# CHAPTER 5: Derivatives of Exponential and Trigonometric Functions

## Review of Prerequisite Skills, pp. 224–225

**1. a.** $3^{-2} = \frac{1}{3^2}$

$= \frac{1}{9}$

**b.** $32^{\frac{2}{5}} = (\sqrt[5]{32})^2$

$= 2^2$

$= 4$

**c.** $27^{-\frac{2}{3}} = \frac{1}{(\sqrt[3]{27})^2}$

$= \frac{1}{3^2}$

$= \frac{1}{9}$

**d.** $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2$

$= \frac{9}{4}$

**2. a.** $\log_5 625 = 4$

**b.** $\log_4 \frac{1}{16} = -2$

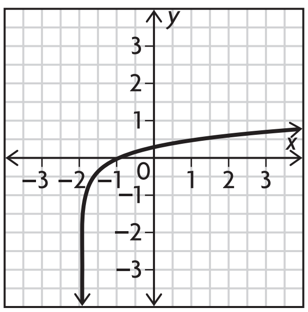
**c.** $\log_x 3 = 3$

**d.** $\log_{10} 450 = w$

**e.** $\log_3 z = 8$

**f.** $\log_a T = b$

**3. a.**



The $x$-intercept occurs where $y = 0$.

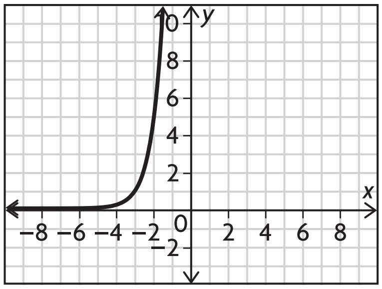
$0 = \log_{10}(x + 2)$

$10^0 = x + 2$

$x = -1$

The $x$-intercept is $(-1, 0)$.

**b.**



An exponential function is always positive, so there is no $x$-intercept.

**4. a.** $\sin\theta = \frac{y}{r}$

**b.** $\cos\theta = \frac{x}{r}$

**c.** $\tan\theta = \frac{y}{x}$

**5.** To convert to radian measure from degree measure, multiply the degree measure by $\frac{\pi}{180°}$.

**a.** $360° \times \frac{\pi}{180°} = 2\pi$

**b.** $45° \times \frac{\pi}{180°} = \frac{\pi}{4}$

**c.** $-90° \times \frac{\pi}{180°} = -\frac{\pi}{2}$

**d.** $30° \times \frac{\pi}{180°} = \frac{\pi}{6}$

**e.** $270° \times \frac{\pi}{180°} = \frac{3\pi}{2}$

**f.** $-120° \times \frac{\pi}{180°} = -\frac{2\pi}{3}$

**g.** $225° \times \frac{\pi}{180°} = \frac{5\pi}{4}$

**h.** $330° \times \dfrac{\pi}{180°} = \dfrac{11\pi}{6}$

**6.** For the unit circle, sine is associated with the $y$-coordinate of the point where the terminal arm of the angle meets the circle, and cosine is associated with the $x$-coordinate.

**a.** $\sin\theta = b$

**b.** $\tan\theta = \dfrac{b}{a}$

**c.** $\cos\theta = a$

**d.** $\sin\left(\dfrac{\pi}{2} - \theta\right) = a$

**e.** $\cos\left(\dfrac{\pi}{2} - \theta\right) = b$

**f.** $\sin(-\theta) = -b$

**7. a.** The angle is in the second quadrant, so cosine and tangent will be negative.

$\cos\theta = -\dfrac{12}{13}$

$\tan\theta = -\dfrac{5}{12}$

**b.** The angle is in the third quadrant, so sine will be negative and tangent will be positive.

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta + \dfrac{4}{9} = 1$

$\sin^2\theta = \dfrac{5}{9}$

$\sin\theta = -\dfrac{\sqrt{5}}{3}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$= \dfrac{\sqrt{5}}{2}$

**c.** The angle is in the fourth quadrant, so cosine will be positive and sine will be negative.
Because $\tan\theta = -2$, the point $(1, -2)$ is on the terminal arm of the angle. The reference triangle for this angle has a hypotenuse of $\sqrt{2^2 + 1^2}$ or $\sqrt{5}$.

$\sin\theta = -\dfrac{2}{\sqrt{5}}$

$\cos\theta = \dfrac{1}{\sqrt{5}}$

**d.** The sine is only equal to 1 for one angle between 0 and $\pi$, so $\theta = \frac{\pi}{2}$.

$\cos\dfrac{\pi}{2} = 0$

$\tan\dfrac{\pi}{2}$ is undefined

**8. a.** The period is $\frac{2\pi}{2}$ or $\pi$. The amplitude is 1.

**b.** The period is $\dfrac{2\pi}{\frac{1}{2}}$ or $4\pi$. The amplitude is 2.

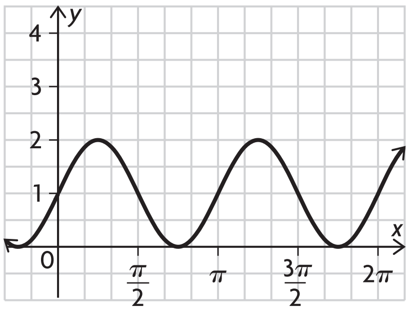
**c.** The period is $\frac{2\pi}{\pi}$ or 2. The amplitude is 3.

