.
Local maximum is at $(-1, -2)$ and local minimum is at $(1, 2)$.

t	$t < -1$	$t = -1$	$-1 < t < 0$	$0 < t < 1$	$t = 1$	$t > 1$
$s'(t)$	+	0	−	−	0	+
Graph	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing



e. $g(x) = \dfrac{2x^2 + 5x + 2}{x + 3}$

Discontinuity is at $x = -3$.

$$\frac{2x^2 + 5x + 2}{x + 3} = 2x - 1 + \frac{5}{x + 3}$$

Oblique asymptote is at $y = 2x - 1$.

$\lim_{x \to -3^+} g(x) = +\infty$, $\lim_{x \to -3^-} g(x) = -\infty$

$$g'(x) = \frac{(4x + 5)(x + 3) - (2x^2 + 5x + 2)}{(x + 3)^2}$$

$$= \frac{2x^2 + 12x + 13}{(x + 3)^2}$$

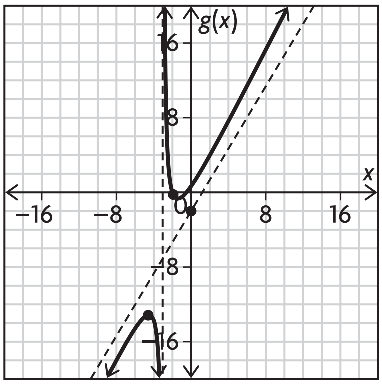
Let $g'(x) = 0$, therefore, $2x^2 + 12x + 13 = 0$:

$$x = \frac{-12 \pm \sqrt{144 - 104}}{4}$$

$x = -1.4$ or $x = -4.6$.

t	$x < -4.6$	-4.6	$-4.6 < x < -3$	-3	$-3 < x < -1.4$	$x = 1.4$	$x > -1.4$
$s'(t)$	+	0	−	Undefined	−	0	+
Graph	Increasing	Local Max	Decreasing	Vertical Asymptote	Decreasing	Local Min	Increasing

Local maximum is at $(-4.6, -10.9)$ and local minimum is at $(-1.4, -0.7)$.



f. $s(t) = \dfrac{t^2 + 4t - 21}{t - 3}, t \geq -7$

$$= \frac{(t + 7)(t - 3)}{(t - 3)}$$

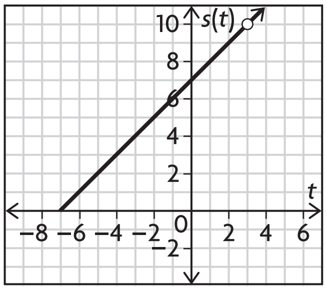
Discontinuity is at $t = 3$.

$$\lim_{x \to 3^+} \frac{(t + 7)(t - 3)}{(t - 3)} = \lim_{x \to 3^+} (t + 7)$$

$$= 10$$

$$\lim_{x \to 3^-} (t + 7) = 10$$

There is no vertical asymptote. The function is the straight line $s = t + 7, t \geq -7$.



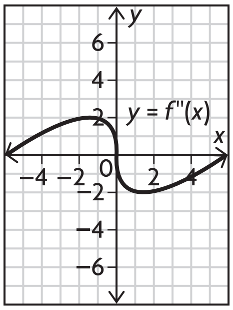
11. a. The horizontal asymptote occurs at $y = \frac{a}{c}$.

b. The vertical asymptote occurs when $cx + d = 0$ or $x = -\frac{d}{c}$.

---

12. a. Since f is defined for all values of x, f' and f'' are also defined for all values of x. f has a horizontal asymptote at $y = 0$. So f' and f'' will have horizontal asymptotes there. f has a local maximum at (0, 1) so f' will be 0 when $x = 0$. f has a point of inflection at $(-0.7, 0.6)$ and $(0.7, 0.6)$, so f'' will be 0 at $x = \pm 0.7$. At $x = 0.7$, f changes from concave up to concave down, so the sign of f'' changes from positive to negative. At $x = 0.7$, f changes from concave down to concave up, so the sign of f'' changes from negative to positive there. f is increasing for $x < 0$, so f' will be positive. f is decreasing for $x > 0$, so f' will be negative. The graph of f is concave up for $x < -0.7$ and $x > 0.7$, so f'' is positive for $x < -0.7$ and $x > 0.7$. The graph of f is concave down for $-0.7 < x < 0.7$, so f'' is negative for $-0.7 < x < 0.7$.

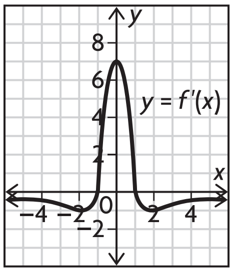
Also, since f'' is 0 at $x = \pm 0.7$, the graph of f' will have a local minimum or local maximum at these points. Since the sign of f'' changes from negative to positive at $x = 0.7$, it must be a local minimum point. Since the sign of f'' changes from positive to negative at $x = -0.7$, it must be a local maximum point.



b. Since f is defined for all values of x, f' and f'' are also defined for all values of x. f has a horizontal asymptote at $y = 0$ so f' and f'' will have a horizontal asymptote there. f has a local maximum at $(1, 3.5)$ so f' will be 0 when $x = 1$. f has a local minimum at $(-1, -3.5)$ so f' will be 0 when $x = -1$. f has a point of inflection at $(-1.7, -3)$, $(1.7, 3)$ and $(90, 0)$ so f'' will be 0 at $x = \pm 1.7$ and $x = 0$. At $x = 0$, f changes from concave up to concave down, so the sign of f'' changes from positive to negative.

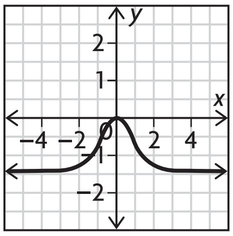
At $x = -1.7$, f changes from concave down to concave up, so the sign of f'' changes from negative to positive. At $x = 1.7$, f changes from concave down to concave up, so the sign of f'' changes from negative to positive. f is decreasing for $x < -1$ and $x > 1$, so f' will be negative. The graph of f is concave up for $-1.7 < x < 0$ and $x > 1.7$, so f'' is positive for $-1.7 < x < 0$ and $x > 1.7$. The graph of f is concave down for $x < -1.7$ and $0 < x < 1.7$, so f'' is negative for $x < -1.7$ and $0 < x < 1.7$.

Also, since f'' is 0 when $x = 0$ and $x = \pm 1.7$, the graph of f' will have a local maximum or minimum at these points. Since the sign of f'' changes from negative to positive at $x = -1.7$, f' has a local minimum at $x = -1.7$. Since the sign of f'' changes from positive to negative at $x = 0$, it must be a local maximum point. Since the sign of f'' changes from negative to positive at $x = 1.7$, it must be a local minimum point.

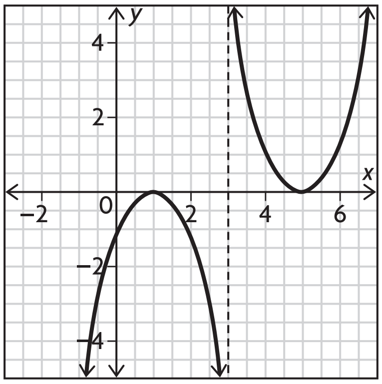


13. a. If $f(x)$ is increasing, then $f'(x) > 0$. From the graph of f', $f'(x) > 0$ for $x < 0$. If $f(x)$ is decreasing, then $f'(x) < 0$. From the graph of f', $f'(x) < 0$ for $x > 0$. At a stationary point, $f'(x) = 0$. From the graph, the zero of $f'(x)$ occurs at $(0, 0)$. At $x = 0$, The graph changes from positive to negative, so f has a local maximum there. If the graph of f is concave up, $f''(x)$ is positive. From the slope of f', the graph of f is concave up for $x < -0.6$ and $x > 0.6$. If the graph of f is concave down, $f''(x)$ is negative and concave down for $-0.6 < x < 0.6$. Graphs will vary slightly.

An example showing the shape of the curve is illustrated.



b. If $f(x)$ is increasing, then $f'(x) > 0$. From the graph of f', $f''(x) > 0$ for $x < 1$ and $x > 5$. If $f(x)$ is decreasing, then $f'(x) < 0$. From the graph of f', $f'(x) < 0$ for $1 < x < 3$ and $3 < x < 5$. At a stationary point, $f'(x) = 0$. From the graph, the zeros of $f'(x)$ occur at $x = 1$ and $x = 5$. At $x = 1$, the graph changes from positive to negative, so f has a local maximum there. At $x = 5$, the graph changes from negative to positive, so f has a local minimum there. If the graph of f is concave up, $f''(x)$ is positive. From the slope of f', the graph is concave up for $x > 3$. If the graph of f is concave down, $f''(x)$ is negative. From the slope of f', the graph of f is concave down for $x < 3$. There is a vertical asymptote at $x = 3$ since f' is not defined there. Graphs will vary slightly. An example showing the shape of the curve is illustrated.



14. a. $f(x)$ and $r(x)$: $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} r(x)$ exist.

b. $h(x)$: the highest degree of x in the numerator is exactly one degree higher than the highest degree of x in the denominator.

c. $h(x)$: the denominator is defined for all $x \in \mathbf{R}$.

$f(x) = \dfrac{-x-3}{(x-7)(x+2)}$ has vertical asymptotes at $x = 7$ and $x = -2$. $f(-2.001) = -110.99$ so as $x \to -2^-$, $f(x) \to -\infty$

$f(-1.999) = 111.23$ so as $x \to -2^+$, $f(x) \to \infty$

$f(6.999) = 111.12$ so as $x \to 7^-$, $f(x) \to \infty$

$f(7.001) = -111.10$ so as $x \to 7^+$, $f(x) \to -\infty$

$f(x)$ has a horizontal asymptote at $y = 0$.

$g(x)$ has a vertical asymptote at $x = 3$.

$g(2.999) = 23\,974.009$ so as $x \to 3^-$, $g(x) \to \infty$

$g(3.001) = -24\,026.009$ so as $x \to 3^+$, $g(x) \to -\infty$

By long division, $h(x) = x + \left(\dfrac{-4x-1}{x^2+1}\right)$ so $y = x$ is an oblique asymptote.

$r(x) = \dfrac{(x+3)(x-2)}{(x-4)(x+4)}$ has vertical asymptotes at $x = -4$ and $x = 4$.

$r(-4.001) = 750.78$ so as $x \to -4^-$, $r(x) \to \infty$

$r(-3.999) = -749.22$ so as $x \to -4^+$, $r(x) \to -\infty$

$r(3.999) = -1749.09$ so as $x \to 4^-$, $r(x) \to -\infty$

$r(4.001) = 1750.91$ so as $x \to 4^+$, $r(x) \to \infty$

$r(x)$ has a horizontal asymptote at $y = 1$.

15. $f(x) = \dfrac{ax+5}{3-bx}$

Vertical asymptote is at $x = -4$.

Therefore, $3 - bx = 0$ at $x = -5$.

That is, $3 - b(-5) = 0$

$$b = \frac{3}{5}.$$

Horizontal asymptote is at $y = -3$.

$$\lim_{x\to\infty}\left(\frac{ax+5}{3-bx}\right) = -3$$

$$\lim_{x\to\infty}\left(\frac{ax+5}{3-bx}\right) = \lim_{x\to\infty}\left(\frac{a+\frac{5}{x}}{\frac{3}{x}-b}\right) = \frac{-a}{b}$$

But $-\frac{a}{b} = -3$ or $a = 3b$.

But $b = \frac{3}{5}$, then $a = \frac{9}{5}$.

16. a.
$$\lim_{x\to\infty}\frac{x^2+1}{x+1} = \lim_{x\to\infty}\frac{x+\frac{1}{x}}{1+\frac{1}{x}} = \infty$$

$$\lim_{x\to\infty}\frac{x^2+2x+1}{x+1} = \lim_{x\to\infty}\frac{(x+1)(x+1)}{(x+1)} = \lim_{x\to\infty}(x+1) = \infty$$

b.
$$\lim_{x\to\infty}\left[\frac{x^2+1}{x+1} - \frac{x^2+2x+1}{x+1}\right]$$
$$= \lim_{x\to\infty}\frac{x^2+1-x^2-2x-1}{x+1}$$
$$= \lim_{x\to\infty}\frac{-2x}{x+1}$$
$$= \lim_{x\to\infty}\frac{-2}{1+\frac{1}{x}} = -2$$

17. $f(x) = \dfrac{2x^2-2x}{x^2-9}$

Discontinuity is at $x^2 - 9 = 0$ or $x = \pm 3$.

$\lim_{x\to 3^+} f(x) = +\infty$

$\lim_{x\to 3^-} f(x) = -\infty$

$\lim_{x\to -3^+} f(x) = -\infty$

$\lim_{x\to -3^-} f(x) = +\infty$

Vertical asymptotes are at $x = 3$ and $x = -3$.

Horizontal asymptote:

$\lim_{x\to\infty} f(x) = 2$ (from below)

$\lim_{x\to\infty} f(x) = 2$ (from above)

Horizontal asymptote is at $y = 2$.

$$f'(x) = \frac{(4x-2)(x^2-9) - 2x(2x^2-2x)}{(x^2-9)^2}$$
$$= \frac{4x^3 - 2x^2 - 36x + 18 - 4x^3 + 4x^2}{(x^2-9)^2}$$
$$= \frac{2x^2 - 36x + 18}{(x^2-9)^2}$$

Let $f'(x) = 0$,

$2x^2 - 36x + 18 = 0$ or $x^2 - 18x + 9 = 0$.

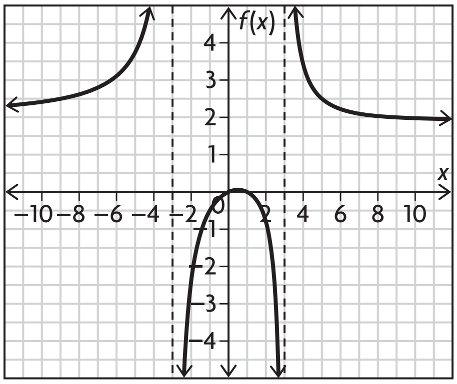
$$x = \frac{18 \pm \sqrt{18^2 - 36}}{2}$$

$x = 0.51$ or $x = 17.5$

$y = 0.057$ or $y = 1.83$.

Local maximum is at $(0.51, 0.057)$ and local minimum is at $(17.5, 1.83)$.

t	$-3 < x < 0.51$	0.51	$0.51 < x < 3$	$3 < x < 17.5$	17.5	$x > 17.5$
$s'(t)$	+	0	−	−	0	+
Graph	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing



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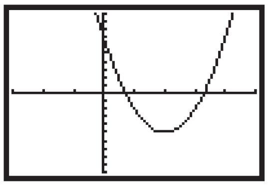
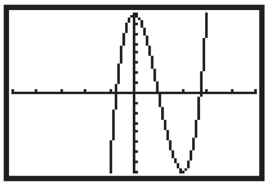
1. a.



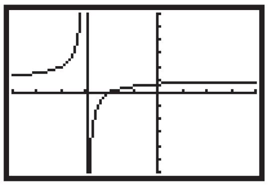
The function appears to be decreasing on $(-\infty, 2)$ and increasing on $(2, \infty)$.

b.



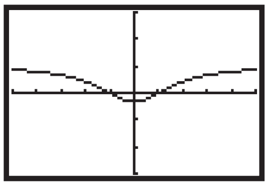
The function appears to be increasing on $(-\infty, 0)$ and $(2, \infty)$ and decreasing on $(0, 2)$.

c.



The function is increasing on $(-\infty, -3)$ and $(-3, \infty)$.

d.



The function appears to be decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

2. The slope of a general tangent to the graph $g(x) = 2x^3 - 3x^2 - 12x + 15$ is given by $\frac{dg}{dx} = 6x^2 - 6x - 12$. We first determine values of x for which $\frac{dg}{dx} = 0$.

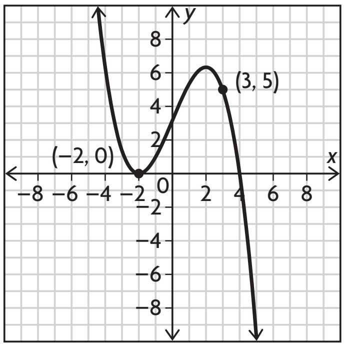
So $6x^2 - 6x - 12 = 0$

$$6(x^2 - x - 2) = 0$$
$$6(x + 1)(x - 2) = 0$$
$$x = -1, x = 2$$

Since $\frac{dg}{dx}$ is defined for all values of x, and since $\frac{dg}{dx} = 0$ only at $x = -1$ and $x = 2$, it must be either positive or negative for all other values of x. Consider the intervals between $x < -1$, $-1 < x < 2$, and $x > 2$.

Value of x	$x < -1$	$-1 < x < 2$	$x > 2$
Value of $\frac{dg}{dx} = 6x^2 - 6x - 12$	$\frac{dg}{dx} > 0$	$\frac{dg}{dx} < 0$	$\frac{dg}{dx} > 0$
Slope of Tangents	positive	negative	positive
y-values Increasing or Decreasing	increasing	decreasing	increasing

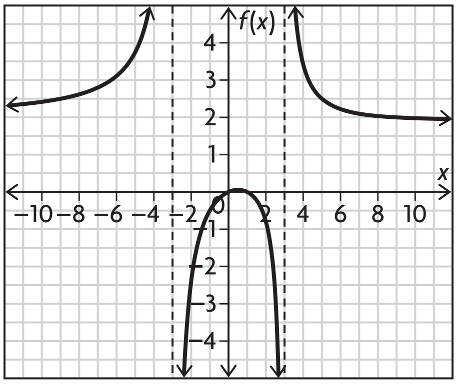
3.



4. The critical numbers can be found when $\frac{dy}{dx} = 0$.

a. $\frac{dy}{dx} = -4x + 16$. When $\frac{dy}{dx} = 0$,

$$\frac{dy}{dx} = -4(x + 4) = 0$$
$$x = -4$$



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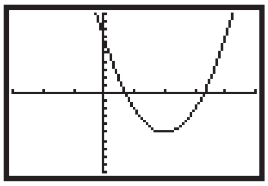
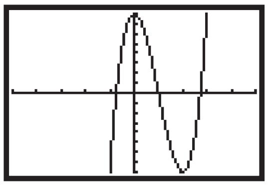
1. a.



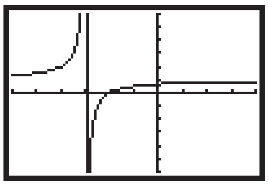
The function appears to be decreasing on $(-\infty, 2)$ and increasing on $(2, \infty)$.

b.



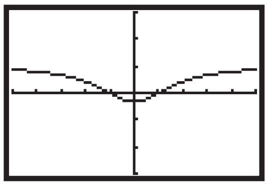
The function appears to be increasing on $(-\infty, 0)$ and $(2, \infty)$ and decreasing on $(0, 2)$.

c.



The function is increasing on $(-\infty, -3)$ and $(-3, \infty)$.

d.



The function appears to be decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

2. The slope of a general tangent to the graph $g(x) = 2x^3 - 3x^2 - 12x + 15$ is given by $\frac{dg}{dx} = 6x^2 - 6x - 12$. We first determine values of x for which $\frac{dg}{dx} = 0$.