**d.** The period is $\frac{2\pi}{12}$ or $\frac{\pi}{6}$. The amplitude is $\frac{2}{7}$.

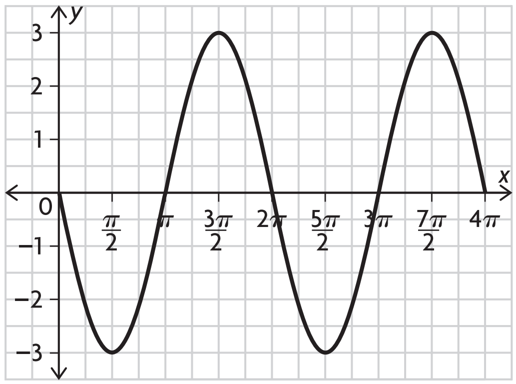
**e.** The period is $2\pi$. The amplitude is 5.

**f.** The period is $2\pi$. Because of the absolute value being taken, the amplitude is $\frac{3}{2}$.

**9. a.** The period is $\frac{2\pi}{2}$ or $\pi$. Graph the function from $x = 0$ to $x = 2\pi$.




**b.** The period is $2\pi$, so graph the function from $x = 0$ to $x = 4\pi$.




**10. a.** $\tan x + \cot x = \sec x \csc x$

$LS = \tan x + \cot x$

$= \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}$

$= \dfrac{\sin^2 x + \cos^2 x}{\cos x + \sin x}$

$$= \frac{1}{\cos x + \sin x}$$

$$\text{RS} = \sec x + \csc x$$

$$= \frac{1}{\cos x} \cdot \frac{1}{\sin x}$$

$$= \frac{1}{\cos x \sin x}$$

Therefore, $\tan x + \cot x = \sec x \csc x$.

**b.** $\frac{\sin x}{1 - \sin^2 x} = \tan x + \sec x$

$$\text{LS} = \frac{\sin x}{1 - \sin^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$\text{RS} = \tan x \sec x$$

$$= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

$$= \frac{\sin x}{\cos^2 x}$$

Therefore, $\frac{\sin x}{1 - \sin^2 x} = \tan x \sec x$.

**11. a.** $3 \sin x = \sin x + 1$

$$2 \sin x = 1$$

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

**b.** $\cos x - 1 = -\cos x$

$$2 \cos x = 1$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

## 5.1 Derivatives of Exponential Functions, $y = e^x$, pp. 232–234

**1.** You can only use the power rule when the term containing variables is in the base of the exponential expression. In the case of $y = e^x$, the exponent contains a variable.