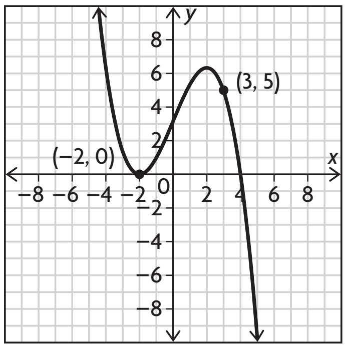
So $6x^2 - 6x - 12 = 0$

$$6(x^2 - x - 2) = 0$$
$$6(x + 1)(x - 2) = 0$$
$$x = -1, x = 2$$

Since $\frac{dg}{dx}$ is defined for all values of x, and since $\frac{dg}{dx} = 0$ only at $x = -1$ and $x = 2$, it must be either positive or negative for all other values of x. Consider the intervals between $x < -1$, $-1 < x < 2$, and $x > 2$.

Value of x	$x < -1$	$-1 < x < 2$	$x > 2$
Value of $\frac{dg}{dx} = 6x^2 - 6x - 12$	$\frac{dg}{dx} > 0$	$\frac{dg}{dx} < 0$	$\frac{dg}{dx} > 0$
Slope of Tangents	positive	negative	positive
y-values Increasing or Decreasing	increasing	decreasing	increasing

3.



4. The critical numbers can be found when $\frac{dy}{dx} = 0$.

a. $\frac{dy}{dx} = -4x + 16$. When $\frac{dy}{dx} = 0$,

$$\frac{dy}{dx} = -4(x + 4) = 0$$
$$x = -4$$

	$x < 1$	$1 < x < 2$	$x > 2$
$x - 1$	$-$	$+$	$+$
$x - 2$	$-$	$-$	$+$
$(x - 1)(x - 2)$	$(-)(-) = +$	$(+)(-) = -$	$(+)(+) = +$
$\frac{dy}{dx}$	> 0	< 0	> 0
$g(x) = 2x^3 - 9x^2 + 12x$	increasing	decreasing	increasing

b. $\frac{dy}{dx} = x^3 - 27x$. When $\frac{dy}{dx} = 0$,

$\frac{dy}{dx} = x(x^2 - 27) = 0$

$x = 0, x = \pm 3\sqrt{3}$

c. $\frac{dy}{dx} = 4x^3 - 8x$. When $\frac{dy}{dx} = 0$,

$\frac{dy}{dx} = 4x(x^2 - 2) = 0$

$x = 0, x = \pm\sqrt{2}$

d. $\frac{dy}{dx} = 15x^4 - 75x^2 + 60$. When $\frac{dy}{dx} = 0$,

$\frac{dy}{dx} = 15(x^4 - 5x^2 + 4) = 0$

$\frac{dy}{dx} = 15(x^2 - 1)(x^2 - 4) = 0$

$x = \pm 1, x = \pm 2$

e. $\frac{dy}{dx} = \frac{2x(x^2 + 1) - (x^2 - 1)(2x)}{(x^2 + 1)^2}$. When $\frac{dy}{dx} = 0$, the numerator equals 0. So $\frac{dy}{dx} = 2x(x^2 + 1) - (x^2 - 1)(2x) = 0$. After simplifying, $\frac{dy}{dx} = 4x = 0$.
$x = 0$

f. $\frac{dy}{dx} = \frac{(x^2 + 2) - x(2x)}{(x^2 + 2)^2}$. When $\frac{dy}{dx} = 0$, the numerator equals 0. So after simplifying,

$\frac{dy}{dx} = -x^2 + 2 = 0$.

$x = \pm\sqrt{2}$

5. a. $\frac{dg}{dx} = 6x^2 - 18x + 12$

To find the critical numbers, set $\frac{dg}{dx} = 0$. So

$6x^2 - 18x + 12 = 0$

$6(x - 1)(x - 2) = 0$

$x = 1, x = 2$

From the table above, $x = 1$ is the local maximum and $x = 2$ is the local minimum.

b. $\frac{dg}{dx} = 3x^2 - 4x - 4$

To find the critical numbers, set $\frac{dg}{dx} = 0$.

$3x^2 - 4x - 4 = 0$

$(3x + 2)(x - 2) = 0$

$x = -\frac{2}{3}$ or $x = 2$

	$x < -\frac{2}{3}$	$-\frac{2}{3} < x < 2$	$x > 2$
$3x + 2$	$-$	$+$	$+$
$x - 2$	$-$	$-$	$+$
$\frac{dg}{dx}$	$+$	$-$	$+$
$g(x)$	increasing	decreasing	increasing

The function has a local maximum at $x = -\frac{2}{3}$ and a local minimum at $x = 2$

6. $\frac{df}{dx} = 2x + k$

To have a local minimum value, $\frac{df}{dx} = 0$. This occurs when $x = -\frac{k}{2}$. So $f\left(-\frac{k}{2}\right) = 1$.

$\frac{k^2}{4} - \frac{k^2}{2} + 2 = 1$

$-\frac{k^2}{4} + 2 = 1$

$-\frac{k^2}{4} = -1$

$k^2 = 4$

$k = \pm 2$

7. $f'(x) = 4x^3 - 32$

To find the critical numbers, set $f'(x) = 0$.

$$4x^3 - 32 = 0$$
$$4(x^3 - 8) = 0$$
$$x = 2$$

	$x < 2$	$x > 2$
$f'(x) = 4x^3 - 32$	$-$	$+$
$f(x)$	decreasing	increasing

The function has a local minimum at $x = 2$.

8. a. Since $x + 2 = 0$ for $x = -2$, $x = -2$ is a vertical asymptote. Large and positive to left of asymptote, large and negative to right of asymptote.

b. Since $9 - x^2 = 0$ for $x = \pm 3$, $x = -3$ and $x = 3$ are vertical asymptotes. For $x = -3$: large and negative to left of asymptote, large and positive to right of asymptote.

c. Since $3x + 9 = 0$ for $x = -3$, $x = -3$ is a vertical asymptote. Large and negative to left of asymptote, large and positive to right of asymptote.

d. Since $3x^2 - 13x - 10 = 0$ when $x = -\frac{2}{3}$ and $x = 5$, $x = -\frac{2}{3}$ and $x = 5$ are vertical asymptotes. For $x = -\frac{2}{3}$ large and positive to left of asymptote, large and negative to right of asymptote. For $x = 3$: large and positive to left of asymptote, large and negative to right of asymptote.

9. a. $$f(x) = \frac{3x - 1}{x + 5} = \frac{3x\left(1 - \frac{1}{3x}\right)}{x\left(1 + \frac{5}{x}\right)}$$
$$= \frac{3\left(1 - \frac{1}{3x}\right)}{1 + \frac{5}{x}}$$
$$\lim_{x\to+\infty} f(x) = \frac{3\left[\lim_{x\to-\infty}\left(1 - \frac{1}{3x}\right)\right]}{\lim_{x\to-\infty}\left(1 + \frac{5}{x}\right)}$$
$$= \frac{3(1 - 0)}{(1 + 0)}$$
$$= 3$$

So the horizontal asymptote is $y = 3$. Similarly, we can prove $\lim_{x\to-\infty} f(x) = 3$. If x is large and positive, for example, if $x = 1000$, $f(x) = \frac{2999}{1005}$, which is smaller than 3. If x is large and negative, for example, if $x = -1000$, $f(x) = \frac{-3001}{-995}$, which is larger than 3. So $f(x)$ approaches $y = 3$ from below when x is large and positive and approached $y = 3$ from above when x is large and negative.

b. $$f(x) = \frac{x^2 + 3x - 2}{(x - 1)^2} = \frac{x^2\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{x^2\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}$$
$$= \frac{\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}$$
$$\lim_{x\to+\infty} \frac{\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)} = \frac{(1 + 0 - 0)}{(1 - 0 + 0)}$$
$$= 1$$

So the horizontal asymptote is 1. Similarly, we can prove $\lim_{x\to-\infty} f(x) = 1$. If x is large and positive, for example, $x = 1000$, $f(x) = \frac{1000^2 + 3(1000) - 2}{(1000 - 1)^2} = \frac{9\,96\,998}{10\,02\,001}$, which is greater than 1. If x is large and negative, for example, $x = -1000$, $f(x) = \frac{(-1000)^2 + 3(-1000) - 2}{(-1000 - 1)^2} = \frac{9\,96\,998}{10\,02\,001}$, which is less than 1. So $f(x)$ approaches $y = 1$ from above when x is large and positive and approaches $y = 1$ from below when x is large and negative.

10. a. Since $(x - 5)^2 = 0$ when $x = 5$, $x = 5$ is a vertical asymptote.

$$f(x) = \frac{x}{(x - 5)^2} = \frac{x}{x^2\left(1 - \frac{10}{x} + \frac{25}{x^2}\right)}$$
$$= \frac{1}{x\left(1 - \frac{10}{x} + \frac{25}{x^2}\right)}$$
$$\lim_{x\to+5} f(x) = \frac{\lim_{x\to+5}(1)}{\lim_{x\to+5}\left(x\left(1 - \frac{10}{x} + \frac{25}{x^2}\right)\right)} = +\infty$$

This limit gets larger as it approaches 5 from the right. Similarly, we can prove that the limit goes to $+\infty$ as it approaches 5 from the left. For example, if $x = 1000$ $f(x) = \frac{1}{1000\left(1 - \frac{10}{1000} + \frac{25}{1000^2}\right)}$, which gets larger as x gets larger. Thus, $f(x)$ approaches $+\infty$ on both sides of $x = 5$.

b. There are no discontinuities because $x^2 + 9$ never equals zero.

c. Using the quadratic formula, we find that $x^2 - 12x + 12 = 0$ when $x = 6 \pm 2\sqrt{6}$. So $x = 6 \pm 2\sqrt{6}$ are vertical asymptotes.

$$f(x) = \frac{x - 2}{x^2 - 12x + 12} = \frac{x\left(1 - \frac{2}{x}\right)}{x^2\left(1 - \frac{12}{x} + \frac{12}{x^2}\right)}$$

$$\lim_{x\to+6+2\sqrt{6}} f(x) = \frac{\lim_{x\to+6+2\sqrt{6}} x\left(1 - \frac{2}{x}\right)}{\lim_{x\to+6+2\sqrt{6}} x^2\left(1 - \frac{12}{x} + \frac{12}{x^2}\right)}.$$

This limit gets smaller as it approaches $6 + 2\sqrt{6}$ from the right and get larger as it approaches $6 + 2\sqrt{6}$ from the left. Similarly, we can show that the limit gets smaller as it approaches $6 - 2\sqrt{6}$ from the left and gets larger as it approaches from the right.

11. a. $f'(x) > 0$ implies that $f(x)$ is increasing.

b. $f'(x) < 0$ implies that $f(x)$ is decreasing.

12. a. $h(t) = -4.9t^2 + 9.5t + 2.2$

Note that $h(0) = 2.2 < 3$ because when the diver dives, the board is curved down.

$h'(t) = -9.8t + 9.5$

Set $h'(t) = 0$

$0 = -9.8t + 9.5$

$t \doteq 0.97$

	$0 < t < 0.97$	$t > 0.97$
$-9.8t + 9.5$	+	−
Sign of $h'(t)$	+	−
Behaviour of $h(t)$	increasing	decreasing

b. $h'(t) = v(t)$

$v(t) = -9.8t + 9.5$

$v'(t) = -9.8 < 0$

The velocity is decreasing all the time.

13. $C(t) = \frac{t}{4} + 2t^{-2}$

$C'(t) = \frac{1}{4} - 4t^{-3}$

Set $C'(t) = 0$

$0 = \frac{1}{4} - 4t^{-3}$

$\frac{1}{4} = 4t^{-3}$

$t^3 = 16$

$t \doteq 2.5198$

	$t < 2.5198$	$t > 2.5198$
$\frac{1}{4} - 4t^{-3}$	−	+
Sign of $C'(t)$	−	+
Behaviour of $C(t)$	decreasing	increasing

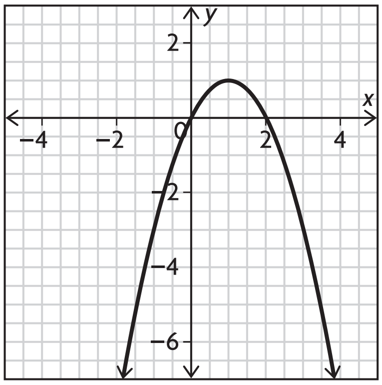
14. For $f(x)$ the derivative function $f'(0) = 0$ and $f'(2) = 0$.

Therefore, $f'(x)$ passes through (0, 0) and (2, 0).

When $x < 0$, $f(x)$ is decreasing, therefore, $f'(x) < 0$.

When $0 < x < 2$, $f(x)$ is increasing, therefore, $f'(x) > 0$.

When $x > 2$, $f(x)$ is decreasing, therefore, $f'(x) < 0$.



15. a. $f(x) = x^2 - 7x - 18$

i. $f'(x) = 2x - 7$

Set $f'(x) = 0$

$0 = 2x - 7$

$x = \frac{7}{2}$

ii.

	$x < \frac{7}{2}$	$x > \frac{7}{2}$
$2x - 7$	−	+
Sign of $f'(x)$	−	+
Behaviour of $f(x)$	decreasing	increasing

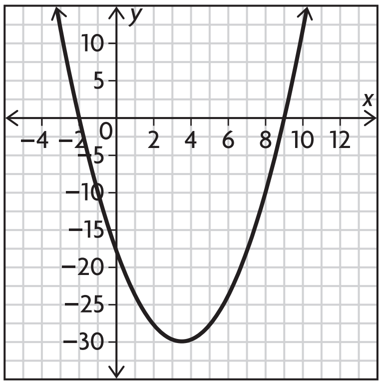
iii. From ii., there is a minimum at $x = \frac{7}{2}$.

$f\left(\frac{7}{2}\right) = \left(\frac{7}{2}\right)^2 - 7\left(\frac{7}{2}\right) - 18$

$f\left(\frac{7}{2}\right) = \frac{49}{4} - \frac{49}{2} - 18$

$f\left(\frac{7}{2}\right) = -\frac{121}{4}$

iv.



b. $f(x) = -2x^3 + 9x^2 + 3$

i. $f'(x) = -6x^2 + 18x$

Set $f'(x) = 0$

$$0 = -6x^2 + 18x$$
$$0 = -6x(x - 3)$$
$$x = 0 \text{ or } x = 3$$

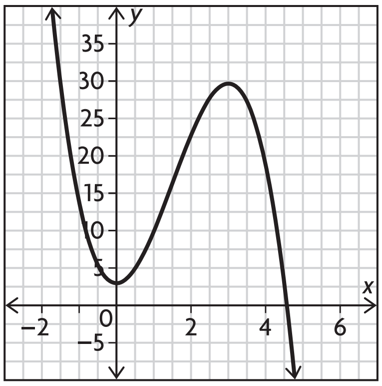
ii.

	$x < 0$	$0 < x < 3$	$x > 3$
$-6x$	+	−	−
$x - 3$	−	−	+
Sign of $f'(x)$	$(+)(-) = -$	$(-)(-) = +$	$(-)(+) = -$
Behaviour of $f(x)$	decreasing	increasing	decreasing

iii. From ii., there is a minimum at $x = 0$ and a maximum at $x = 3$.

$f(0) = -2(0)^3 + 9(0)^2 + 3$
$f(0) = 3$
$f(3) = -2(3)^3 + 9(3)^2 + 3$
$f(3) = -54 + 81 + 3$
$f(3) = 30$

iv.



c. $f(x) = 2x^4 - 4x^2 + 2$

i. $f'(x) = 8x^3 - 8x$

$f'(x) = 0$

$$0 = 8x^3 - 8x$$
$$0 = 8x(x^2 - 1)$$
$$0 = 8x(x - 1)(x + 1)$$
$$x = -1 \text{ or } x = 0 \text{ or } x = 1$$

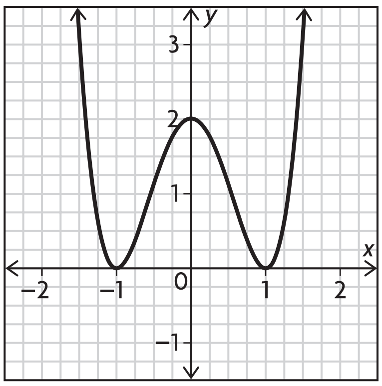
ii.

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$x > 1$
$8x$	−	−	+	+
$x - 1$	−	−	−	+
$x + 1$	−	+	+	+
Sign of $f'(x)$	$(-)(-)(-) = -$	$(-)(-)(+) = +$	$(+)(-)(+) = -$	$(+)(+)(+) = +$
Behaviour of $f(x)$	decreasing	increasing	decreasing	increasing

iii. From ii., there is a maximum at $x = 0$ and minima at $x = -1$ and $x = 1$

$f(-1) = 2(-1)^4 - 4(-1)^2 + 2$
$f(-1) = 2 - 4 + 2$
$f(-1) = 0$
$f(0) = 2(0)^4 - 4(0)^2 + 2$
$f(0) = 2$
$f(1) = 2(1)^4 - 4(1)^2 + 2$
$f(1) = 2 - 4 + 2$
$f(1) = 0$

iv.



d. $f(x) = x^5 - 5x$

i. $f'(x) = 5x^4 - 5$

Set $f'(x) = 0$

$$0 = 5x^4 - 5$$
$$0 = 5(x^4 - 1)$$
$$0 = 5(x^2 - 1)(x^2 + 1)$$
$$0 = 5(x - 1)(x + 1)(x^2 + 1)$$
$$x = -1 \text{ or } x = 1$$

ii.

	$x < -1$	$-1 < x < 1$	$x > 1$
5	+	+	+
$x - 1$	−	−	+
$x + 1$	−	+	+
$x^2 + 1$	+	+	+
Sign of $f'(x)$	$(+)(-)(-)(+) = +$	$(+)(-)(+)(+) = -$	$(+)(+)(+)(+) = +$
Behaviour of $f(x)$	increasing	decreasing	increasing

iii. From ii., there is a maximum at $x = -1$ and a minimum at $x = 1$

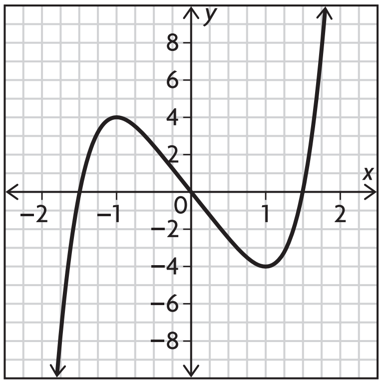
$f(-1) = (-1)^5 - 5(-1)$

$f(-1) = -1 + 5$

$f(-1) = 4$

$f(1) = (1)^5 - 5(1)$

$f(1) = -4$



16. a. vertical asymptote: $x = -\frac{1}{2}$, horizontal asymptote $y = \frac{1}{2}$; as x approaches $\frac{1}{2}$ from the left, graph approaches infinity; as x approaches $\frac{1}{2}$ from the right, graph approaches negative infinity.

b. vertical asymptote: $x = -2$, horizontal asymptote: $y = 1$; as x approaches -2 from the left, graph approaches infinity; as x approaches -2 from the right, graph decreases to $(-0.25, -1.28)$ and then approaches to infinity.

c. vertical asymptote: $x = -3$, horizontal asymptote: $y = -1$; as x approaches -3 from the left, graph approaches infinity; as x approaches -3 from the right, graph approaches infinity

d. vertical asymptote: $x = -4$, no horizontal asymptote; as x approaches -4 from the left, graph increases to $(-7.81, -30.23)$ and then decreases to -4; as x approaches -4 from the right, graph decreases to $(-0.19, 0.23)$ then approaches infinity.