**2. a.** $y = e^{3x}$

$$\frac{dy}{dx} = 3e^{3x}$$

**b.** $s = e^{3t-5}$

$$\frac{ds}{dt} = 3e^{3t-5}$$

**c.** $y = 2e^{10t}$

$$\frac{dy}{dt} = 20e^{10t}$$

**d.** $y = e^{-3x}$

$$\frac{dy}{dx} = -3e^{-3x}$$

**e.** $y = e^{5-6x+x^2}$

$$\frac{dy}{dx} = (-6 + 2x)e^{5-6x+x^2}$$

**f.** $y = e^{\sqrt{x}}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}e^{\sqrt{x}}$$

**3. a.** $y = 2e^{x^3}$

$$\frac{dy}{dx} = 2(3x^2)e^{x^3}$$

$$= 6x^2e^{x^3}$$

**b.** $\frac{dy}{dx} = \frac{d(xe^{3x})}{dx}$

$$= (x)(3e^{3x}) + (e^{3x})(1)$$

$$= 3xe^{3x} + e^{3x}$$

$$= e^{3x}(3x + 1)$$

**c.** $f(x) = \frac{e^{-x^3}}{x}$

$$f'(x) = \frac{-3x^2e^{-x^3}(x) - e^{-x^3}}{x^2}$$

**d.** $f(x) = \sqrt{x}e^x$

$$f'(x) = \sqrt{x}e^x + e^x\left(\frac{1}{2\sqrt{x}}\right)$$

**e.** $h(t) = e^{t^2} + 3e^{-t}$

$$h'(t) = 2te^{t^2} - 3e^{-t}$$

**f.** $g(t) = \frac{e^{2t}}{1 + e^{2t}}$

$$g'(t) = \frac{2e^{2t}(1 + e^{2t}) - 2e^{2t}(e^{2t})}{(1 + e^{2t})^2}$$

$$= \frac{2e^{2t}}{(1 + e^{2t})^2}$$

**4. a.** $f'(x) = \frac{1}{3}(3e^{3x} - 3e^{-3x})$

$$= e^{3x} - e^{-3x}$$

$$f'(1) = e^3 - e^{-3}$$

**b.** $f(x) = e^{-\frac{1}{x+1}}$

$$f'(x) = e^{-\frac{1}{x+1}}\left(\frac{1}{(x + 1)^2}\right)$$

$$f'(0) = e^{-1}(1)$$

$$= \frac{1}{e}$$

**c.** $h'(z) = 2z(1 + e^{-z}) + z^2(-e^{-z})$

$$h'(-1) = 2(-1)(1 + e) + (-1)^2(-e^1)$$
$$= -2 - 2e - e$$
$$= -2 - 3e$$

**5. a.** $y = \dfrac{2e^x}{1 + e^x}$

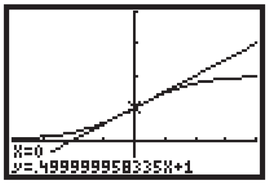
$$\frac{dy}{dx} = \frac{(1 + e^x)2e^x - 2e^x(e^x)}{(1 + e^x)^2}$$
$$\frac{dy}{dx} = \frac{2(2) - 2(1)(1)}{2^2}$$
$$= \frac{1}{2}$$

When $x = 0$,

the slope of the tangent is $\frac{1}{2}$.

The equation of the tangent is $y = \frac{1}{2}x + 1$, since the $y$-intercept was given as $(0, 1)$.

**b.**



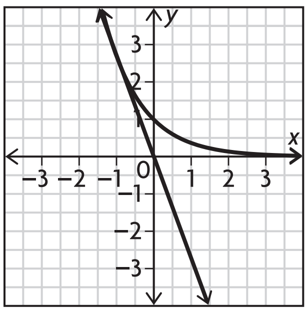

**c.** The answers agree very well; the calculator does not show a slope of exactly 0.5, due to internal rounding.

**6.** $y = e^{-x}$

$$\frac{dy}{dx} = -e^{-x}$$

When $x = -1$, $\frac{dy}{dx} = -e$. And when $x = -1$, $y = e$.

The equation of the tangent is $y - e = -e(x + 1)$ or $ex + y = 0$.




**7.** The slope of the tangent line at any point is given by

$$\frac{dy}{dx} = (1)(e^{-x}) + x(-e^{-x})$$
$$= e^{-x}(1 - x).$$

At the point $(1, e^{-1})$, the slope is $e^{-1}(0) = 0$. The equation of the tangent line at the point $A$ is

$y - e^{-1} = 0(x - 1)$ or $y = \frac{1}{e}$.

**8.** The slope of the tangent line at any point on the curve is $\dfrac{dy}{dx} = 2xe^{-x} + x^2(e^{-x})$

$$= (2x - x^2)(e^{-x})$$
$$= \frac{2x - x^2}{e^x}.$$

Horizontal lines have slope equal to 0.

We solve $\dfrac{dy}{dx} = 0$

$$\frac{x(2 - x)}{e^x} = 0.$$

Since $e^x > 0$ for all $x$, the solutions are $x = 0$ and $x = 2$. The points on the curve at which the tangents are horizontal are $(0, 0)$ and $\left(2, \frac{4}{e^2}\right)$.

**9.** If $y = \dfrac{5}{2}(e^{\frac{x}{5}} + e^{-\frac{x}{5}})$, then

$$y' = \frac{5}{2}\left(\frac{1}{5}e^{\frac{x}{5}} - \frac{1}{5}e^{-\frac{x}{5}}\right), \text{ and}$$
$$y'' = \frac{5}{2}\left(\frac{1}{25}e^{\frac{x}{5}} + \frac{1}{25}e^{-\frac{x}{5}}\right)$$
$$= \frac{1}{25}\left[\frac{5}{2}(e^{\frac{x}{5}} + e^{-\frac{x}{5}})\right]$$
$$= \frac{1}{25}y.$$

**10. a.** $y = e^{-3x}$

$$\frac{dy}{dx} = -3e^{-3x}$$
$$\frac{d^2y}{dx^2} = 9e^{-3x}$$
$$\frac{d^3y}{dx^3} = -27e^{-3x}$$

**b.** $\dfrac{d^ny}{dx^n} = (-1)^n(3^n)e^{-3x}$

**11. a.** $\dfrac{dy}{dx} = \dfrac{d(-3e^x)}{dx}$

$$= -3e^x$$
$$\frac{d^2y}{dx^2} = -3e^x$$

**b.** $\dfrac{dy}{dx} = \dfrac{d(xe^{2x})}{dx}$

$$= (x)(2e^{2x}) + (e^{2x})(1)$$
$$= 2xe^{2x} + e^{2x}$$
$$= e^{2x}(2x + 1)$$

$$\frac{d^2y}{dx^2} = e^{2x}(2) + (2x + 1)(2e^{2x})$$
$$= 4xe^{2x} + 4e^{2x}$$

**c.** $\frac{dy}{dx} = \frac{d(e^x(4 - x))}{dx}$
$$= (e^x)(-1) + (4 - x)(e^x)$$
$$= -e^x + 4e^x - xe^x$$
$$= 3e^x - xe^x$$
$$= e^x(3 - x)$$
$$\frac{d^2y}{dx^2} = e^x(-1) + (3 - x)(e^x)$$
$$= 2e^x - xe^x$$
$$= e^x(2 - x)$$