17. a. $\lim_{x\to\infty} \frac{3 - 2x}{3x}$

$= \lim_{x\to\infty} \frac{\frac{3}{x} - \frac{2x}{x}}{\frac{3x}{x}}$

$= \frac{0 - 2}{3}$

$= -\frac{2}{3}$

b. $\lim_{x\to\infty} \frac{x^2 - 2x + 5}{6x^2 + 2x - 1}$

$= \lim_{x\to\infty} \frac{\frac{x^2}{x^2} - \frac{2x}{x^2} + \frac{5}{x^2}}{\frac{6x^2}{x^2} + \frac{2x}{x^2} - \frac{1}{x^2}}$

$= \frac{1 - 0 + 0}{6 + 0 - 0}$

$= \frac{1}{6}$

c. $\lim_{x\to\infty} \frac{7 + 2x^2 - 3x^3}{x^3 - 4x^2 + 3x}$

$= \lim_{x\to\infty} \frac{\frac{7}{x^3} + \frac{2x^2}{x^3} - \frac{3x^3}{x^3}}{\frac{x^3}{x^3} - \frac{4x^2}{x^3} + \frac{3x}{x^3}}$

$= \frac{0 + 0 - 3}{1 - 0 + 0}$

$= -3$

d. $\lim_{x\to\infty} \frac{5 + 2x^3}{x^4 - 4x}$

$= \lim_{x\to\infty} \frac{\frac{5}{x^4} - \frac{2x^3}{x^4}}{\frac{x^4}{x^4} - \frac{4x}{x^4}}$

$= \frac{0 - 0}{1 - 0}$

$= 0$

e. $\lim_{x\to\infty} \frac{2x^5 - 1}{3x^4 - x^2 - 2} = \lim_{x\to\infty} \left(\frac{2}{3}x + \frac{\frac{2}{3}x^3 + \frac{4}{3}x - 1}{3x^4 - x^2 - 2} \right)$

$= \lim_{x\to\infty} \frac{2}{3}x + \lim_{x\to\infty} \frac{\frac{\frac{2}{3}x^3}{x^4} + \frac{\frac{4}{3}x}{x^4} - \frac{1}{x^4}}{\frac{3x^4}{x^4} - \frac{x^2}{x^4} - \frac{2}{x^4}}$

$= \infty$

f. $\lim_{x\to\infty} \frac{x^2 + 3x - 18}{(x-3)^2}$

$= \lim_{x\to\infty} \frac{x^2 + 3x - 18}{x^2 - 6x + 9}$

$= \lim_{x\to\infty} \frac{\frac{x^2}{x^2} + \frac{3x}{x^2} - \frac{18}{x^2}}{\frac{x^2}{x^2} - \frac{6x}{x^2} + \frac{9}{x^2}}$

$= \frac{1 + 0 - 0}{1 - 0 + 0}$

$= 1$

g. $\lim_{x\to\infty} \frac{x^2 - 4x - 5}{x^2 - 1}$

$= \lim_{x\to\infty} \frac{\frac{x^2}{x^2} - \frac{4x}{x^2} - \frac{5}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$

$= \lim_{x\to\infty} \frac{1 - 0 - 0}{1 - 0}$

$= 1$

h. $\lim_{x\to\infty} \left(5x + 4 - \frac{7}{x+3}\right)$

$= \lim_{x\to\infty} 5x + \lim_{x\to\infty} 4 - \lim_{x\to\infty} \frac{\frac{7}{x}}{\frac{x}{x} + \frac{3}{x}}$

$= \infty$

4.4 Concavity and Points of Inflection, pp. 205–206

1. a. A: negative; B: negative; C: positive; D: positive

b. A: negative; B: negative; C: positive; D: negative

2. a. $y = x^3 - 6x^2 - 15x + 10$

$\frac{dy}{dx} = 3x^2 - 12x - 15$

For critical values, we solve $\frac{dy}{dx} = 0$:

$3x^2 - 12x - 15 = 0$

$x^2 - 4x - 5 = 0$

$(x - 5)(x + 1) = 0$

$x = 5$ or $x = -1$

The critical points are $(5, -105)$ and $(-1, 20)$.

Now, $\frac{d^2y}{dx^2} = 6x - 12$.

At $x = 5$, $\frac{d^2y}{dx^2} = 18 > 0$. There is a local minimum at this point.

At $x = -1$, $\frac{d^2y}{dx^2} = -18 < 0$. There is a local maximum at this point.

The local minimum is $(5, -105)$ and the local maximum is $(-1, 20)$

b. $y = \frac{25}{x^2 + 48}$

$\frac{dy}{dx} = -\frac{50x}{(x^2 + 48)^2}$

For critical values, solve $\frac{dy}{dx} = 0$ or $\frac{dy}{dx}$ does not exist.

Since $x^2 + 48 > 0$ for all x, the only critical point is $\left(0, \frac{25}{48}\right)$

$\frac{d^2y}{dx^2} = -50(x^2 + 48)^{-2} + 100x(x^2 + 48)^{-3}(2x)$

$= -\frac{50}{(x^2 + 48)^2} + \frac{200x^2}{(x^2 + 48)^3}$

At $x = 0$, $\frac{d^2y}{dx^2} = -\frac{50}{48^2} < 0$. The point $\left(0, \frac{25}{48}\right)$ is a local maximum.

c. $s = t + t^{-1}$

$\frac{ds}{dt} = 1 - \frac{1}{t^2}, t \neq 0$

For critical values, we solve $\frac{ds}{dt} = 0$:

$1 - \frac{1}{t^2} = 0$

$t^2 = 1$

$t = \pm 1.$

The critical points are $(-1, -2)$ and $(1, 2)$

$\frac{d^2s}{dt^2} = \frac{2}{t^3}$

At $t = -1$, $\frac{d^2s}{dt} = -2 < 0$. The point $(-1, -2)$ is a local maximum. At $t = 1$, $\frac{d^2s}{dt^2} = 2 > 0$. The point $(1, 2)$ is a local minimum.

d. $y = (x - 3)^3 + 8$

$\frac{dy}{dx} = 3(x - 3)^2$

$x = 3$ is a critical value.

The critical point is $(3, 8)$

$\frac{d^2y}{dx^2} = 6(x - 3)$

At $x = 3$, $\frac{d^2y}{dx^2} = 0$.

The point $(3, 8)$ is neither a relative (local) maximum or minimum.

3. a. For possible point(s) of inflection, solve $\frac{d^2y}{dx^2} = 0$:

$6x - 8 = 0$

$x = \frac{4}{3}.$

f. $\lim_{x\to\infty} \frac{x^2 + 3x - 18}{(x-3)^2}$

$= \lim_{x\to\infty} \frac{x^2 + 3x - 18}{x^2 - 6x + 9}$

$= \lim_{x\to\infty} \frac{\frac{x^2}{x^2} + \frac{3x}{x^2} - \frac{18}{x^2}}{\frac{x^2}{x^2} - \frac{6x}{x^2} + \frac{9}{x^2}}$

$= \frac{1 + 0 - 0}{1 - 0 + 0}$

$= 1$

g. $\lim_{x\to\infty} \frac{x^2 - 4x - 5}{x^2 - 1}$

$= \lim_{x\to\infty} \frac{\frac{x^2}{x^2} - \frac{4x}{x^2} - \frac{5}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$

$= \lim_{x\to\infty} \frac{1 - 0 - 0}{1 - 0}$

$= 1$

h. $\lim_{x\to\infty} \left(5x + 4 - \frac{7}{x+3}\right)$

$= \lim_{x\to\infty} 5x + \lim_{x\to\infty} 4 - \lim_{x\to\infty} \frac{\frac{7}{x}}{\frac{x}{x} + \frac{3}{x}}$

$= \infty$

4.4 Concavity and Points of Inflection, pp. 205–206

1. a. A: negative; B: negative; C: positive; D: positive
b. A: negative; B: negative; C: positive; D: negative
2. a. $y = x^3 - 6x^2 - 15x + 10$

$\frac{dy}{dx} = 3x^2 - 12x - 15$

For critical values, we solve $\frac{dy}{dx} = 0$:

$3x^2 - 12x - 15 = 0$
$x^2 - 4x - 5 = 0$
$(x - 5)(x + 1) = 0$
$x = 5$ or $x = -1$

The critical points are $(5, -105)$ and $(-1, 20)$.

Now, $\frac{d^2y}{dx^2} = 6x - 12$.

At $x = 5$, $\frac{d^2y}{dx^2} = 18 > 0$. There is a local minimum at this point.

At $x = -1$, $\frac{d^2y}{dx^2} = -18 < 0$. There is a local maximum at this point.

The local minimum is $(5, -105)$ and the local maximum is $(-1, 20)$

b. $y = \frac{25}{x^2 + 48}$

$\frac{dy}{dx} = -\frac{50x}{(x^2 + 48)^2}$

For critical values, solve $\frac{dy}{dx} = 0$ or $\frac{dy}{dx}$ does not exist.
Since $x^2 + 48 > 0$ for all x, the only critical point is $\left(0, \frac{25}{48}\right)$

$\frac{d^2y}{dx^2} = -50(x^2 + 48)^{-2} + 100x(x^2 + 48)^{-3}(2x)$

$= -\frac{50}{(x^2 + 48)^2} + \frac{200x^2}{(x^2 + 48)^3}$

At $x = 0$, $\frac{d^2y}{dx^2} = -\frac{50}{48^2} < 0$. The point $\left(0, \frac{25}{48}\right)$ is a local maximum.

c. $s = t + t^{-1}$

$\frac{ds}{dt} = 1 - \frac{1}{t^2}, t \neq 0$

For critical values, we solve $\frac{ds}{dt} = 0$:

$1 - \frac{1}{t^2} = 0$
$t^2 = 1$
$t = \pm 1.$

The critical points are $(-1, -2)$ and $(1, 2)$

$\frac{d^2s}{dt^2} = \frac{2}{t^3}$

At $t = -1$, $\frac{d^2s}{dt} = -2 < 0$. The point $(-1, -2)$ is a local maximum. At $t = 1$, $\frac{d^2s}{dt^2} = 2 > 0$. The point $(1, 2)$ is a local minimum.

d. $y = (x - 3)^3 + 8$

$\frac{dy}{dx} = 3(x - 3)^2$

$x = 3$ is a critical value.
The critical point is $(3, 8)$

$\frac{d^2y}{dx^2} = 6(x - 3)$

At $x = 3$, $\frac{d^2y}{dx^2} = 0$.

The point $(3, 8)$ is neither a relative (local) maximum or minimum.

3. a. For possible point(s) of inflection, solve $\frac{d^2y}{dx^2} = 0$:

$6x - 8 = 0$
$x = \frac{4}{3}.$

Interval	$x < \frac{4}{3}$	$x = \frac{4}{3}$	$x > \frac{4}{3}$
$f''(x)$	<0	$=0$	>0
Graph of $f(x)$	Concave Down	Point of Inflection	Concave Up

The point $\left(\frac{4}{3}, -14\frac{20}{27}\right)$is point of inflection.

b. For possible point(s) of inflection, solve

$\frac{d^2y}{dx^2} = 0$:

$200x^2 - 50x^2 - 2400 = 0$

$150x^2 = 2400.$

Since $x^2 + 48 > 0$:

$x = \pm 4.$

Interval	$x < -4$	$x = -4$	$-4 < x < 4$	$x = 4$	$x > 4$
$f''(x)$	>0	$=0$	<0	$=0$	>0
Graph of $f(x)$	Concave Up	Point of Inflection	Concave Down	Point of Inflection	Concave Up

$\left(-4, \frac{25}{64}\right)$ and $\left(4, \frac{25}{64}\right)$ are points of inflection.

c. $\frac{d^2s}{dt^2} = \frac{3}{t^2}$

Interval	$t < 0$	$t = 0$	$t > 0$
$f''(t)$	<0	Undefined	>0
Graph of $f(t)$	Concave Down	Undefined	Concave Up

The graph does not have any points of inflection.

d. For possible points of inflection, solve

$\frac{d^2y}{dx^2} = 0$:

$6(x - 3) = 0$

$x = 3.$

Interval	$x < 3$	$x = 3$	$x > 3$
$f''(x)$	<0	$=0$	>0
Graph of $f(x)$	Concave Down	Point of Inflection	Concave Up

(3, 8) is a point of inflection.

4. a. $f(x) = 2x^3 - 10x + 3$ at $x = 2$

$f'(x) = 6x^2 - 10$

$f''(x) = 12x$

$f''(2) = 24 > 0$

The curve lies above the tangent at $(2, -1)$.

b. $g(x) = x^2 - \frac{1}{x}$ at $x = -1$

$g'(x) = 2x + \frac{1}{x^2}$

$g''(x) = 2 - \frac{2}{x^3}$

$g''(-1) = 2 + 2 = 4 > 0$

The curve lies above the tangent line at $(-1, 2)$.

c. $p(w) = \frac{w}{\sqrt{w^2 + 1}}$ at $w = 3$

$p(w) = w(w^2 + 1)^{\frac{1}{2}}$

$$\frac{dp}{dw} = (w^2 + 1)^{\frac{1}{2}} + w\left(-\frac{1}{2}\right)(w^2 + 1)^{\frac{3}{2}}(2w)$$

$$= (w^2 + 1)^{\frac{1}{2}} - w^2(w^2 + 1)^{\frac{3}{2}}$$

$$\frac{d^2p}{dw^2} = -\frac{1}{2}(w^2 + 1)^{\frac{3}{2}}(2w) - 2w(w^2 + 1)^{\frac{3}{2}} + w^2\left(\frac{3}{2}\right)(w^2 + 1)^{\frac{5}{2}}(2w)$$

At $w = 3$, $\frac{d^2p}{dw^2} = -\frac{3}{10\sqrt{10}} - \frac{6}{10\sqrt{10}} + \frac{81}{100\sqrt{10}}$

$$= -\frac{9}{100\sqrt{10}} < 0.$$

The curve is below the tangent line at $\left(3, \frac{3}{\sqrt{10}}\right)$.

d. The first derivative is

$$s'(t) = \frac{(t - 4)(2) - (2t)(1)}{(t - 4)^2}$$

$$= \frac{-8}{(t - 4)^2}$$

The second derivative is

$$s''(t) = \frac{(t - 4)^2(0) - (-8)2(t - 4)^1}{(t - 4)^4}$$

$$= \frac{16}{(t - 4)^3}$$

So $s''(-2) = \frac{16}{(-2 - 4)^3}$

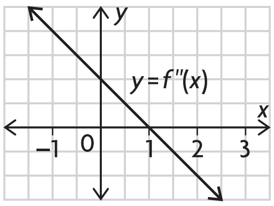
$$= -\frac{16}{216} = -\frac{2}{27}$$

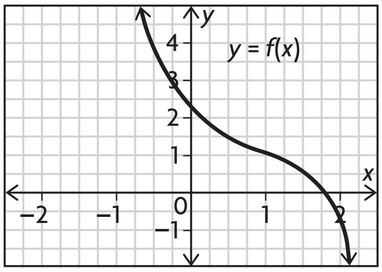
Since the second derivative is negative at this point, the function lies below the tangent there.

5. For the graph on the left: **i.** $f''(x) > 0$ for $x < 1$ Thus, the graph of $f(x)$ is concave up on $x < 1$. $f''(x) \leq 0$ for $x > 1$. The graph of $f(x)$ is concave down on $x > 1$.

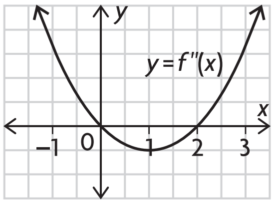
ii. There is a point of inflection at $x = 1$.

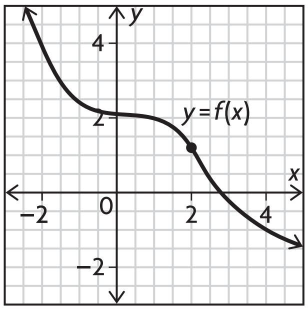
iii.





For the graph on the right: **i.** $f''(x) > 0$ for $x < 0$ or $x > 2$
The graph of $f(x)$ is concave up on $x < 0$ or $x > 2$.
The graph of $f(x)$ is concave down on $0 < x < 2$.
ii. There are points of inflection at $x = 0$ and $x = 2$.
iii.





6. For any function $y = f(x)$, find the critical points, i.e., the values of x such that $f'(x) = 0$ or $f'(x)$ does not exist. Evaluate $f''(x)$ for each critical value.
If the value of the second derivative at a critical point is positive, the point is a local minimum. If the value of the second derivative at a critical point is negative, the point is a local maximum.
7. Step 4: Use the first derivative test or the second derivative test to determine the type of critical points that may be present.
8. a. $f(x) = x^4 + 4x^3$
i. $f'(x) = 4x^3 + 12x^2$
$f''(x) = 12x^2 + 24x$

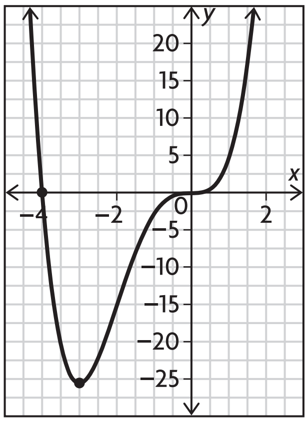
For possible points of inflection, solve $f''(x) = 0$:
$12x^2 + 24x = 0$
$12x(x + 2) = 0$
$x = 0$ or $x = -2$.

Interval	$x < -2$	$x = -2$	$-2 < x < 0$	$x = 0$	$x > 0$
$f''(x)$	>0	$=0$	<0	$=0$	>0
Graph of $f(x)$	Concave Up	Point of Inflection	Concave Down	Point of Inflection	Concave Up

The points of inflection are $(-2, -16)$ and $(0, 0)$.
ii. If $x = 0$, $y = 0$.
For critical points, we solve $f'(x) = 0$:
$4x^3 + 12x^2 = 0$
$4x^2(x + 3) = 0$
$x = 0$ and $x = -3$.

Interval	$x < -3$	$x = -3$	$-3 < x < 0$	$x = 0$	$x > 0$
$f'(x)$	<0	$=0$	>0	$=0$	>0
Graph of $f(x)$	Decreasing	Local Min	Increasing		Increasing

If $y = 0$, $x^4 + 4x^3 = 0$
$x^3(x + 4) = 0$
$x = 0$ or $x = -4$
The x-intercepts are 0 and -4.



b. d. $g(w) = \dfrac{4w^2 - 3}{w^3}$
$= \dfrac{4}{3} - \dfrac{3}{w^3}, w \neq 0$

i. $g'(w) = -\dfrac{4}{w^2} + \dfrac{9}{w^4}$
$= \dfrac{9 - 4w^2}{w^4}$

$g''(w) = \dfrac{8}{w^3} - \dfrac{36}{w^3}$
$= \dfrac{8w^2 - 36}{w^5}$

For possible points of inflection, we solve
$g''(w) = 0$:
$8w^2 - 36 = 0$, since $w^5 \neq 0$

$$w^2 = \frac{9}{2}$$

$$w = \pm\frac{3}{\sqrt{2}}.$$

Interval	$w < -\frac{3}{\sqrt{2}}$	$w = -\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}} < w < 0$	$< w < 0 < \frac{3}{\sqrt{2}}$	$w = \frac{3}{\sqrt{2}}$	$w > \frac{3}{\sqrt{2}}$
$g'(w)$	<0	$=0$	>0	<0	0	>0
Graph of $g(w)$	Concave Down	Point of Inflection	Concave Up	Concave Down	Point of Inflection	Concave Up

The points of inflection are $\left(-\frac{3}{\sqrt{2}}, -\frac{8\sqrt{2}}{9}\right)$ and $\left(\frac{3}{\sqrt{2}}, -\frac{8\sqrt{2}}{9}\right)$.

ii. There is no y-intercept.

The x-intercept is $\pm\frac{3}{\sqrt{2}}$.
For critical values, we solve $g'(w) = 0$:
$9 - 4w^2 = 0$ since $w^4 \neq 0$

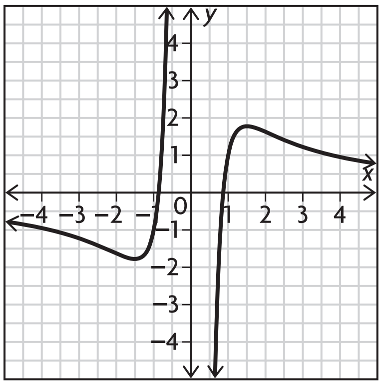
$$w = \pm\frac{3}{2}.$$

Interval	$w < -\frac{3}{2}$	$w = -\frac{3}{2}$	$-\frac{3}{2} < w < 0$	$< 0 < w < \frac{3}{2}$	$w = \frac{3}{2}$	$w > \frac{3}{2}$
$g'(w)$	<0	$=0$	>0	>0	0	<0
Graph of $g(w)$	Decreasing Down	Local Min	Increasing	Increasing	Local Max	Decreasing

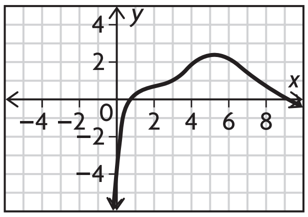
$$\lim_{w\to 0^-}\frac{4w^2-3}{w^3} = \infty,\ \lim_{w\to 0^+}\frac{4w^2-3}{w^3} = -\infty$$

$$\lim_{w\to -\infty}\left(\frac{4}{w} - \frac{3}{w^3}\right) = 0,\ \lim_{w\to \infty}\left(\frac{4}{w} - \frac{3}{w^3}\right) = 0$$

Thus, $y = 0$ is a horizontal asymptote and $x = 0$ is a vertical asymptote.



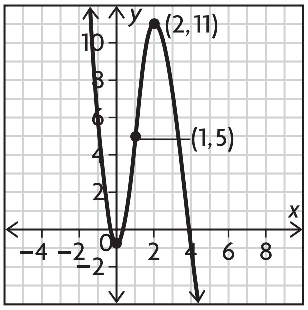
9. The graph is increasing when $x < 2$ and when $2 < x < 5$.
The graph is decreasing when $x > 5$.
The graph has a local maximum at $x = 5$.
The graph has a horizontal tangent line at $x = 2$.
The graph is concave down when $x < 2$ and when $4 < x < 7$.
The graph is concave up when $2 < x < 4$ and when $x > 7$.
The graph has points of inflection at $x = 2$, $x = 4$, and $x = 7$.
The y-intercept of the graph is -4.



10. $f(x) = ax^3 + bx^2 + c$
$f'(x) = 3ax^2 + 2bx$
$f''(x) = 6ax + 2b$
Since (2, 11) is a relative extremum,
$f(2) = 12a + 4b = 0.$
Since (1, 5) is an inflection point,
$f''(1) = 6a + 2b = 0.$
Since the points are on the graph, $a + b + c = 5$ and
$8a + 4b + c = 11$

$$7a + 3b = 6$$
$$9a + 3b = 0$$
$$2a = -6$$
$$a = -3$$
$$b = 9$$
$$c = -1.$$

Thus, $f(x) = -3x^3 + 9x^2 - 1.$



11. $f(x) = (x + 1)^{\frac{1}{2}} + bx^{-1}$

$$f'(x) = \frac{1}{2}(x + 1)^{\frac{1}{2}} - bx^{-2}$$

$$f''(x) = -\frac{1}{4}(x + 1)^{\frac{3}{2}} + 2bx^{-3}$$

Since the graph of $y = f(x)$ has a point of inflection at $x = 3$:

$$-\frac{1}{4}(4)^{\frac{3}{2}} + \frac{2b}{27} = 0$$

$$-\frac{1}{32} + \frac{2b}{27} = 0$$

$$b = \frac{27}{64}.$$

12. $f(x) = ax^4 + bx^3$

$f'(x) = 4ax^3 + 3bx^2$

$f''(x) = 12ax^2 + 6bx$

For possible points of inflection, we solve $f''(x) = 0$:

$12ax^2 + 6bx = 0$

$6x(2ax + b) = 0$

$x = 0$ or $x = -\frac{b}{2a}$.

The graph of $y = f''(x)$ is a parabola with x-intercepts 0 and $-\frac{b}{2a}$.

We know the values of $f''(x)$ have opposite signs when passing through a root. Thus at $x = 0$ and at $x = -\frac{b}{2a}$, the concavity changes as the graph goes through these points. Thus, $f(x)$ has points of inflection at $x = 0$ and $x = -\frac{b}{2a}$. To find the x-intercepts, we solve $f(x) = 0$

$x^3(ax + b) = 0$

$x = 0$ or $x = -\frac{b}{a}$.

The point midway between the x-intercepts has x-coordinate $-\frac{b}{2a}$.

The points of inflection are (0, 0) and $\left(-\frac{b}{2a}, -\frac{b}{16a^3}\right)$.

13. a. $y = \frac{x^3 - 2x^2 + 4x}{x^2 - 4} = x - 2 + \frac{8x - 8}{x^2 - 4}$ (by division of polynomials). The graph has discontinuities at $x = \pm 2$.

$$\left.\begin{aligned}\lim_{x\to -2^-}\left(x - 2 + \frac{8x-8}{x^2-4}\right) = -\infty \\ \lim_{x\to -2^+}\left(x - 2 + \frac{8x-8}{x^2-4}\right) = -\infty\end{aligned}\right\}$$

$$\left.\begin{aligned}\lim_{x\to -2^-}\left(x - 2 + \frac{8x-8}{x^2-4}\right) = -\infty \\ \lim_{x\to -2^+}\left(x - 2 + \frac{8x-8}{x^2-4}\right) = -\infty\end{aligned}\right\}$$

When $x = 0$, $y = 0$.

Also, $y = \frac{x(x^2 - 2x + 4)}{x^2 - 4} = \frac{x[(x-1)^2 + 3]}{x^2 - 4}$.

Since $(x - 1)^2 + 3 > 0$, the only x-intercept is $x = 0$.

Since $\lim_{x\to\infty}\frac{8x - 8}{x^2 - 4} = 0$, the curve approaches the value $x - 2$ as $x \to \infty$. This suggests that the line $y = x - 2$ is an oblique asymptote. It is verified by the limit $\lim_{x\to\infty}[x - 2 - f(x)] = 0$. Similarly, the curve approaches $y = x - 2$ as $x \to -\infty$.

$$\frac{dy}{dx} = 1 + \frac{8(x^2 - 4) - 8(x - 1)(2x)}{(x^2 - 4)^2}$$

$$= 1 - \frac{8(x^2 - 2x + 4)}{(x^2 - 4)^2}$$

We solve $\frac{dy}{dx} = 0$ to find critical values:

$8x^2 - 16x + 32 = x^4 - 8x^2 + 16$

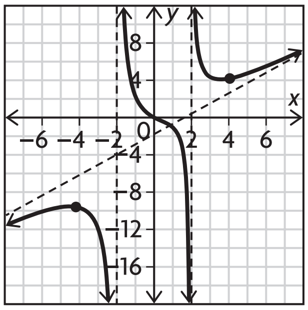
$x^4 - 16x^2 - 16 = 0$

$x^2 = 8 + 4\sqrt{5}$ $\left(8 - 4\sqrt{5}\text{ is inadmissible}\right)$

$x \doteq \pm 4.12.$

$\lim_{x\to\infty} y = \infty$ and $\lim_{x\to -\infty} y = -\infty$

Interval	$x < -4.12$	$x = -4.12$	$-4.12 < x < 2$	$-2 < x < 2$	$2 < x -4.12$	$x = 4.12$	$x > 4.12$
$\frac{dy}{dx}$	>0	$=0$	<0	<0	<0	0	>0
Graph of y	Increasing	Local Max	Decreasing	Decreasing	Decreasing	Local Min	Increasing



b. Answers may vary. For example, there is a section of the graph that lies between the two sections of the graph that approach the asymptote.

14. For the various values of n, $f(x) = (x - c)^n$ has the following properties:

n	n = 1	n = 2	n = 3	n = 4
$f'(x)$	1	$2(x-c)$	$3(x-c)^2$	$4(x-c)^3$
$f''(x)$	0	2	$6(x-c)$	$12(x-c)^2$
Infl. Pt.	None	None	$x = c$	$x = c$

It appears that the graph of f has an inflection point at $x = c$ when $n \geq 3$.

4.5 An Algorithm for Curve Sketching, pp. 212–213

1. A cubic polynomial that has a local minimum must also have a local maximum. If the local minimum is to the left of the local maximum, then $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$. If the local minimum is to the right of the local maximum, then $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

2. Since each local maximum and minimum of a function corresponds to a zero of its derivative, the number of zeroes of the derivative is the maximum number of local extreme values that the function can have. For a polynomial of degree n, the derivative has degree $n - 1$, so it has at most $n - 1$ zeroes, and thus at most $n - 1$ local extremes. A polynomial of degree three has at most 2 local extremes. A polynomial of degree four has at most 3 local extremes.

3. a. This function is discontinuous when

$x^2 + 4x + 3 = 0$

$(x + 3)(x + 1) = 0$

$x = -3$ or $x = -1$. Since the numerator is non-zero at both of these points, they are both equations of vertical asymptotes.

b. This function is discontinuous when

$x^2 - 6x + 12$

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(12)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{-12}}{2}$$

This equation has no real solutions, so the function has no vertical asymptotes.

c. This function is discontinuous when

$x^2 - 6x + 9 = 0$

$(x - 3)^2 = 0$

$x = 3$. Since the numerator is non-zero at this point, it is the equation of a vertical asymptote.

4. a. $y = x^3 - 9x^2 + 15x + 30$

We know the general shape of a cubic polynomial with leading coefficient positive. The local extrema will help refine the graph.

$$\frac{dy}{dx} = 3x^2 - 18x + 15$$

Set $\frac{dy}{dx} = 0$ to find the critical values:

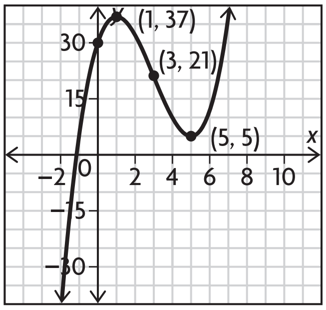
$3x^2 - 18x + 15 = 0$

$x^2 - 6x + 5 = 0$

$(x - 1)(x - 5) = 0$

$x = 1$ or $x = 5$.

The local extrema are (1, 37) and (5, 5).



b. $f(x) = 4x^3 + 18x^2 + 3$

The graph is that of a cubic polynomial with leading coefficient negative. The local extrema will help refine the graph.

$$\frac{dy}{dx} = 12x^2 + 36x$$

To find the critical values, we solve $\frac{dy}{dx} = 0$:

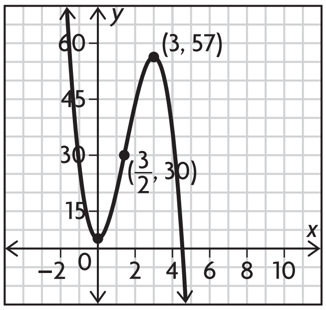
$-12x(x - 3) = 0$

$x = 0$ or $x = 3$.

The local extrema are (0, 3) and (3, 57).

$$\frac{d^2y}{dx^2} = -24x + 36$$

The point of inflection is $\left(\frac{3}{2}, 30\right)$.



n	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$f'(x)$	1	$2(x - c)$	$3(x - c)^2$	$4(x - c)^3$
$f''(x)$	0	2	$6(x - c)$	$12(x - c)^2$
Infl. Pt.	None	None	$x = c$	$x = c$

It appears that the graph of f has an inflection point at $x = c$ when $n \geq 3$.

4.5 An Algorithm for Curve Sketching, pp. 212–213

1. A cubic polynomial that has a local minimum must also have a local maximum. If the local minimum is to the left of the local maximum, then $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$. If the local minimum is to the right of the local maximum, then $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

2. Since each local maximum and minimum of a function corresponds to a zero of its derivative, the number of zeroes of the derivative is the maximum number of local extreme values that the function can have. For a polynomial of degree n, the derivative has degree $n - 1$, so it has at most $n - 1$ zeroes, and thus at most $n - 1$ local extremes. A polynomial of degree three has at most 2 local extremes. A polynomial of degree four has at most 3 local extremes.

3. a. This function is discontinuous when

$$x^2 + 4x + 3 = 0$$
$$(x + 3)(x + 1) = 0$$

$x = -3$ or $x = -1$. Since the numerator is non-zero at both of these points, they are both equations of vertical asymptotes.

b. This function is discontinuous when

$x^2 - 6x + 12$

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(12)}}{2(1)}$$
$$= \frac{6 \pm \sqrt{-12}}{2}$$

This equation has no real solutions, so the function has no vertical asymptotes.

c. This function is discontinuous when

$$x^2 - 6x + 9 = 0$$
$$(x - 3)^2 = 0$$

$x = 3$. Since the numerator is non-zero at this point, it is the equation of a vertical asymptote.

4. a. $y = x^3 - 9x^2 + 15x + 30$

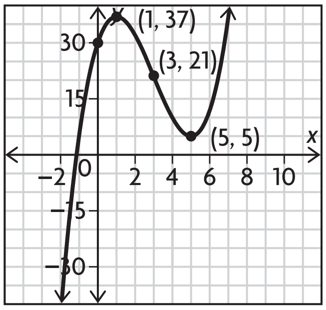
We know the general shape of a cubic polynomial with leading coefficient positive. The local extrema will help refine the graph.

$$\frac{dy}{dx} = 3x^2 - 18x + 15$$

Set $\frac{dy}{dx} = 0$ to find the critical values:

$$3x^2 - 18x + 15 = 0$$
$$x^2 - 6x + 5 = 0$$
$$(x - 1)(x - 5) = 0$$
$$x = 1 \text{ or } x = 5.$$

The local extrema are (1, 37) and (5, 5).



b. $f(x) = 4x^3 + 18x^2 + 3$

The graph is that of a cubic polynomial with leading coefficient negative. The local extrema will help refine the graph.

$$\frac{dy}{dx} = 12x^2 + 36x$$

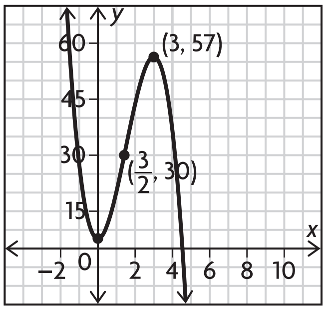
To find the critical values, we solve $\frac{dy}{dx} = 0$:

$$-12x(x - 3) = 0$$
$$x = 0 \text{ or } x = 3.$$

The local extrema are (0, 3) and (3, 57).

$$\frac{d^2y}{dx^2} = -24x + 36$$

The point of inflection is $\left(\frac{3}{2}, 30\right)$.

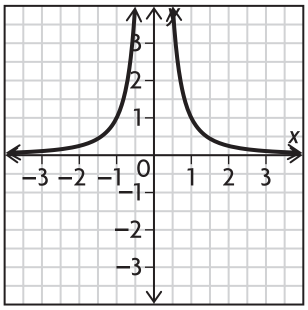


c. $y = 3 + \frac{1}{(x+2)^2}$

We observe that $y = 3 + \frac{1}{(x+2)^2}$ is just a translation of $y = \frac{1}{x^2}$.

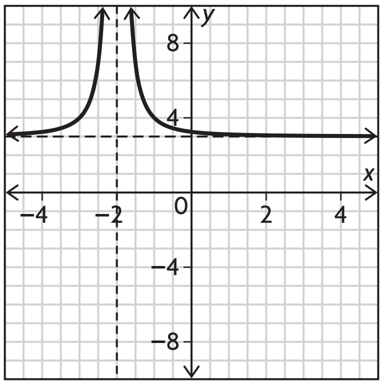
The graph of $y = \frac{1}{x^2}$ is

The reference point (0, 0) for $y = \frac{1}{x^2}$ becomes the point $(-2, 3)$ for $y = 3 + \frac{1}{(x+2)^2}$. The vertical asymptote is $x = -2$, and the horizontal asymptote is $y = 3$.



$\frac{dy}{dx} = -\frac{2}{(x+2)^3}$, hence there are no critical points.

$\frac{d^2y}{dx^2} = \frac{6}{(x+2)^4} > 0$, hence the graph is always concave up.



d. $f(x) = x^4 - 4x^3 - 8x^2 + 48x$

We know the general shape of a fourth degree polynomial with leading coefficient positive. The local extrema will help refine the graph.

$f'(x) = 4x^3 - 12x^2 - 16x + 48$

For critical values, we solve $f'(x) = 0$

$x^3 - 3x^2 - 4x + 12 = 0$.

Since $f'(2) = 0$, $x - 2$ is a factor of $f'(x)$.

The equation factors are

$(x-2)(x-3)(x+2) = 0$.

The critical values are $x = -2, 2, 3$.

$f''(x) = 12x^2 - 24x - 16$

Since $f''(-2) = 80 > 0$, $(-2, -80)$ is a local minimum.

Since $f''(2) = -16 < 0$, $(2, 48)$ is a local maximum.

Since $f''(3) = 20 > 0$, $(3, 45)$ is a local minimum.

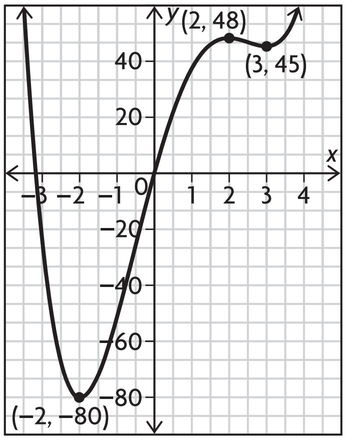
The graph has x-intercepts 0 and -3.2

The points of inflection can be found by solving $f''(x) = 0$:

$3x^2 - 6x - 4 = 0$

$x = \frac{6 \pm \sqrt{84}}{6}$

$x \doteq -\frac{1}{2}$ or $\frac{5}{2}$.



e. $y = \frac{2x}{x^2 - 25}$

There are discontinuities at $x = -5$ and $x = 5$.

$\lim_{x \to 5^-} \left(\frac{2x}{x^2 - 25}\right) = -\infty$ and $\lim_{\to -5^+} \left(\frac{2x}{x^2 - 25}\right) = \infty$

$\lim_{x \to 5^-} \left(\frac{2x}{x^2 - 25}\right) = -\infty$ and $\lim_{\to -5^+} \left(\frac{2x}{x^2 - 25}\right) = \infty$

$x = -5$ and $x = 5$ are vertical asymptotes.