**12. a.** When $t = 0$, $N = 1000[30 + e^0] = 31\,000$.

**b.** $\frac{dN}{dt} = 1000\left[0 - \frac{1}{30}e^{\frac{-t}{30}}\right] = -\frac{100}{3}e^{-\frac{t}{30}}$

**c.** When $t = 20h$, $\frac{dN}{dt} = -\frac{100}{3}e^{-\frac{2}{3}} \doteq -17$ bacteria/h.

**d.** Since $e^{-\frac{t}{30}} > 0$ for all $t$, there is no solution to $\frac{dN}{dt} = 0$.
Hence, the maximum number of bacteria in the culture occurs at an endpoint of the interval of domain.
When $t = 50$, $N = 1000[30 + e^{-\frac{5}{3}}] \doteq 30\,189$.
The largest number of bacteria in the culture is 31 000 at time $t = 0$.

**e.** The number of bacteria is constantly decreasing as time passes.

**13. a.** $v = \frac{ds}{dt} = 160\left(\frac{1}{4} - \frac{1}{4}e^{-\frac{t}{4}}\right)$
$$= 40(1 - e^{-\frac{t}{4}})$$

**b.** $a = \frac{dv}{dt} = 40\left(\frac{1}{4}e^{-\frac{t}{4}}\right) = 10e^{-\frac{t}{4}}$

From **a.**, $v = 40(1 - e^{-\frac{t}{4}})$, which gives $e^{-\frac{t}{4}} = 1 - \frac{v}{40}$.

Thus, $a = 10\left(1 - \frac{v}{40}\right) = 10 - \frac{1}{4}v$.

**c.** $v_T = \lim_{t\to\infty} v$
$$v_T = \lim_{t\to\infty} 40(1 - e^{-\frac{t}{4}})$$
$$= 40 \lim_{t\to\infty}\left(1 - \frac{1}{e^{\frac{t}{4}}}\right)$$
$$= 40(1), \text{ since } \lim_{t\to\infty}\frac{1}{e^{\frac{t}{4}}} = 0$$

The terminal velocity of the skydiver is 40 m/s.

**d.** 95% of the terminal velocity is
$\frac{95}{100}(40) = 38$ m/s.
To determine when this velocity occurs, we solve
$40(1 - e^{-\frac{t}{4}}) = 38$
$$1 - e^{-\frac{t}{4}} = \frac{38}{40}$$
$$e^{-\frac{t}{4}} = \frac{1}{20}$$
$$e^{\frac{t}{4}} = 20$$
$$\text{and } \frac{t}{4} = \ln 20,$$
which gives $t = 4 \ln 20 \doteq 12$ s.
The skydiver's velocity is 38 m/s, 12 s after jumping.
The distance she has fallen at this time is
$S = 160(\ln 20 - 1 + e^{-20})$
$$= 160\left(\ln 20 - 1 + \frac{1}{20}\right)$$
$$\doteq 327.3 \text{ m.}$$

**14. a. i.** Let $f(x) = \left(1 + \frac{1}{x}\right)^x$. Then,

| $x$ | $f(x)$ |
|---|---|
| 1 | 2 |
| 10 | 2.5937 |
| 100 | 2.7048 |
| 1000 | 2.7169 |
| 10 000 | 2.7181 |

So, from the table one can see that $\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e$.

**ii.** Let $f(x) = (1 + x)^{\frac{1}{x}}$.

| $x$ | $f(x)$ |
|---|---|
| −0.1 | 2.8680 |
| −0.01 | 2.7320 |
| −0.001 | 2.7196 |
| −0.0001 | 2.7184 |
| ? | ? |
| 0.0001 | 2.7181 |
| 0.001 | 2.7169 |
| 0.01 | 2.7048 |
| 0.1 | 2.5937 |

So, from the table one can see that $\lim_{x\to 0}(1 + x)^{\frac{1}{x}} = e$.
That is, the limit approaches the value of
$e = 2.718\,281\,828\ldots$

**b.** The limits have the same value because as $x \to \infty$, $\frac{1}{x} \to 0$.

**15. a.** The given limit can be rewritten as

$$\lim_{h\to 0}\frac{e^h - 1}{h} = \lim_{h\to 0}\frac{e^{0+h} - e^0}{h}$$

This expression is the limit definition of the derivative at $x = 0$ for $f(x) = e^x$.

$$\left[f'(0) = \lim_{h\to 0}\frac{e^{0+h} - e^0}{h}\right]$$

Since $f'(x) = \frac{de^x}{dx} = e^x$, the value of the given limit is $e^0 = 1$.