$\frac{dy}{dx} = \frac{2(x^2 - 25) - 2x(2x)}{(x^2 - 25)^2} = -\frac{2x^2 + 50}{(x^2 - 25)^2} < 0$ for all x in the domain. The graph is decreasing throughout the domain.

$$\lim_{x \to \infty} \left(\frac{2x}{x^2 - 25}\right) = \lim_{x \to \infty} \left(\frac{\frac{2}{x}}{1 - \frac{25}{x^2}}\right) = 0$$

$$\lim_{x \to -\infty} \left(\frac{\frac{2}{x}}{1 - \frac{25}{x^2}}\right) = 0$$

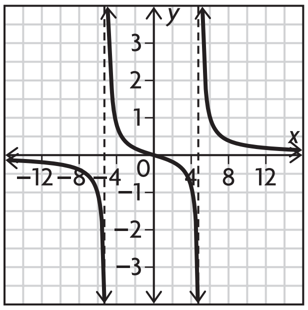
$y = 0$ is a horizontal asymptote.

$$\frac{d^2y}{dx^2} = -\frac{4x(x^2-25)^2 - (2x^2+50)(2)(x^2-25)(2x)}{(x^2-25)^4}$$

$$= \frac{4x^3+300x}{(x^2-25)^3} = \frac{4x(x^2+75)}{(x^2-25)^3}$$

There is a possible point of inflection at $x = 0$.

Interval	$x < -5$	$-5 < x < 0$	$x = 0$	$0 < x < 5$	$x > 5$
$\frac{d^2y}{dx^2}$	<0	>0	$=0$	<0	>0
Graph of y	Concave Down	Point of Up	Concave Inflection	Point of Down	Concave Up



f. This function is discontinuous when

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0$ or $x = 4$. The numerator is non-zero at these points, so the function has vertical asymptotes at both of them. The behaviour of the function near these asymptotes is:

x-values	1	x	$x - 4$	$f(x)$	$\lim_{x\to c} f(x)$
$x \to 0^-$	> 0	< 0	< 0	> 0	$+\infty$
$x \to 0^+$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 4^-$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 4^+$	> 0	> 0	> 0	> 0	$+\infty$

To check for a horizontal asymptote:

$$\lim_{x\to\infty} \frac{1}{x^2-4x} = \lim_{x\to\infty} \frac{1}{x^2\left(1-\frac{4}{x}\right)}$$

$$= \frac{\lim_{x\to\infty}(1)}{\lim_{x\to\infty}\left(x^2\left(1-\frac{4}{x}\right)\right)}$$

$$= \frac{\lim_{x\to\infty}(1)}{\lim_{x\to\infty}(x^2) \times \lim_{x\to\infty}\left(1-\frac{4}{x}\right)}$$

$$= \lim_{x\to\infty}\frac{1}{x^2} \times \frac{1}{1+0}$$

$$= 0$$

Similarly, $\lim_{x\to-\infty} \frac{1}{x^2-4x} = 0$, so $y = 0$ is a horizontal asymptote of the function.

Since $y = 0$ and $x = 0$ are both asymptotes of the function, it has no x- or y- intercepts.

The derivative is

$$f'(x) = \frac{(x^2-4x) - (1)(2x-4)}{(x^2-4x)^2}$$

$$= \frac{4-2x}{(x^2-4x)^2}, \text{ and the second derivative is}$$

$$f''(x) = \frac{(x^2-4x)^2(-2) - (4-2x)(2(x^2-4x)(2x-4))}{(x^2-4x)^4}$$

$$= \frac{-2x^2+8x+8x^2-32x+32}{(x^2-4x)^3}$$

$$= \frac{6x^2-24x+32}{(x^2-4x)^3}$$

Letting $f'(x) = 0$ shows that $x = 2$ is a critical point of the function. The inflection points can be found by letting $f''(x) = 0$, so

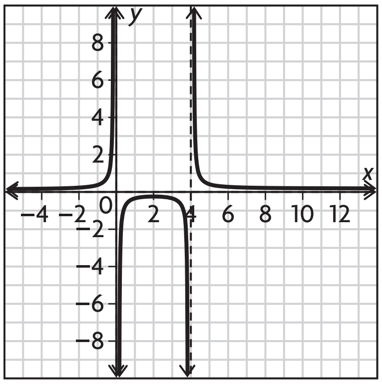
$2(3x^2 - 12x + 16) = 0$

$$x = \frac{12 \pm \sqrt{(-12)^2 - 4(3)(16)}}{2(3)}$$

$$= \frac{12 \pm \sqrt{-48}}{6}$$

This equation has no real solutions, so the graph of f has no inflection points.

x	$x < 0$	$0 < x < 2$	$x = 0$	$2 < x < 4$	$x > 4$
$f'(x)$	+	+	0	−	−
Graph	Inc.	Inc.	Local Max	Dec.	Dec.
$f''(x)$	+	−	−	−	+
Concavity	Up	Down	Down	Down	Up



g. $y = \dfrac{6x^2 - 2}{x^3}$

$= \dfrac{6}{x} - \dfrac{2}{x^3}$

There is a discontinuity at $x = 0$.

$\lim\limits_{x \to 0^-} \dfrac{6x^2 - 2}{x^3} = \infty$ and $\lim\limits_{x \to 0^+} \dfrac{6x^2 - 2}{x^3} = -\infty$

The y-axis is a vertical asymptote. There is no y-intercept. The x-intercept is a vertical asymptote. There is no y-intercept. The x-intercept is $\pm\frac{1}{\sqrt{3}}$.

$$\frac{dy}{dx} = -\frac{6}{x^2} + \frac{6}{x^4} = \frac{-6x^2 + 6}{x^4}$$

$\dfrac{dy}{dx} = 0$ when $6x^2 = 6$

$x = \pm 1$

Interval	$x < -1$	$x = -1$	$-1 < x < 0$	$0 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	<0	$=0$	>0	>0	$=0$	<0
Graph of $y = f(x)$	Decreasing	Local Min	Increasing	Increasing	Local Max	Decreasing

There is a local minimum at $(-1, -4)$ and a local maximum at $(1, 4)$.

$$\frac{d^2y}{dx^2} = \frac{12}{x^3} = \frac{24}{x^3} = \frac{12x^2 - 24}{x^3}$$

For possible points of inflection, we solve $\dfrac{d^2y}{dx^2} = 0$ $(x^3 \neq 0)$:

$12x^2 = 24$

$x = \pm\sqrt{2}.$

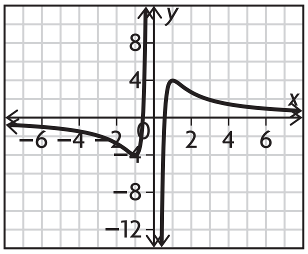
Interval	$x < -\sqrt{2}$	$x < -\sqrt{2}$	$-\sqrt{2} < x < 0$	$0 < x < \sqrt{2}$	$x = \sqrt{2}$	$x > \sqrt{2}$
$\frac{d^2y}{dx^2}$	<0	$=0$	>0	<0	$=0$	>0
Graph of $y = f(x)$	Concave Down	Point of Inflection	Concave Up	Concave Down	Point of Inflection	Convave Up

There are points of inflection at $\left(-\sqrt{2}, -\frac{5}{\sqrt{2}}\right)$ and $\left(\sqrt{2}, \frac{5}{\sqrt{2}}\right)$.

$$\lim_{x \to \infty} \frac{6x^2 - 2}{x^3} = \lim_{x \to \infty} \frac{\frac{6}{x} - \frac{2}{x^3}}{1} = 0$$

$$\lim_{x = -\infty} \frac{\frac{6}{x} - \frac{2}{x^3}}{x} = 0$$

The x-axis is a horizontal asymptote.



h. $y = \dfrac{x + 3}{x^2 - 4}$

There are discontinuities at $x = -2$ and at $x = 2$.

$\lim\limits_{x \to -2^-} \left(\dfrac{x + 3}{x^2 - 4}\right) = \infty$ and $\lim\limits_{x \to 2^-} \left(\dfrac{x + 3}{x^2 - 4}\right) = -\infty$

$\lim\limits_{x \to 2^-} \left(\dfrac{x + 3}{x^2 - 4}\right) = -\infty$ and $\lim\limits_{x \to 2^-} \left(\dfrac{x + 3}{x^2 - 4}\right) = \infty$

There are vertical asymptotes at $x = -2$ and $x = 2$.

When $x = 0$, $y = -\frac{3}{4}$. The x-intercept is -3.

$$\frac{dy}{dx} = \frac{(1)(x^2 - 4) - (x + 3)(2x)}{(x^2 - 4)^2}$$

$$= \frac{-x^2 - 6x - 4}{(x^2 - 4)^2}$$

For critical values, we solve $\dfrac{dy}{dx} = 0$:

$x^2 + 6x + 4 = 0$

$$x = -\frac{6 \pm \sqrt{36 - 16}}{2}$$

$$= -3 \pm \sqrt{5}$$

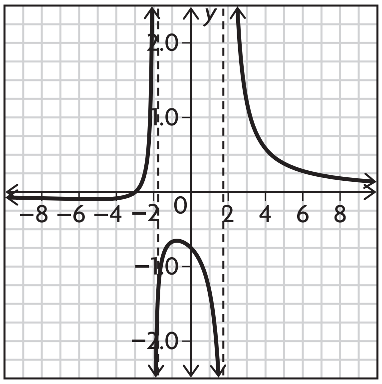
$$\doteq -5.2 \text{ or } -0.8.$$

Interval	$x < -5.2$	$x = -5.2$	$-5.2 < x < -2$	$-2 < x < -0.8$	$x = -0.8$	$-0.8 < x < 2$	$x > 2$
$\frac{dy}{dx}$	< 0	$= 0$	> 0	> 0	$= 0$	< 0	< 0
Graph of y	Decreasing	Local Min	Increasing	Increasing	Local Max	Decreasing	Decreasing

$$\lim_{x \to \infty} y = \lim_{x \to \infty} \left(\frac{\frac{1}{x} + \frac{3}{x^2}}{1 - \frac{4}{x^2}}\right) = 0$$

$$\lim_{x \to \infty} \left(\frac{\frac{1}{x} + \frac{3}{x^2}}{1 - \frac{4}{x^2}}\right) = 0$$

The x-axis is a horizontal asymptote.



i. $y = \dfrac{x^2 - 3x + 6}{x - 1}$

$= x - 2 + \dfrac{4}{x - 1}$

$$\begin{array}{r} x - 2 \\ x - 1\overline{)x^2 - 3x + 6} \\ \underline{x^2 - x} \\ -2x + 6 \\ \underline{-2x + 2} \\ 4 \end{array}$$

There is a discontinuity at $x = 1$.

$$\lim_{x \to 1^-} \left(\frac{x^2 - 3x + 6}{x - 1} \right) = -\infty$$

$$\lim_{x \to 1^+} \left(\frac{x^2 - 3x + 6}{x - 1} \right) = \infty$$

Thus, $x = 1$ is a vertical asymptote.

The y-intercept is -6.

There are no x-intercepts ($x^2 - 3x + 6 > 0$ for all x in the domain).

$$\frac{dy}{dx} = 1 - \frac{4}{(x - 1)^2}$$

For critical values, we solve $\dfrac{dy}{dx} = 0$:

$$1 - \frac{4}{(x - 1)^2} = 0$$

$$(x - 1)^2 = 4$$

$$x - 1 = \pm 2$$

$$x = -1 \text{ or } x = 3.$$

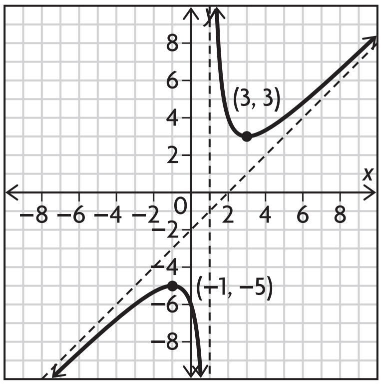
Interval	$x < -1$	$x = -1$	$-1 < x < 1$	$1 < x < 3$	$x = 3$	$x > 3$
$\frac{dy}{dx}$	>0	$=0$	<0	<0	$=0$	>0
Graph of y	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing

$$\frac{d^2y}{d^2x} = \frac{8}{(x - 1)^3}$$

For $x < 1$, $\dfrac{d^2y}{dx^2} < 0$ and y is always concave down.

For, $x > 1$, $\dfrac{d^2y}{dx^2} > 0$ and y is always concave up.

The line $y = x - 2$ is an oblique asymptote.



j. This function is continuous everywhere, so it has no vertical asymptotes. It also has no horizontal asymptote, because

$\lim_{x \to \infty} (x - 4)^{\frac{2}{3}} = \infty$ and $\lim_{x \to -\infty} (x - 4)^{\frac{2}{3}} = \infty$.

The x-intercept of the function is found by letting $f(x) = 0$, which gives

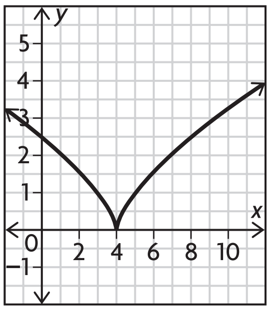
$$(x - 4)^{\frac{2}{3}} = 0$$

$$x = 4$$

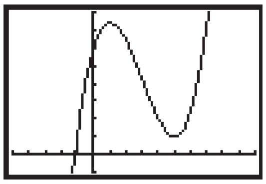
The y-intercept is found by letting $x = 0$, which gives $f(0) = (0 - 4)^{\frac{2}{3}} \doteq 2.5$.

The derivative of the function is $f'(x) = \left(\frac{2}{3}\right)(x - 4)^{-\frac{1}{3}}$ and the second derivative is $f''(x) = \left(-\frac{2}{9}\right)(x - 4)^{-\frac{4}{3}}$. Neither of these derivatives has a zero, but each is undefined for $x = 4$, so it is a critical value and a possible point of inflection.

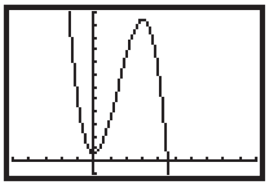
x	$x < 4$	$x = 4$	$x > 4$
$f'(x)$	−	Undefined	+
Graph	Dec.	Local Min	Inc.
$f''(x)$	−	Undefined	−
Concavity	Down	Undefined	Down



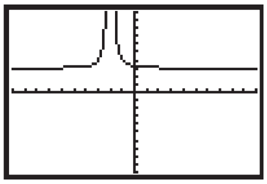
5. a.



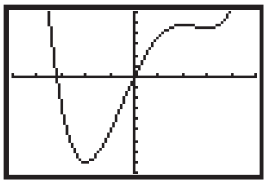
b.



c.



d.



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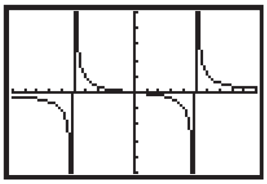
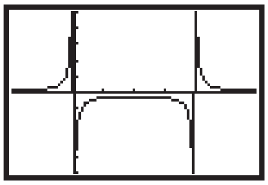
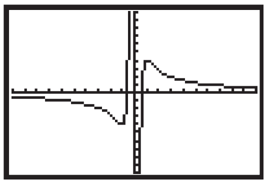


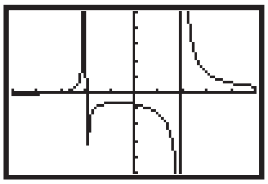
f.



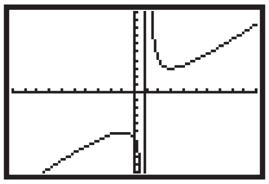
g.



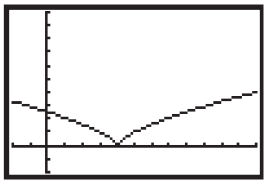
h.



i.



j.



6. $y = ax^3 + bx^2 + cx + d$

Since (0, 0) is on the curve $d = 0$:

$\frac{dy}{dx} = 3ax^2 + 2bx + c$

At $x = 2$, $\frac{dy}{dx} = 0$.

Thus, $12a + 4b + c = 0$.

Since (2, 4) is on the curve, $8a + 4b + 2c = 4$ or $4a + 2b + c = 2$.

$$\frac{d^2y}{dx^2} = 6ax + 2b$$

Since $(0, 0)$ is a point of inflection, $\frac{d^2y}{dx^2} = 0$ when $x = 0$.

Thus, $2b = 0$

$b = 0$.

Solving for a and c:

$12a + c = 0$

$4a + c = 2$

$8a = -2$

$a = -\frac{1}{4}$

$c = 3$.

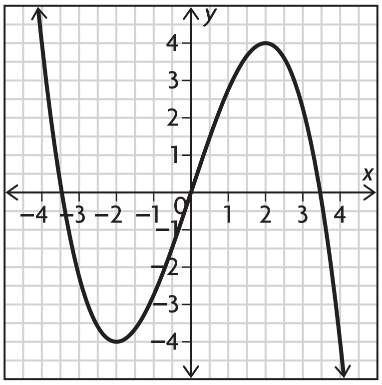
The cubic polynomial is $y = -\frac{1}{4}x^3 + 3x$.

The y-intercept is 0. The x-intercepts are found by setting $y = 0$:

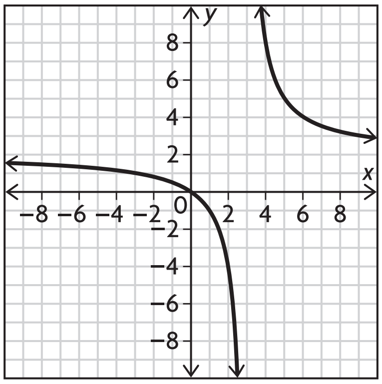
$-\frac{1}{4}x(x^2 - 12) = 0$

$x = 0$, or $x = \pm 2\sqrt{3}$.

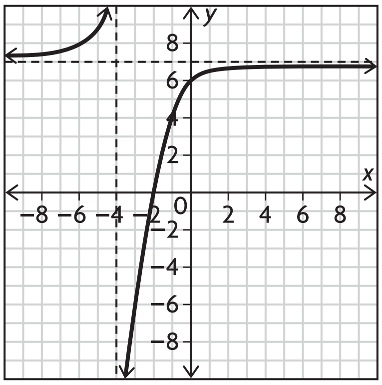
Let $y = f(x)$. Since $f(-x) = \frac{1}{4}x^3 - 3x = -f(x)$, $f(x)$ is an odd function. The graph of $y = f(x)$ is symmetric when reflected in the origin.



7. a. Answers may vary. For example:



b. Answers may vary. For example:



8. $f(x) = \dfrac{k - x}{k^2 + x^2}$

There are no discontinuities.

The y-intercept is $\dfrac{1}{k}$ and the x-intercept is k.

$$f'(x) = \frac{(-1)(k^2 + x^2) - (k - x)(2x)}{(k^2 + x^2)^2}$$

$$= \frac{x^2 - 2kx - k^2}{(k^2 + x^2)^2}$$

For critical points, we solve $f'(x) = 0$:

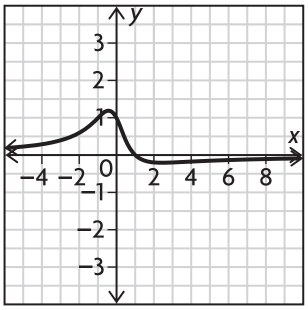
$$x^2 - 2kx - k^2 = 0$$
$$x^2 - 2kx - k^2 = 2k^2$$
$$(x - k)^2 = 2k^2$$
$$x - k = \pm\sqrt{2}k$$
$$x = \left(1 + \sqrt{2}\right)k \text{ or } x = \left(1 - \sqrt{2}\right)k.$$

Interval	$x < -0.41k$	$x = 0.41k$	$-0.41k < x < 2.41k$	$x = 2.41k$	$x > 2.41k$
$f(x)$	>0	<0	<0	$=0$	>0
Graph of $f(x)$	Increasing	Local Max	Decreasing	Local Min	Increasing

$$\lim_{x\to\infty}\left(\frac{k - x}{k^2 + x^2}\right) = \lim_{x\to\infty}\left(\frac{\frac{k}{x^2} - \frac{1}{x}}{\frac{k^2}{x^2} + 1}\right) = 0$$

$$\lim_{x\to\infty}\left(\frac{\frac{k}{x^2} - \frac{1}{x}}{\frac{k^2}{x^2} + 1}\right) = 0$$

Hence, the x-axis is a horizontal asymptote.