**b.** Again, $\lim_{h\to 0}\frac{e^{2+h} - e^2}{h}$ is the derivative of $e^x$ at $x = 2$.

Thus, $\lim_{h\to 0}\frac{e^{2+h} - e^2}{h} = e^2$.

**16.** For $y = Ae^{ex}$, $\frac{dy}{dt} = Ame^{ex}$ and $\frac{d^2y}{dt^2} = Am^2e^{ex}$.

Substituting in the differential equation gives

$$Am^2e^{ex} + Ame^{ex} - 6Ae^{ex} = 0$$
$$Ae^{ex}(m^2 + m - 6) = 0.$$

Since $Ae^{ex} \neq 0$, $m^2 + m - 6 = 0$

$$(m + 3)(m - 2) = 0$$
$$m = -3 \text{ or } m = 2.$$

**17. a.** $\frac{d}{dx}\sinh x = \frac{d}{dx}\left[\frac{1}{2}(e^x - e^{-x})\right]$

$$= \frac{1}{2}(e^x + e^{-x})$$
$$= \cosh x$$

**b.** $\frac{d}{dx}\cosh x = \frac{1}{2}(e^x - e^{-x})$

$$= \sinh x$$

**c.** Since $\tanh x = \frac{\sinh x}{\cosh x}$,

$\frac{d}{dx}\tanh x$

$$= \frac{\left(\frac{d}{dx}\sinh x\right)(\cosh x) - (\sinh x)\left(\frac{d}{dx}\cosh x\right)}{(\cosh x)^2}$$

$$= \frac{\frac{1}{2}(e^x + e^{-x})\left(\frac{1}{2}\right)(\cosh x)^2\,(e^x + e^{-x})}{(\cosh x)^2}$$

$$- \frac{\frac{1}{2}(e^x - e^{-x})\left(\frac{1}{2}\right)(e^x - e^{-x})}{(\cosh x)^2}$$

$$= \frac{\frac{1}{4}\left[(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})\right]}{(\cosh x)^2}$$

$$= \frac{\frac{1}{4}(4)}{(\cosh x)^2}$$

$$= \frac{1}{(\cosh x)^2}$$

**18. a.** Four terms:

$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} = 2.666\,66\overline{6}$

Five terms:

$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 2.708\,33\overline{3}$

Six terms:

$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = 2.716\,66\overline{6}$

Seven terms:

$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} = 2.718\,05\overline{5}$

**b.** The expression for $e$ in part **a.** is a special case of $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots.$ in that it is the case when $x = 1$. Then $e^x = e^1 = e$ is in fact $e^1 = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots.$ The value of $x$ is 1.

## 5.2 Derivatives of the General Exponential Function, $y = b^x$, p. 240

**1. a.** $\frac{dy}{dx} = \frac{d(2^{3x})}{dx}$

$$= 3(2^{3x})\ln 2$$

**b.** $\frac{dy}{dx} = \frac{d(3.1^x + x^3)}{dx}$

$$= \ln 3.1(3.1)^x + 3x^2$$

**c.** $\frac{ds}{dt} = \frac{d(10^{3t-5})}{dt}$

$$= 3(10^{3t-5})\ln 10$$

**d.** $\frac{dw}{dn} = \frac{d(10^{5-6n+n^2})}{dn}$

$$= (-6 + 2n)(10^{5-6n+n^2})\ln 10$$

**e.** $\frac{dy}{dx} = \frac{d(3^{x^2+2})}{dx}$

$$= 2x(3^{x^2+2})\ln 3$$

**f.** $\frac{dy}{dx} = \frac{d(400(2)^{x+3})}{dx}$

$$= 400(2)^{x+3}\ln 2$$

**2. a.** $\frac{dy}{dx} = \frac{d(x^5 \times (5)^x)}{dx}$

$= (x^5)((5)^x(\ln 5)) + ((5)^x)(5x^4)$

$= 5^x[(x^5 \times \ln 5) + 5x^4]$

**b.** $\frac{dy}{dx} = \frac{d(x(3)^{x^2})}{dx}$

$= (x)(2x(3)^{x^2} \ln 3) + (3)^{x^2}(1)$

$= (3)^{x^2}[(2x^2 \ln 3) + 1]$

**c.** $v = (2^t)(t^{-1})$

$\frac{dv}{dt} = \frac{d((2^t)(t^{-1}))}{dx}$

$= (2^t)(-1t^{-2}) + (t^{-1})(2^t \ln 2)$

$= -\frac{2^t}{t^2} + \frac{2^t \ln 2}{t}$

**d.** $f(x) = \frac{3^{\frac{x}{2}}}{x^2}$

$f'(x) = \frac{\frac{1}{2}\ln 3(3^{\frac{x}{2}})(x^2) - 2x(3^{\frac{x}{2}})}{x^4}$

$= \frac{x \ln 3(3^{\frac{x}{2}}) - 4(3^{\frac{x}{2}})}{x^4}$

$= \frac{3^{\frac{x}{2}}[x \ln 3 - 4]}{x^3}$

**3.** $f(t) = 10^{3t-5} \cdot e^{2t^2}$

$f'(t) = (10^{3t-5})(4te^{2t^2}) + (e^{2t^2})(3(10)^{3t-5} \ln 10)$

$= 10^{3t-5}e^{2t^2}(4t + 3 \ln 10)$

Now, set $f'(t) = 0$.