9. $g(x) = x^{\frac{1}{3}}(x + 3)^{\frac{2}{3}}$

There are no discontinuities.

$$g'(x) = \frac{1}{3}x^{\frac{2}{3}}(x + 3)^{\frac{2}{3}} + x^{\frac{1}{3}}\left(\frac{2}{3}\right)(x + 3)^{\frac{1}{3}}(1)$$

$$= \frac{x + 3 + 2x}{3x^{\frac{2}{3}}(x + 3)^{\frac{1}{3}}} = \frac{3(x + 1)}{3x^{\frac{2}{3}}(x + 3)^{\frac{1}{3}}}$$

$$= \frac{x + 1}{3x^{\frac{2}{3}}(x + 3)^{\frac{1}{3}}}$$

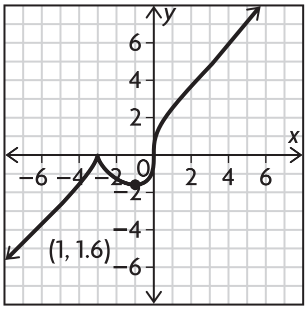
$g'(x) = 0$ when $x = -1$.

$g'(x)$ doesn't exist when $x = 0$ or $x = -3$.

Interval	$x < -3$	$x = -3$	$-3 < x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$x < 0$
$g^2(x)$	>0	Does not Exist	<0	$=0$	>0	Does not Exist	>0
Graph of $g(x)$	Increasing	Local Max	Decreasing	Local Min	Increasing		Increasing

There is a local maximum at $(-3, 0)$ and a local minimum at $(-1, -1.6)$. The second derivative is algebraically complicated to find.

Interval	$x < -3$	$x = -3$	$-3 < x < 0$	$x = 0$	$x > 0$
$g''(x)$	>0	Does Not Exist	>0	Does Not Exist	>0
Graph $g''(x)$	Concave Down	Cusp	Concave Up	Point of Inflection	Concave Down



10. a. $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}$

$$= \frac{x}{|x|\sqrt{1 + \frac{1}{x^2}}}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x\sqrt{1 + \frac{1}{x^2}}}, \text{ since } x > 0$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1 + \frac{1}{x^2}}}$$

$$= 1$$

$y = 1$ is a horizontal asymptote to the right-hand branch of the graph.

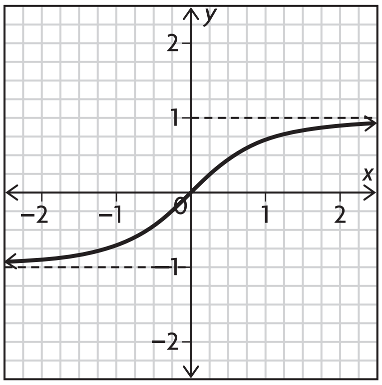
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{-x\sqrt{1 + \frac{1}{x^2}}}, \text{ since } |x| = -x$$

for $x < 0$

$$= \lim_{x\to-\infty} \frac{x}{-\sqrt{1 + \frac{1}{x^2}}}$$

$$= -1$$

$y = -1$ is a horizontal asymptote to the left-hand branch of the graph.



b. $g(t) = \sqrt{t^2 + 4t} - \sqrt{t^2 + t}$

$$= \frac{(\sqrt{t^2 + 4t} - \sqrt{t^2 + t})(\sqrt{t^2 + 4t} + \sqrt{t^2 + t})}{\sqrt{t^2 + 4t} + \sqrt{t^2 + t}}$$

$$= \frac{3t}{\sqrt{t^2 + 4t} + \sqrt{t^2 + t}}$$

$$= \frac{3t}{|t|\sqrt{1 + \frac{4}{t}} + |t|\sqrt{1 + \frac{1}{t}}}$$

$\lim_{x\to\infty} g(t) = 3 = \frac{3}{2}$, since $|t| = t$ for $t > 0$

$\lim_{x\to-\infty} g(t) = \frac{3}{-1 - 1} = -\frac{3}{2}$, since $|t| = -t$ for $t < 0$

$y = \frac{3}{2}$ and $y = -\frac{3}{2}$ are horizontal asymptotes.

11. $y = ax^3 + bx^2 + cx + d$

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

$$\frac{d^2y}{dx^2} = 6ax + 2b = 6a\left(x + \frac{b}{3a}\right)$$

For possible points of inflection, we solve $\frac{d^2y}{dx^2} = 0$:

$x = -\dfrac{b}{3a}$.

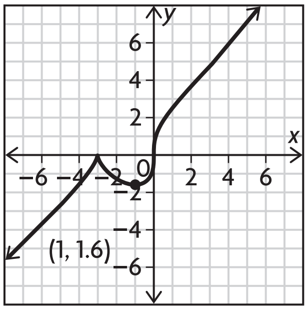
The sign of $\frac{d^2y}{dx^2}$ changes as x goes from values less than $-\frac{b}{3a}$ to values greater than $-\frac{b}{3a}$. Thus, there is a point of inflection at $x = -\frac{b}{3a}$.

At $x = \dfrac{b}{3a}$, $\dfrac{dy}{dx} = 3a\left(-\dfrac{b}{3a}\right)^2 + 2b\left(-\dfrac{b}{3a}\right) + c$

$$= c - \frac{b^2}{3a}.$$

Review Exercise, pp. 216–219

1. a. i. $x < 1$
ii. $x > 1$
iii. (1, 20)
b. i. $x < -3$, $-3 < x < 1$, $x > 6.5$
ii. $1 < x < 3$, $3 < x < 6.5$
iii. $(1, -1)$, $(6.5, -1)$
2. No. A counter example is sufficient to justify the conclusion. The function $f(x) = x^3$ is always increasing yet the graph is concave down for $x < 0$ and concave up for $x > 0$.



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$$= \frac{x}{|x|\sqrt{1 + \frac{1}{x^2}}}$$

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$$= \lim_{x\to\infty} \frac{1}{\sqrt{1 + \frac{1}{x^2}}}$$

$$= 1$$

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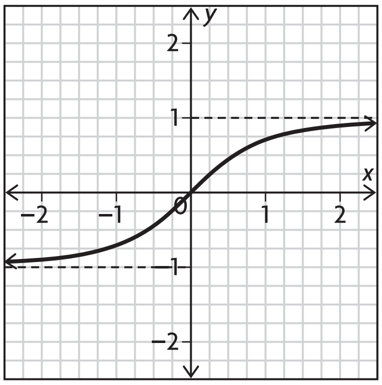
for $x < 0$

$$= \lim_{x\to-\infty} \frac{x}{-\sqrt{1 + \frac{1}{x^2}}}$$

$$= -1$$

$y = -1$ is a horizontal asymptote to the left-hand branch of the graph.



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$$= \frac{(\sqrt{t^2 + 4t} - \sqrt{t^2 + t})(\sqrt{t^2 + 4t} + \sqrt{t^2 + t})}{\sqrt{t^2 + 4t} + \sqrt{t^2 + t}}$$

$$= \frac{3t}{\sqrt{t^2 + 4t} + \sqrt{t^2 + t}}$$

$$= \frac{3t}{|t|\sqrt{1 + \frac{4}{t}} + |t|\sqrt{1 + \frac{1}{t}}}$$

$\lim\limits_{x\to\infty} g(t) = \frac{3}{} = \frac{3}{2}$, since $|t| = t$ for $t > 0$

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At $x = \dfrac{b}{3a}$, $\dfrac{dy}{dx} = 3a\left(-\dfrac{b}{3a}\right)^2 + 2b\left(-\dfrac{b}{3a}\right) + c$

$$= c - \frac{b^2}{3a}.$$

Review Exercise, pp. 216–219

1. a. i. $x < 1$
ii. $x > 1$
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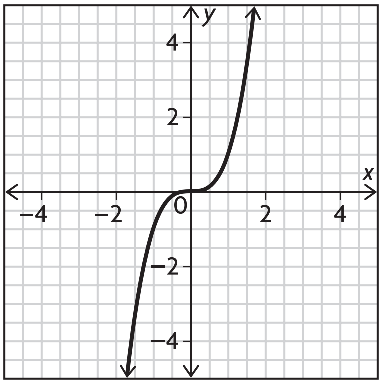


3. a. $f(x) = -2x^3 + 9x^2 + 20$
$f'(x) = -6x^2 + 18x$
For critical values, we solve:
$f'(x) = 0$
$-6x(x - 3) = 0$
$x = 0$ or $x = 3$.
$f''(x) = -12x + 18$
Since $f''(0) = 18 > 0$, (0, 20) is a local minimum point. The tangent to the graph of $f(x)$ is horizontal at (0, 20). Since $f''(3) = -18 < 0$, (3, 47) is a local maximum point. The tangent to the graph of $f(x)$ is horizontal at (3, 47).

b. $f(x) = x^4 - 8x^3 + 18x^2 + 6$
$f(x) = 4x^3 - 24x^2 + 36x$
$f(x) = 4x(x^2 - 6x + 9)$
$f(x) = 4x(x - 3)^2$
Let $f(x) = 0$:
$4x(x - 3)^2 = 0$
$x = 0$ or $x = 3$
The critical points are $(0, 6)$ and $(3, 33)$.

x	$x < 0$	0	$0 < x < 3$	3	$x > 3$
$\frac{dy}{dx}$	−	0	+	0	+
Graph	Dec	Local Min	Inc		Inc

Local minimum at $(0, 6)$
$(3, 33)$ is neither a local maximum nor a local minimum.
The tangent is parallel to the x-axis at both points because the derivative is defined at both points.

c. $h(x) = \dfrac{x - 3}{x^2 + 7}$

$$h(x) = \frac{(1)(x^2 + 7) - (x - 3)(2x)}{(x^2 + 7)^2}$$
$$= \frac{7 + 6x - x^2}{(x^2 + 7)^2}$$
$$= \frac{(7 - x)(1 + x)}{(x^2 + 7)^2}$$

Since $x^2 + 7 > 0$ for all x, the only critical values occur when $h'(x) = 0$. The critical values are $x = 7$ and $x = -1$.

Interval	$x < -1$	$x = -1$	$-1 < x < 7$	$x = 7$	$x > 7$
$h'(x)$	< 0	$= 0$	> 0	$= 0$	< 0
Graph of $h(t)$	Decreasing	Local Min	Increasing	Local Max	Decreasing

There is a local minimum at $(-1, -\frac{1}{2})$ and a local maximum at $(7, \frac{1}{14})$. At both points, the tangents are parallel to the x-axis.

d) $g(x) = (x - 1)^{\frac{1}{3}}$
$g'(x) = \dfrac{1}{3}(x - 1)^{\frac{-2}{3}}$
Let $g'(x) = 0$:
$\dfrac{1}{3}(x - 1)^{\frac{-2}{3}} = 0$
There are no solutions, but $g'(x)$ is undefined for $x = 1$, so the point $(1, 0)$ is a critical point.

x	$x < 1$	1	$x > 1$
$f'(x)$	+	Undefined	+
Graph	Inc.		Inc.

$(1, 0)$ is neither a local maximum nor a local minimum.
The tangent is not parallel to the x-axis because it is not defined for $x = 1$.

4. a. $a < x < b$, $x > e$
b. $b < x < c$
c. $x < a$, $d < x < e$
d. $c < x < d$

5. a. $y = \dfrac{2x}{x - 3}$
There is a discontinuity at $x = 3$.
$\lim_{x \to 3^-} \left(\dfrac{2x}{x - 3}\right) = -\infty$ and $\lim_{x \to 3^+} \left(\dfrac{2x}{x - 3}\right) = \infty$
Therefore, $x = 3$ is a vertical asymptote.

b. $g(x) = \dfrac{x - 5}{x + 5}$
There is a discontinuity at $x = -5$.
$\lim_{x \to -5^-} \left(\dfrac{x - 5}{x + 5}\right) = \infty$ and $\lim_{x \to -5^+} \left(\dfrac{x - 5}{x + 5}\right) = -\infty$
Therefore, $x = -5$ is a vertical asymptote.

c. $f(x) = \dfrac{x^2 - 2x - 15}{x + 3}$

$= \dfrac{(x+3)(x-5)}{x+3}$

$= x - 5, x \neq -3$

There is a discontinuity at $x = -3$.

$\lim_{x \to -3^+} f(x) = -8$ and $\lim_{x \to -3^-} f(x) = -8$

There is a hole in the graph of $y = f(x)$ at $(-3, -8)$.

d. $g(x) = \dfrac{5}{x^2 - x - 20}$

$g(x) = \dfrac{5}{(x-5)(x+4)}$

To find vertical asymptotes, set the denominator equal to 0:

$(x-5)(x+4) = 0$

$x = -4$ or $x = 5$

Vertical asymptotes at $x = -4$ and $x = 5$

$$\lim_{x \to -4^-} \frac{5}{(x-5)(x+4)} = \infty$$

$$\lim_{x \to -4^+} \frac{5}{(x-5)(x+4)} = -\infty$$

$$\lim_{x \to 5^-} \frac{5}{(x-5)(x+4)} = -\infty$$

$$\lim_{x \to 5^+} \frac{5}{(x-5)(x+4)} = \infty$$

6. $y = x^3 + 5$

$y' = 3x^2$

$y'' = 6x$

Let $y'' = 0$

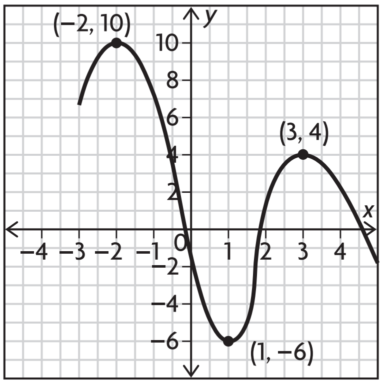
$6x = 0$

$x = 0$

The point of inflection is $(0, 5)$

Since the derivative is 0 at $x = 0$, the tangent line is parallel to the x-axis at that point. Because the derivative is always positive, the function is always increasing and therefore must cross the tangent line instead of just touching it.

7.

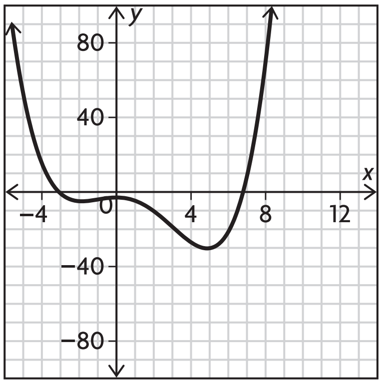


8. a. i. Concave up: $-1 < x < 3$

Concave down: $x < -1, 3 < x$

ii. Points of inflection at $x = -1$ and $x = 3$

iii.

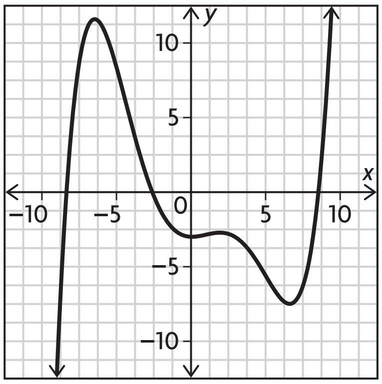


b. i. Concave up: $-4.5 < x < 1, 5 < x$

Concave down: $x < -4.5, 1 < x < 5$

ii. Points of inflection at $x = -4.5$, $x = 1$, and $x = 5$

iii.



9. a. $g(x) = \dfrac{ax + b}{(x-1)(x-4)}$

$= \dfrac{ax + b}{x^2 - 5x + 4}$

$g'(x) = \dfrac{a(x^2 - 5x + 4) - (ax + b)(2x - 5)}{(x^2 - 5x + 4)^2}$

Since the tangent at $(2, -1)$ has slope 0, $g'(2) = 0$.

Hence, $\dfrac{-2a + 2a + b}{4} = 0$ and $b = 0$.

Since $(2, -1)$ is on the graph of $g(x)$:

$-1 = \dfrac{2a + b}{-2}$

$2a + 0 = 2$

$a = 1$.

Therefore $g(x) = \dfrac{x}{(x-1)(x-4)}$.

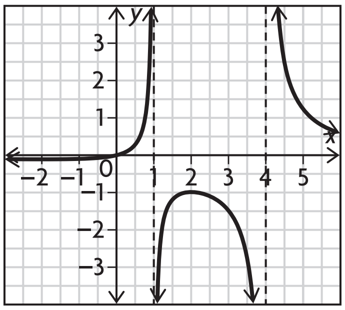
b. There are discontinuities at $x = 1$ and $x = 4$.
$\lim_{x \to 1^-} g(x) = \infty$ and $\lim_{x \to 1^+} g(x) = -\infty$
$\lim_{x \to 4^-} g(x) = -\infty$ and $\lim_{x \to 4^+} g(x) = \infty$
$x = 1$ and $x = 4$ are vertical asymptotes.
The y-intercept is 0.

$$g'(x) = \frac{4 - x^2}{(x^2 - 5x + 4)^2}$$

$g'(x) = 0$ when $x = \pm 2$.

Interval	$x < -2$	$x = -2$	$-2 < x < 1$	$1 < x < 2$	$x = 2$	$2 < x < 4$	$x > 4$
$g'(x)$	< 0	0	> 0	> 0	0	< 0	< 0
Graph of $g(x)$	Decreasing	Local Min	Increasing	Increasing	Local Max	Decreasing	Decreasing

There is a local minimum at $\left(-2, -\frac{1}{9}\right)$ and a local maximum at $(2, -1)$.



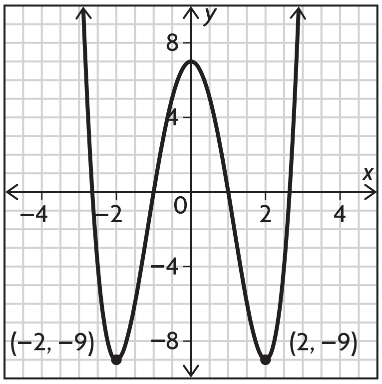
10. a. $y = x^4 - 8x^2 + 7$
This is a fourth degree polynomial and is continuous for all x. The y-intercept is 7.

$$\frac{dy}{dx} = 4x^3 - 16x$$
$$= 4x(x - 2)(x + 2)$$

The critical values are $x = 0$, -2 and 2.