So, $f'(t) = 0 = 10^{3t-5}e^{2t^2}(4t + 3\ln 10)$

So $10^{3t-5}e^{2t^2} = 0$ and $4t + 3 \ln 10 = 0$.

The first equation never equals zero because solving would force one to take the natural log of both sides, but ln 0 is undefined. So the first equation does not produce any values for which $f'(t) = 0$. The second equation does give one value.

$4t + 3 \ln 10 = 0$

$4t = -3 \ln 10$

$t = -\frac{3 \ln 10}{4}$

**4.** When $x = 3$, the function $y = f(x)$ evaluated at 3 is $f(3) = 3(2^3) = 3(8) = 24$. Also,

$\frac{dy}{dx} = \frac{d(3(2)^x)}{dx}$

$= 3(2^x)\ln 2$

So, at $x = 3$,

$\frac{dy}{dx} = 3(2^3)(\ln 2) = 24(\ln 2) \doteq 16.64$

Therefore, $y - 24 = 16.64(x - 3)$

$y - 24 = 16.64x - 49.92$

$-16.64x + y + 25.92 = 0$

**5.** $\frac{dy}{dx} = \frac{d(10^x)}{dx}$

$= 10^x \ln 10$

So, at $x = 1$,

$\frac{dy}{dx} = 10^1 \ln 10 = 10(\ln 10) \doteq 23.03$

Therefore, $y - 10 = 23.03(x - 1)$

$y - 10 = 23.03x - 23.03$

$-23.03x + y + 13.03 = 0$

**6. a.** The half-life of the substance is the time required for half of the substance to decay. That is, it is when 50% of the substance is left, so $P(t) = 50$.

$50 = 100(1.2)^{-t}$

$\frac{1}{2} = (1.2)^{-t}$

$\frac{1}{2} = \frac{1}{(1.2)^t}$

$(1.2)^t = 2$

$t(\ln 1.2) = \ln 2$

$t = \frac{\ln 1.2}{\ln 2}$

$t \doteq 3.80$ years

Therefore, the half-life of the substance is about 3.80 years.

**b.** The problem asks for the rate of change when $t \doteq 3.80$ years.

$P'(t) = -100(1.2)^{-t}(\ln 1.2)$

$P'(3.80) = -100(1.2)^{-(3.80)}(\ln 1.2)$

$\doteq -9.12$

So, the substance is decaying at a rate of about $-9.12$ percent/year at the time 3.80 years where the half-life is reached.

**7.** $P = 0.5(10^9)e^{0.200\,15t}$

**a.** $\frac{dP}{dt} = 0.5(10^9)(0.200\,15)e^{0.200\,15t}$

In 1968, $t = 1$ and $\frac{dP}{dt} = 0.5(10^9)(0.200\,15)e^{0.200\,15} \doteq 0.122\,25 \times 10^9$ dollars/annum

In 1978, $t = 11$ and

$\frac{dP}{dt} = 0.5(10^9)(0.200\,15)e^{11 \times 0.200\,15}$

$\doteq 0.904\,67 \times 10^9$ dollars/annum.

In 1978, the rate of increase of debt payments was \$904 670 000/annum compared to \$122 250 000/annum in 1968. As a ratio, $\frac{\text{Rate in 1978}}{\text{Rate in 1968}} = \frac{7.4}{1}$. The rate of increase for 1978 is 7.4 times larger than that for 1968.

**b.** In 1988, $t = 21$ and

$$\frac{dP}{dt} = 0.5(10^9)(0.200\,15)e^{21 \times 0.200\,15}$$
$$\doteq 6.694\,69 \times 10^9 \text{ dollars/annum}$$

In 1998, $t = 31$ and

$$\frac{dP}{dt} = 0.5(10^9)(0.200\,15)e^{31 \times 0.200\,15}$$
$$\doteq 49.541\,69 \times 10^9 \text{ dollars/annum}$$

As a ratio, $\frac{\text{Rate in 1998}}{\text{Rate in 1988}} = \frac{7.4}{1}$. The rate of increase for 1998 is 7.4 times larger than that for 1988.

**c.** Answers may vary. For example, data from the past are not necessarily good indicators of what will happen in the future. Interest rates change, borrowing may decrease, principal may be paid off early.

**8.** When $x = 0$, the function $y = f(x)$ evaluated at 0 is $f(0) = 2^{-0^2} = 2^0 = 1$. Also,

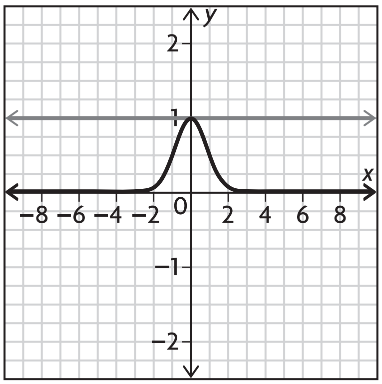
$$\frac{dy}{dx} = \frac{d(2^{-x^2})}{dx}$$
$$= -2x(2^{-x^2})\ln 2$$

So, at $x = 0$,

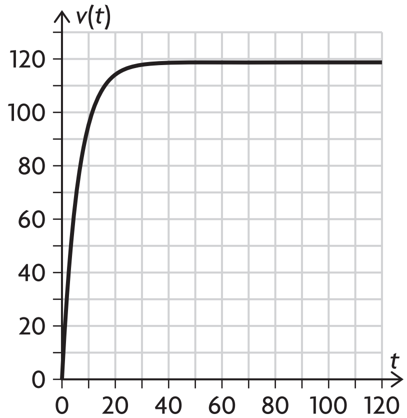
$$\frac{dy}{dx} = -2(0)(2^{-0^2})\ln 2 = 0$$

Therefore, $y - 1 = 0(x - 0)$

So, $y - 1 = 0$ or $y = 1$.



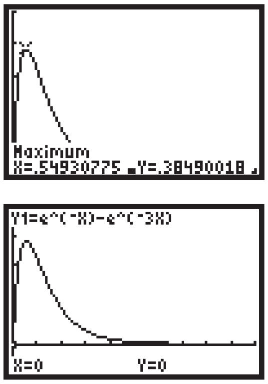

**9.**




From the graph, one can notice that the values of $v(t)$ quickly rise in the range of about $0 \le t \le 15$. The slope for these values is positive and steep. Then as the graph nears $t = 20$ the steepness of the slope decreases and seems to get very close to 0. One can reason that the car quickly accelerates for the first 20 units of time. Then, it seems to maintain a constant acceleration for the rest of the time. To verify this, one could differentiate and look at values where $v'(t)$ is increasing.

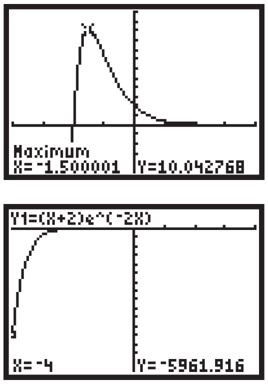
## 5.3 Optimization Problems Involving Exponential Functions, pp. 245–247

**1. a.**




The maximum value is about 0.3849. The minimum value is 0.

**b.**




The maximum value is about 10.043. The minimum value is about −5961.916.

**2. a.** $f(x) = e^{-x} - e^{-3x}$ on $0 \le x \le 10$

$f'(x) = -e^{-x} + 3e^{-3x}$

Let $f'(x) = 0$, therefore $e^{-x} + 3e^{-3x} = 0$.

Let $e^{-x} = w$, when $-w + 3w^3 = 0$.

$w(-1 + 3w^2) = 0$.

Therefore, $w = 0$ or $w^2 = \frac{1}{3}$

$$w = \pm\frac{1}{\sqrt{3}}.$$

But $w \ge 0$, $w = +\frac{1}{\sqrt{3}}$.

When $w = \frac{1}{\sqrt{3}}$, $e^{-x} = \frac{1}{\sqrt{3}}$,

$$-x \ln e = \ln 1 - \ln \sqrt{3}$$
$$x = \frac{\ln \sqrt{3} - \ln 1}{1}$$
$$= \ln \sqrt{3}$$
$$\doteq 0.55.$$

$$f(0) = e^0 - e^0 = 0$$
$$f(0.55) \doteq 0.3849$$
$$f(10) = e^{-10} - e^{-30} \doteq 0.00005$$

Absolute maximum is about 0.3849 and absolute minimum is 0.

$m(x) = (x + 2)e^{-2x}$ on $-4 \le x \le 4$

$m'(x) = e^{-2x} + (-2)(x + 2)e^{-2x}$

Let $m'(x) = 0$.

$e^{-2x} \neq 0$, therefore, $1 + (-2)(x + 2) = 0$

$$x = \frac{-3}{2} = -1.5.$$

$$m(-4) = -2e^8 \doteq -5961$$
$$m(-1.5) = 0.5e^3 \doteq 10$$
$$m(4) = 6e^{-8} \doteq 0.0002$$

The maximum value is about 10 and the minimum value is about $-5961$.

**b.** The graphing approach seems to be easier to use for the functions. It is quicker and it gives the graphs of the functions in a good viewing rectangle. The only problem may come in the second function, $m(x)$, because for $x < 1.5$ the function quickly approaches values in the negative thousands.