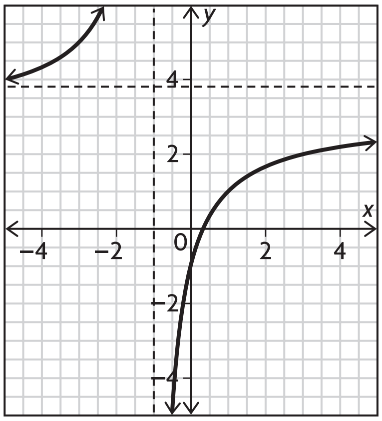
Interval	$x < -2$	$x = -2$	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
$\frac{dy}{dx}$	< 0	$= 0$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of y	Decreasing	Local Min	Increasing	Local Max	Decreasing	Local Min	Increasing

There are local minima at $(-2, -9)$ and at $(2, -9)$, and a local maximum at $(0, 7)$.



b. $f(x) = \frac{3x - 1}{x + 1}$
$= 3 - \frac{4}{x + 1}$

From experience, we know the graph of $y = -\frac{1}{x}$ is



The graph of the given function is just a transformation of the graph of $y = -\frac{1}{x}$. The vertical asymptote is $x = -1$ and the horizontal asymptote is $y = 3$. The y-intercept is -1 and there is an x-intercept at $\frac{1}{3}$.

c. $g(x) = \frac{x^2 + 1}{4x^2 - 9}$
$= \frac{x^2 + 1}{(2x - 3)(2x + 3)}$

The function is discontinuous at $x = -\frac{3}{2}$ and at $x = \frac{3}{2}$.
$\lim_{x \to -\frac{3}{2}^-} g(x) = \infty$

$\lim_{x\to -\frac{3}{2}^+} g(x) = -\infty$
$\lim_{x\to \frac{3}{2}^-} g(x) = -\infty$
$\lim_{x\to -\frac{3}{2}^+} g(x) = \infty$

Hence, $x = -\frac{3}{2}$ and $x = \frac{3}{2}$ are vertical asymptotes.
The y-intercept is $-\frac{1}{9}$.

$$g'(x) = \frac{2x(4x^2 - 9) - (x^2 + 1)(8x)}{(4x^2 - 9)^2} = \frac{-26x}{(4x^2 - 9)^2}$$

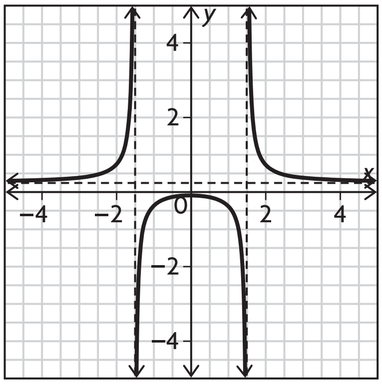
$g'(x) = 0$ when $x = 0$.

Interval	$x < -\frac{3}{2}$	$-\frac{3}{2} < x < 0$	$x = 0$	$0 < x < \frac{3}{2}$	$x > \frac{3}{2}$
$g'(x)$	> 0	> 0	$= 0$	< 0	< 0
Graph $g(x)$	Increasing	Increasing	Local Max	Decreasing	Decreasing

There is a local maximum at $\left(0, -\frac{1}{9}\right)$.

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{1 + \frac{1}{x^2}}{4 - \frac{9}{x^2}} = \frac{1}{4} \text{ and } \lim_{x\to\infty} g(x) = \tfrac{1}{4}$$

Hence, $y = \frac{1}{4}$ is a horizontal asymptote.



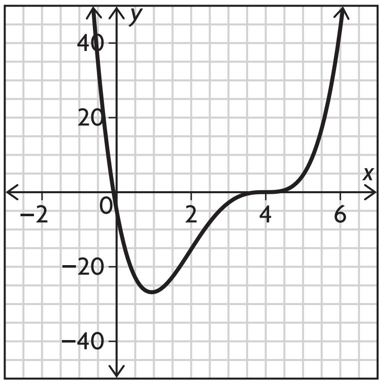
d) $y = x(x - 4)^3$
This is a polynomial function, so there are no discontinuities and no asymptotes. The domain is $\{x \in \mathbf{R}\}$.
x-intercepts at $x = 0$ and $x = 4$
y-intercepts at $y = 0$
$y' = (x - 4)^3 + 3x(x - 4)^2$
$y' = (x - 4)^2(x - 4 + 3x)$
$y' = 4(x - 4)^2(x - 1)$
Let $y' = 0$:
$4(x - 4)^2(x - 1) = 0$
$x = 4$ or $x = 1$
The critical numbers are $(1, -27)$ and $(4, 0)$.

x	$x < 1$	1	$1 < x < 4$	4	$x > 4$
$\frac{dy}{dx}$	$-$	0	$+$	0	$+$
Graph	Dec	Local Min	Inc		Inc

Local minimum at $(1, -27)$
$(4, 0)$ is not a local extremum
$y'' = 4(2(x - 4)(x - 1) + (x - 4)^2)$
$y'' = 4\left(2(x - 4)\left(x - 1 + \frac{x - 4}{2}\right)\right)$
$y'' = 8(x - 4)\left(\frac{3}{2}x - 3\right)$
Let $y'' = 0$:
$8(x - 4)\left(\frac{3}{2}x - 3\right) = 0$
$x = 4$ or $x = 2$
The points of inflection are $(2, -16)$ and $(4, 0)$.

x	$x < 2$	2	$2 < x < 4$	4	$x > 4$
$\frac{dy}{dx}$	$+$	0	$-$	0	$+$
Graph	c. up	point of inflection	c. down	point of inflection	c. up

The graph has a local minimum at $(1, -27)$ and points of inflection at $(2, -16)$ and $(4, 0)$, with x-intercepts of 0 and 4 and a y-intercept of 0.



e. $h(x) = \frac{x}{x^2 - 4x + 4}$
$= \frac{x}{(x - 2)^2} = x(x - 2)^{-2}$
There is a discontinuity at $x = 2$
$\lim_{x\to 2^-} h(x) = \infty = \lim_{x\to 2^+} h(x)$

Thus, $x = 2$ is a vertical asymptote. The y-intercept is 0.

$$h'(x) = (x-2)^{-2} + x(-2)(x-2)^{-3}(1)$$
$$= \frac{x - 2 - 2x}{(x-2)^3}$$
$$= \frac{-2 - x}{(x-2)^3}$$

$h'(x) = 0$ when $x = -2$.

Interval	$x < -2$	$x = -2$	$-2 < x < 2$	$x > 2$
$h'(x)$	< 0	$= 0$	> 0	< 0
Graph of $h(x)$	Decreasing	Local Min	Increasing	Decreasing

There is a local minimum at $\left(-2, -\frac{1}{8}\right)$.

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} = \frac{\frac{1}{x}}{1 - \frac{4}{x} + \frac{4}{x^2}} = 0$$

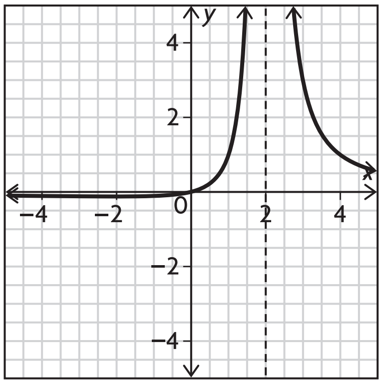
Similarly, $\lim_{x\to\infty} h(x) = 0$

The x-axis is a horizontal asymptote.

$$h''(x) = -2(x-2)^{-3} - 2(x-2)^{-3} + 6x(x-2)^{-4}$$
$$= -4(x-2)^{-3} + 6x(x-2)^{-4}$$
$$= \frac{2x + 8}{(x-2)^4}$$

$h''(x) = 0$ when $x = -4$

The second derivative changes signs on opposite sides $x = -4$, Hence $(-4, -\frac{1}{9})$ is a point of inflection.



f. $f(t) = \dfrac{t^2 - 3t + 2}{t - 3}$

$$= t + \frac{2}{t-3}$$

Thus, $f(t) = t$ is an oblique asymptote. There is a discontinuity at $t = 3$.

$\lim_{t\to 3^-} f(t) = -\infty$ and $\lim_{t\to 3^+} f(t) = \infty$

Therefore, $x = 3$ is a vertical asymptote.

The y-intercept is $-\frac{2}{3}$.

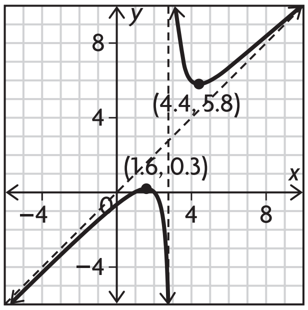
The x-intercepts are $t = 1$ and $t = 2$.

$$f'(t) = 1 - \frac{2}{(t-3)^2}$$

$$f'(t) = 0 \text{ when } 1 - \frac{2}{(t-3)^2} = 0$$
$$(t-3)^2 = 2$$
$$t - 3 = \pm\sqrt{2}$$
$$t = 3 \pm \sqrt{2}.$$

Interval	$t < 3 - \sqrt{2}$	$t = 3 - \sqrt{2}$	$3 - \sqrt{2} < t < 3$	$3 < t < 3 + \sqrt{2}$	$t = 3 + \sqrt{2}$	$t > 3 + \sqrt{2}$
$f'(t)$	> 0	$= 0$	< 0	< 0	$= 0$	> 0
Graph of $f(t)$	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing

(1.6, 0.2) is a local maximum and (4.4, 5.8) is a local minimum.



11. a. $f(x) = \dfrac{2x + 4}{x^2 - k^2}$

$$f'(x) = \frac{2(x^2 - k^2) - (2x+4)(2x)}{(x^2 - k^2)^2}$$
$$= -\frac{2x^2 + 8x + 2k^2}{(x^2 - k^2)^2}$$

For critical values, $f'(x) = 0$ and $x \neq \pm k$:

$x^2 + 4x + k^2 = 0$

$$x = \frac{-4 \pm \sqrt{16 - 4k^2}}{2}.$$

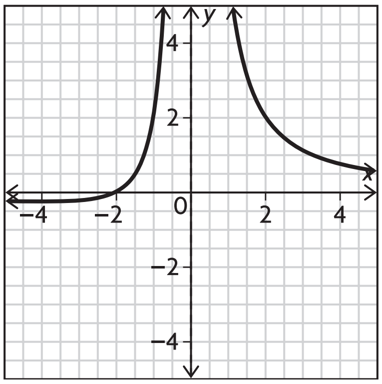
For real roots, $16 - 4k^2 \geq 0$

$$-2 \leq k \leq 2.$$

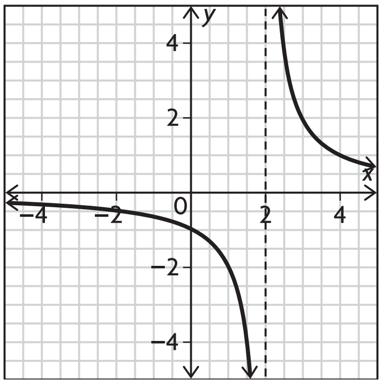
The conditions for critical points to exist are $-2 \leq k \leq 2$ and $x \neq \pm k$.

b. There are three different graphs that results for values of k chosen.

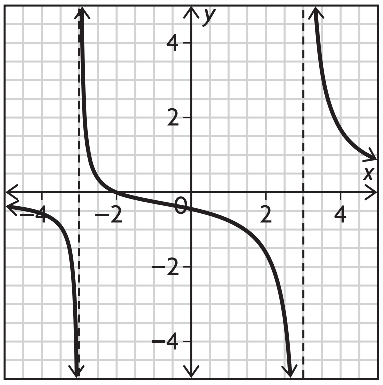
$k = 0$



$k = 2$



For all other values of k, the graph will be similar to that of $1(i)$ in Exercise 9.5.



12. a. $f(x) = \dfrac{2x^2 - 7x + 5}{2x - 1}$

$f(x) = x - 3 + \dfrac{2}{2x - 1}$

The equation of the oblique asymptote is $y = x - 3$.

$$\begin{array}{r} x - 3 \\ 2x - 1\overline{)2x^2 - 7x + 5} \\ \underline{2x^2 - x} \\ -6x + 5 \\ \underline{-6x + 3} \\ 2 \end{array}$$

$$\lim_{x\to\infty}[y - f(x)] = \lim_{x\to\infty}\left[x - 3 - \left(x - 3 + \frac{2}{2x - 1}\right)\right]$$

$$= \lim_{x\to\infty}\left[-\frac{2}{2x - 1}\right] = 0$$

b. $f(x) = \dfrac{4x^3 - x^2 - 15x - 50}{x^2 - 3x}$

$f(x) = 4x + 11 + \dfrac{18x - 50}{x^2 - 3x}$

$$\begin{array}{r} 4x + 11 \\ x^2 - 3x\overline{)4x^3 - x^2 - 15x - 50} \\ \underline{4x^3 - 12x^2} \\ 11x^2 - 15x \\ \underline{11x^2 - 33x} \\ 18x - 50 \end{array}$$

$\lim_{x\to\infty}[y - f(x)]$

$$= \lim_{x\to\infty}\left[4x + 11 - \left(4x + 11 + \frac{18x - 50}{x^2 - 3x}\right)\right]$$

$$= \lim_{x\to\infty}\left[\frac{\frac{18}{x} - \frac{50}{x^2}}{1 - \frac{3}{x}}\right]$$

$= 0$

13. $g(x) = (x^2 - 4)^2$

$g(x) = (x^2 - 4)(x^2 - 4)$

$g'(x) = 2x(x^2 - 4) + 2x(x^2 - 4)$

$g'(x) = 4x(x^2 - 4)$

$g'(x) = 4x(x - 2)(x + 2)$

Set $g'(x) = 0$

$0 = 4x(x - 2)(x + 2)$

$x = -2$ or $x = 0$ or $x = 2$

	$x < -2$	$-2 < x < 0$	$0 < x < 2$	$x > 2$
$4x$	−	−	+	+
$x - 2$	−	−	−	+
$x + 2$	−	+	+	+
Sign of $g'(x)$	$(-)(-)(-) = -$	$(-)(-)(+) = +$	$(+)(-)(+) = -$	$(+)(+)(+) = +$
Behaviour of $g(x)$	decreasing	increasing	decreasing	increasing

14. $f(x) = x^3 + \dfrac{3}{2}x^2 - 7x + 5, -4 \le x \le 3$

$f'(x) = 3x^2 + 3x - 7$

Set $f'(x) = 0$

$0 = 3x^2 + 3x - 7$

$x = \dfrac{-3 \pm \sqrt{(3)^2 - 4(3)(-7)}}{2(3)}$

$$x = \frac{-3 \pm \sqrt{93}}{6}$$

$x \doteq -2.107$ or $x \doteq 1.107$

$f'(x) = 3x^2 + 3x - 7$

$f'(x) = 6x + 3$

When $x = -2.107$,

$f''(-2.107) = 6(-2.107) + 3$

$f''(-2.107) = -9.642$

Since $f''(-2.107) < 0$, a local maximum occurs when $x = -2.107$.

when $x = 1.107$,

$f''(1.107) = 6(1.107) + 3$

$f''(1.107) = 9.642$

Since $f''(1.107) > 0$, a local minimum occurs when $x = (1.107)$.

when $x = -4$,

$$f(-4) = (-4)^3 + \frac{3}{2}(-4)^2 - 7(-4) + 5$$

$f(-4) = -64 + 24 + 28 + 5$

$f(-4) = -7$

when $x = -2.107$,

$$f(-2.107) = (-2.107)^3 + \frac{3}{2}(-2.107)^2 - 7(-2.107) + 5$$

$$f(-2.107) \doteq -9.353\,919 + 6.659\,173\,5 + 14.749 + 5$$

when $x = 1.107$,

$$f(1.107) = (1.107)^3 + \frac{3}{2}(1.107)^2 - 7(1.107) + 5$$

$f(1.107) \doteq 1.356\,572 + 1.838\,173\,5 - 7.749 + 5$

$f(1.107) \doteq 0.446$

when $x = 3$,

$$f(3) = (3)^3 + \frac{3}{2}(3)^2 - 7(3) + 5$$

$f(3) = 27 + 13.5 - 21 + 5$

$f(3) = 24.5$

Local Maximum: $(-2.107, 17.054)$

Local Minimum: $(1.107, 0.446)$

Absolute Maximum: $(3, 24.5)$

Absolute Minimum: $(-4, -7)$

15. $f(x) = 4x^3 + 6x^2 - 24x - 2$

Evaluate $y = 4(0)^3 + 6(0)^2 - 24(0) - 2$

$y = -2$

$f(x) = 4x^3 + 6x^2 - 24x - 2$

$f'(x) = 12x^2 + 12x - 24$

Set $f'(x) = 0$

$0 = 12x^2 + 12x - 24$

$0 = 12(x^2 + x - 2)$

$0 = 12(x - 1)(x + 2)$

$x = -2$ or $x = 1$

	$x < -2$	$-2 < x < 1$	$x > 1$
$12(x - 1)$	−	−	+
$x + 2$	−	+	+
Sign of $f'(x)$	$(-)(-) = +$	$(-)(+) = -$	$(+)(+) = +$
Behaviour of $f(x)$	increasing	decreasing	increasing
	maximum at $x = -2$		minimum at $x = 1$

when $x = -2$,

$f(-2) = 4(-2)^3 + 6(-2)^2 - 24(-2) - 2$

$f(-2) = -32 + 24 + 48 - 2$

$f(-2) = 38$

when $x = 1$,

$f(1) = 4(1)^3 + 6(1)^2 - 24(1) - 2$

$f(1) = 4 + 6 - 24 - 2$

$f(1) = -16$

Maximum: $(-2, 38)$ Minimum: $(1, -16)$

$f'(x) = 12x^2 + 12x - 24$

$f''(x) = 24x + 12$

Set $f''(x) = 0$

$0 = 24x + 12$

$x = -0.5$

	$x < -0.5$	$x > -0.5$
$f''(x) = 24x + 12$	−	+
$f(x)$	concave down	concave up
	point of inflection at $x = -0.5$	

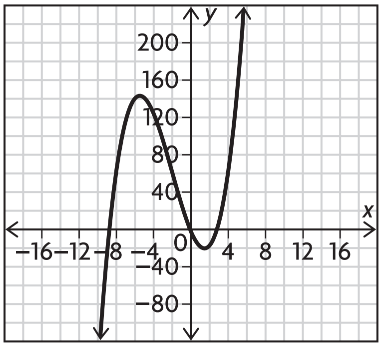
when $x = -0.5$,

$f(-0.5) = 4(-0.5)^3 + 6(-0.5)^2 - 24(-0.5) - 2$

$f(-0.5) = -0.5 + 1.5 + 12 - 2$

$f(-0.5) = 11$

Point of inflection: $(-0.5, 11)$



16. a. $p(x)$: oblique asymptote, because the highest degree of x in the numerator is exactly one degree higher than the highest degree of x in the denominator.
$q(x)$: vertical asymptotes at $x = -1$ and $x = 3$; horizontal asymptote at $y = 0$
$r(x)$: vertical asymptotes at $x = -1$ and $x = 1$; horizontal asymptote at $y = 1$
$s(x)$: vertical asymptote at $y = 2$.

b. $r(x) = \dfrac{x^2 - 2x - 8}{x^2 - 1}$
$= \dfrac{(x - 4)(x + 2)}{(x - 1)(x + 1)}$

The domain is $\{x | x \neq -1, 1, x \in \mathbf{R}\}$.
x-intercepts: -2, 4; y-intercept: 8
r has vertical asymptotes at $x = -1$ and $x = 1$.
$r(-1.001) = -2496.75$, so as $x \rightarrow -1^-$, $r(x) \rightarrow -\infty$
$r(-0.999) = 2503.25$, so as $x \rightarrow -1^+$, $r(x) \rightarrow \infty$
$r(0.999) = 4502.25$, so as $x \rightarrow 1^-$, $r(x) \rightarrow \infty$
$r(1.001) = -4497.75$, so as $x \rightarrow 1^+$, $r(x) \rightarrow -\infty$

$$\lim_{x\to-\infty} \frac{x^2 - 2x - 8}{x^2 - 1} = \lim_{x\to-\infty} \frac{\frac{x^2}{x^2} - \frac{2x}{x^2} - \frac{8}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$$
$$= \lim_{x\to-\infty} \frac{1 - \frac{2}{x} - \frac{8}{x^2}}{1 - \frac{1}{x^2}}$$
$$= \frac{1 - 0 - 0}{1 - 0}$$
$$= 1$$

So, $y = 1$ is a horizontal asymptote on the left.

$$\lim_{x\to\infty} \frac{x^2 - 2x - 8}{x^2 - 1} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2} - \frac{2x}{x^2} - \frac{8}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$$
$$= \lim_{x\to\infty} \frac{1 - \frac{2}{x} - \frac{8}{x^2}}{1 - \frac{1}{x^2}}$$
$$= \frac{1 - 0 - 0}{1 - 0}$$
$$= 1$$

So, $y = 1$ is a horizontal asymptote on the right.

$$r'(x) = \frac{(x^2 - 1)(2x - 2) - (x^2 - 2x - 8)(2x)}{(x^2 - 1)^2}$$
$$= \frac{2x^3 - 2x^2 - 2x + 2 - (2x^3 - 4x^2 - 16x)}{(x^2 - 1)^2}$$
$$= \frac{2x^2 + 14x + 2}{(x^2 - 1)^2}$$
$$= \frac{2(x^2 + 7x + 1)}{(x^2 - 1)^2}$$

r' is defined for all values of x in the domain of r.
$r'(x) = 0$ for $x \doteq -0.15$ and $x \doteq -6.85$. $r'(1)$ and $r'(-1)$ do not exist.

	$x < -6.85$	$x = -6.85$	$-6.85 < x < -1$
$x^2 + 7x + 1$	+	0	−
$r'(x)$	+	0	−
	$x = -1$	$-1 < x < -0.15$	$x = -0.15$
$x^2 + 7x + 1$	−	−	0
$r'(x)$	undefined	−	0
	$-0.15 < x < 1$	$x = 1$	$x > 1$
$x^2 + 7x + 1$	+	+	+
$r'(x)$	+	undefined	+

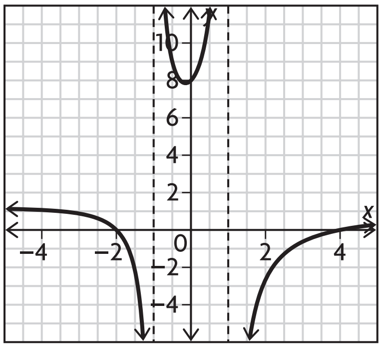
r is increasing when $x < -6.85$, $-0.15 < x < 1$, and $x > 1$. r is decreasing when $-6.85 < x < -1$ and $-1 < x < -0.15$. r has a maximum turning point at $x = -6.85$ and a minimum turning point at $x = -0.15$.

$$r''(x) = \frac{(x^2 - 1)^2(4x + 14)}{(x^2 - 1)^4} - \frac{(2x^2 + 14x + 2)[2(x^2 - 1)(2x)]}{(x^2 - 1)^4}$$
$$= \frac{(x^2 - 1)(4x + 14) - 4x(2x^2 + 14x + 2)}{(x^2 - 1)^3}$$
$$= \frac{4x^3 + 14x^2 - 4x - 14 - 8x^3 - 56x^2 - 8x}{(x^2 - 1)^3}$$
$$= \frac{-4x^3 - 42x^2 - 12x - 14}{(x^2 - 1)^3}$$
$$= \frac{-2(2x^3 + 21x^2 + 6x + 7)}{(x^2 - 1)^3}$$

r'' is defined for all values of x in the domain of r.
$r''(x) = 0$ for $x \doteq -10.24$. This is a possible point of inflection. $r''(1)$ and $r''(-1)$ do not exist.

	$x < -10.24$	$x = 10.24$
$-2(2x^3 + 21x^2 + 6x + 7)$	+	0
$(x^2 - 1)^3$	+	+
$r''(x)$	+	0
	$-10.24 < x < -1$	$x = -1$
$-2(2x^3 + 21x^2 + 6x + 7)$	−	−
$(x^2 - 1)^3$	+	0
$r''(x)$	−	undefined
	$-1 < x < 1$	$x = 1$
$-2(2x^3 + 21x^2 + 6x + 7)$	−	−
$(x^2 - 1)^3$	−	0
$r''(x)$	+	undefined
	$x > 1$	
$-2(2x^3 + 21x^2 + 6x + 7)$	−	
$(x^2 - 1)^3$	+	
$r''(x)$	−	

The graph is concave up for $x < -10.24$ and $-1 < x < 1$. The graph is concave down for $-10.24 < x < -1$ and $x > 1$. The graph changes concavity at $x = -10.24$. This is a point of inflection with coordinates $(-10.24, 1.13)$.
$r(-6.85) = 1.15$ and $r(-0.15) = 7.85$. The graph has a local maximum point at $(-6.85, 1.15)$ and a local minimum point at $(-0.15) = 7.85$.



17. The domain is $\{x | x \neq 0, x \in \mathbf{R}\}$: x-intercept: -2, y-intercept: 8; f has a vertical asymptote at $x = 0$. $f(-0.001) = -7999.99$, so $f(x) \to -\infty$ as $x \to 0^-$. $f(0.001) = 8000.00$, so $f(x) \to \infty$ as $x \to 0^+$. There are no horizontal asymptotes.

$$f'(x) = \frac{x(3x^2) - (x^3 + 8)(1)}{x^2} = \frac{3x^3 - x^3 - 8}{x^2} = \frac{2x^3 - 8}{x^2}$$

$f'(x)$ is defined for all values of x in the domain of f. $f'(x) = 0$ when $x = 1.59$. $f'(0)$ does not exist.

	$x < 0$	$x = 0$	$0 < x < 1.59$
$2x^3 - 8$	−	−	−
x^2	+	0	+
$f'(x)$	−	undefined	−
	$x = 1.59$	$x > 1.59$	
$2x^3 - 8$	0	+	
x^2	+	+	
$f'(x)$	0	+	

f is increasing for $x > 1.59$ and decreasing for $x < 0$ and $0 < x < 1.59$. f has a minimum turning point at $x = 1.59$.

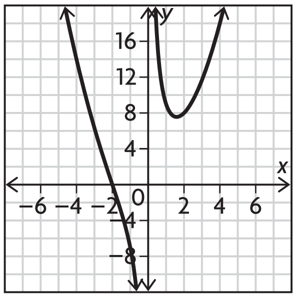
$$f''(x) = \frac{x^2(6x^2) - (2x^3 - 8)(2x)}{x^4} = \frac{x(6x^2) - (2x^3 - 8)2}{x^3} = \frac{6x^3 - 4x^3 + 16}{x^3} = \frac{2x^3 + 16}{x^3}$$

f'' is defined for all values of x in the domain of f. $f''(x) = 0$ when $x = -2$. This is a possible point of inflection. $f(0)$ does not exist.

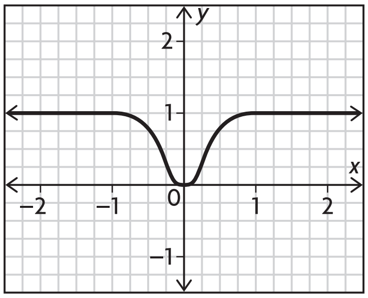
	$x < -2$	$x = -2$	$-2 < x < 0$
$2x^3 + 16$	−	0	+
x^3	−	−	−
$f''(x)$	+	0	−
	$x = 0$	$x > 0$	
$2x^3 + 16$	+	+	
x^3	0	+	
$f''(x)$	undefined	+	

f is concave up when $x < -2$ and $x > 0$. f is concave down when $-2 < x < 0$. The graph changes

concavity where $x = -2$. This is point of inflection with coordinates $(-2, 0)$.
$f(1.59) \doteq 7.56$. The graph has a local minimum at $(1.59, 7.56)$.



18. If $f(x)$ is increasing, then $f'(x) > 0$. From the graph of f', $f'(x) > 0$ for $x > 0$. If $f(x)$ is decreasing, then $f'(x) < 0$. From the graph of f', $f'(x) < 0$ for $x < 0$. At a stationary point, $f'(x) = 0$. From the graph, the zero for $f'(x)$ occurs at $x = 0$. At $x = 0$. $f'(x)$ changes from negative to positive, so f has a local minimum point there.
If the graph of f is concave up, then f'' is positive. From the slope of f', the graph of f is concave up for $-0.6 < x < 0.6$. If the graph of f is concave down, then f'' is negative. From the slope of f', the graph of f is concave down for $x < -0.6$ and $x > 0.6$. Graphs will vary slightly.



19. $f'(x) = \dfrac{(x-1)^2(5) - 5x(2)(x-1)(1)}{(x-1)^4}$

$= \dfrac{5(x-1) - 10x}{(x-1)^3}$

$= \dfrac{-5x - 5}{(x-1)^3}$

$= \dfrac{-5(x+1)}{(x-1)^3}$

$f''(x) = \dfrac{(x-1)^3(-5)}{(x-1)^6} - \dfrac{(-5x-5)(3)(x-1)^2(1)}{(x-1)^6}$

$= \dfrac{(x-1)(-5) - 3(-5x-5)}{(x-1)^4}$

$= \dfrac{10x - 20}{(x-1)^4}$

$= \dfrac{10(x-2)}{(x-1)^4}$

The domain is $\{x | x \neq 1, x \in \mathbf{R}\}$. The x- and y-intercepts are both 0. f has a vertical asymptote at $x = 1$.
$f(0.999) = 4\,995\,000$ so as $x \to 1^-$, $f(x) \to \infty$
$f(1.001) = 5\,005\,000$ so as $x \to 1^+$, $f(x) \to \infty$

$\lim_{x \to -\infty} \dfrac{5x}{x^2 - 2x + 1} = 0$
$y = 0$ is a horizontal asymptote on the right.

$\lim_{x \to -\infty} \dfrac{5x}{x^2 - 2x + 1} = 0$
$y = 0$ is a horizontal asymptote on the left.

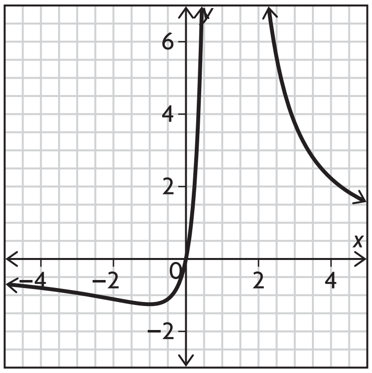
$f'(x)$ is defined for all values of x in the domain of f. $f'(x) = 0$ when $x = -1$. $f(1)$ does not exist.

	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$-5(x + 1)$	+	0	−	−	−
$(x - 1)^3$	−	−	−	0	+
$f'(x)$	−	0	+	undefined	−

f is decreasing when $x < -1$ and $x > 1$. f is increasing when $-1 < x < 1$. f has a minimum turning point at $x = -1$.
$f''(x)$ is defined for all values of x in the domain of f. $f''(x) = 0$ when $x = -3$. This is a possible point of inflection.
$f(1)$ does not exist.

	$x < -2$	$x = -2$	$-2 < x < 1$	$x = 1$	$x > 1$
$x + 2$	−	0	+	+	+
$f''(x)$	−	0	+	undefined	+

The graph is concave down for $x < -2$ and concave up when $-2 < x < 1$ and $x > 1$. It changes concavity at $x = -2$. f has an inflection point at $x = -2$ with coordinates $(-2, -1.11)$.
$f(-1) = -1.25$. f has a local minimum at $(-1, -1.25)$.



20. a. Graph A is f, graph C is f', and graph B is f''. We know this because when you take the derivative, the degree of the denominator increases by one. Graph A has a squared term in the denominator, graph C has a cubic term in the denominator, and graph B has a term to the power of four in the denominator.

b. Graph F is f, graph E is f' and graph D is f''. We know this because the degree of the denominator increases by one degree when the derivative is taken.

Chapter 4 Test, p. 220

1. a. $x < -9$ or $-6 < x < -3$ or $0 < x < 4$ or $x > 8$

b. $-9 < x < -6$ or $-3 < x < 0$ or $4 < x < 8$

c. $(-9, 1)$, $(-6, -2)$, $(0, 1)$, $(8, -2)$

d. $x = -3$, $x = 4$

e. $f''(x) > 0$

f. $-3 < x < 0$ or $4 < x < 8$

g. $(-8, 0)$, $(10, -3)$

2. a. $g(x) = 2x^4 - 8x^3 - x^2 + 6x$

$g'(x) = 8x^3 - 24x^2 - 2x + 6$

To find the critical points, we solve $g'(x) = 0$:

$8x^3 - 24x^2 - 2x + 6 = 0$

$4x^3 - 12x^2 - x + 3 = 0$

Since $g'(3) = 0$, $(x - 3)$ is a factor.

$(x - 3)(4x^2 - 1) = 0$

$x = 3$ or $x = -\frac{1}{2}$ or $x = \frac{1}{2}$.

Note: We could also group to get

$4x^2(x - 3) - (x - 3) = 0$.

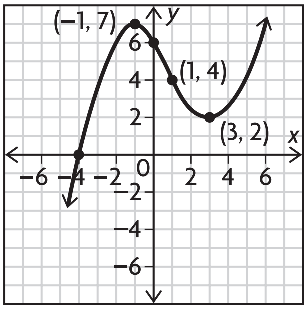
b. $g''(x) = 24x^2 - 48x - 2$

Since $g''\left(-\frac{1}{2}\right) = 28 > 0$, $\left(-\frac{1}{2}, -\frac{17}{8}\right)$ is a local maximum.

Since $g''\left(-\frac{1}{2}\right) = -20 < 0$, $\left(\frac{1}{2}, \frac{15}{8}\right)$ is a local maximum.

Since $g''(3) = 70 > 0$, $(3, -45)$ is a local minimum.

3.



4. $g(x) = \dfrac{x^2 + 7x + 10}{(x - 3)(x + 2)}$

The function $g(x)$ is not defined at $x = -2$ or $x = 3$. At $x = -2$, the value of the numerator is 0. Thus, there is a discontinuity at $x = -2$, but $x = -2$ is not a vertical asymptote.

At $x = 3$, the value of the numerator is 40. $x = 3$ is a vertical asymptote.

$$g(x) = \frac{(x + 2)(x + 5)}{(x - 3)(x + 2)} = \frac{x + 5}{x - 3}, x \neq -2$$

$$\lim_{x \to -2^-} g(x) = \lim_{x \to -2^-} \left(\frac{x + 5}{x - 3}\right) = -\frac{3}{5}$$

$$\lim_{x \to -2^+} g(x) = \lim_{x \to -2^+} \left(\frac{x + 5}{x - 3}\right) = -\frac{3}{5}$$

There is a hole in the graph of $g(x)$ at $\left(-2, -\frac{3}{5}\right)$.

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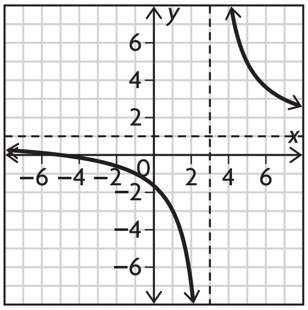
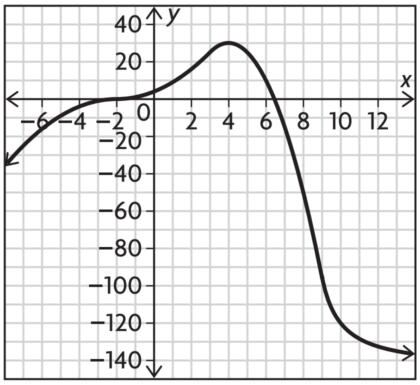
There is a vertical asymptote at $x = 3$.

Also, $\lim_{x \to \infty} g(x) = \lim_{x \to -\infty} g(x) = 1$.

Thus, $y = 1$ is a horizontal asymptote.



5.



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Chapter 4 Test, p. 220

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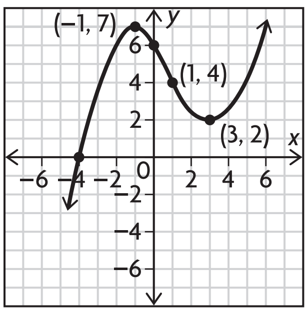

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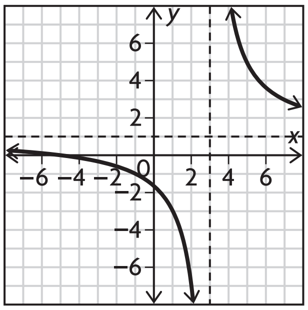
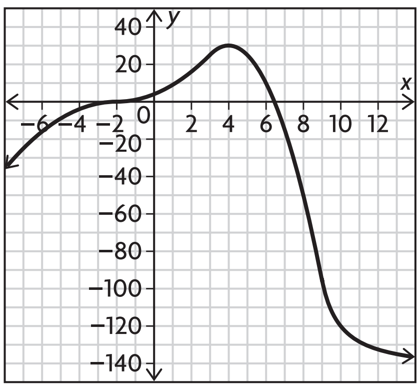


5.


6. $f(x) = \dfrac{2x + 10}{x^2 - 9}$

$= \dfrac{2x + 10}{(x - 3)(x + 3)}$

There are discontinuities at $x = -3$ and at $x = 3$.

$\left.\begin{array}{l}\lim\limits_{x\to 3^-} f(x) = \infty \\ \lim\limits_{x\to 3^+} f(x) = -\infty\end{array}\right\} x = -3$ is a vertical asymptote.

$\left.\begin{array}{l}\lim\limits_{x\to 3^-} f(x) = -\infty \\ \lim\limits_{x\to 3^+} f(x) = \infty\end{array}\right\} x = 3$ is a vertical asymptote.

The y-intercept is $-\frac{10}{9}$ and $x = -5$ is an x-intercept.

$f'(x) = \dfrac{2(x^2 - 9) - (2x + 10)(2)}{(x^2 - 9)^2}$

$= \dfrac{-2x^2 - 20x - 18}{(x^2 - 9)^2}$

For critical values, we solve $f'(x) = 0$:

$x^2 + 10x + 9 = 0$

$(x + 1)(x + 9) = 0$

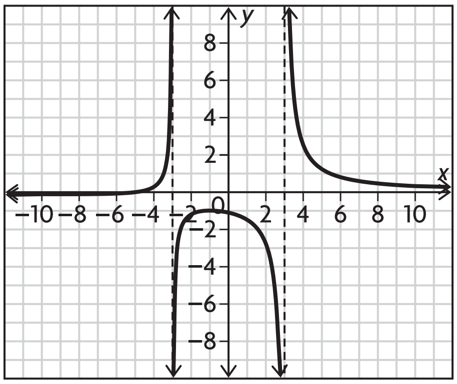
$x = -1$ or $x = -9$.

$\left(-9, -\frac{1}{9}\right)$ is a local minimum and $(-1, -1)$ is a local maximum.

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{\frac{2}{x} + \frac{10}{x^2}}{1 - \frac{9}{x^2}} = 0$ and

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \left(\dfrac{\frac{2}{x} + \frac{10}{x^2}}{1 - \frac{9}{x^2}}\right) = 0$

$y = 0$ is a horizontal asymptote.



7. $f(x) = x^3 + bx^2 + c$

$f'(x) = 3x^2 + 2bx$

Since $f'(-2) = 0$, $12 - 4b = 0$

$b = 3$.

Also, $f(-2) = 6$.

Thus, $-8 + 12 + c = 6$

$c = 2$.

$f'(x) = 3x^2 + 6x$

$= 3x(x + 2)$

The critical points are $(-2, 6)$ and $(0, 2)$.

$f''(x) = 6x + 6$

Since $f''(-2) = -6 < 0$, $(-2, 6)$ is a local maximum.

Since $f'(0) = 6 > 0$, $(0, 2)$ is a local minimum.