**3. a.** $P(t) = \frac{20}{1 + 3e^{-0.02t}}$

$$P(0) = \frac{20}{1 + 3e^{-0.02(0)}} = \frac{20}{1 + 3e^0} = \frac{20}{4} = 5$$

So, the population at the start of the study when $t = 0$ is 500 squirrels.

**b.** The question asks for $\lim_{t \to \infty} P(t)$.

As $t$ approaches $\infty$, $e^{-0.02t} = \frac{1}{e^{0.02t}}$ approaches 0.

So, $\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \frac{20}{1 + 3e^{-0.02t}}$

$$= \frac{20}{1 + 3(0)} = 20.$$

Therefore, the largest population of squirrels that the forest can sustain is 2000 squirrels.

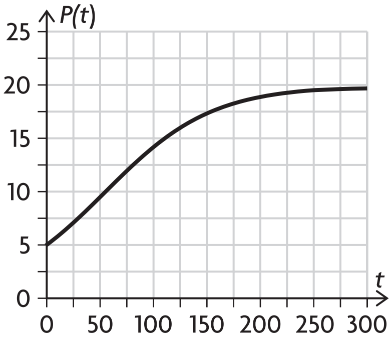
**c.** A point of inflection can only occur when $P''(t) = 0$ and concavity changes around the point.

$$P(t) = \frac{20}{1 + 3e^{-0.02t}}$$
$$P(t) = 20(1 + 3e^{-0.02t})^{-1}$$
$$P'(t) = 20(-(1 + 3e^{-0.02t})^{-2}(-0.06e^{-0.02t}))$$
$$= (1.2e^{-0.02t})(1 + 3e^{-0.02t})^{-2}$$
$$P''(t) = [(1.2e^{-0.02t})(-2(1 + 3e^{-0.02t})^{-3}(-0.06e^{-0.02t})] + (1 + 3e^{-0.02t})^{-2}(-0.024e^{-0.02t})$$
$$= \frac{0.144e^{-0.04t}}{(1 + 3e^{-0.02t})^3} - \frac{0.024e^{-0.02t}}{(1 + 3e^{-0.02t})^2}$$

$P''(0)$ when $\frac{0.144e^{-0.04t}}{(1 + 3e^{-0.02t})^3} - \frac{0.024e^{-0.02t}}{(1 + 3e^{-0.02t})^2} = 0$

Solving for $t$ after setting the second derivative equal to 0 is very tedious. Use a graphing calculator to determine the value of $t$ for which the second derivative is 0, 54.9. Evaluate $P(54.9)$. The point of inflection is (54.9, 10).

**d.**




**e.** $P$ grows exponentially until the point of inflection, then the growth rate decreases and the curve becomes concave down.

**4. a.** $P(x) = 10^6[1 + (x - 1)e^{-0.001x}]$, $0 \le x \le 2000$

Using the Algorithm for Extreme Values, we have

$$P(0) = 10^6[1 - 1] = 0$$
$$P(2000) = 10^6[1 + 1999e^{-2}] \doteq 271.5 \times 10^6.$$

Now,

$$P'(x) = 10^6[(1)e^{-0.001x} + (x - 1)(-0.001)e^{-0.001x}]$$
$$= 10^6e^{-0.001x}(1 - 0.001x + 0.001)$$

Since $e^{-0.001x} > -$ for all $x$,
$P'(x) = 0$ when $1.001 - 0.001x = 0$

$$x = \frac{1.001}{0.001} = 1001.$$

$P(1001) = 10^6[1 + 1000e^{-1.001}] \doteq 368.5 \times 10^6$
The maximum monthly profit will be 368.5 × 106 dollars when 1001 items are produced and sold.

**b.** The domain for $P(x)$ becomes $0 \le x \le 500$.
$P(500) = 10^6[1 + 499e^{-0.5}] = 303.7 \times 10^6$
Since there are no critical values in the domain, the maximum occurs at an endpoint. The maximum monthly profit when 500 items are produced and sold is $303.7 \times 10^6$ dollars.

**5.** $R(x) = 40x^2e^{-0.4x} + 30, 0 \le x \le 8$
We use the Algorithm for Extreme Values:

$$R'(x) = 80xe^{-0.4x} + 40x^2(-0.4)e^{-0.4x}$$
$$= 40xe^{-0.4x}(2 - 0.4x)$$

Since $e^{-0.4x} > 0$ for all $x$, $R'(x) = 0$ when $x = 0$ or $2 - 0.4x = 0$

$$x = 5.$$

$R(0) = 30$
$R(5) \doteq 165.3$
$R(8) \doteq 134.4$
The maximum monthly revenue of 165.3 thousand dollars is achieved when 500 units are produced and sold.

**6.** $P(t) = 100(e^{-t} - e^{-4t}), 0 \le t \le 3$

$$P'(t) = 100(-e^{-t} + 4e^{-4t})$$
$$= 100e^{-t}(-1 + 4e^{-3t})$$

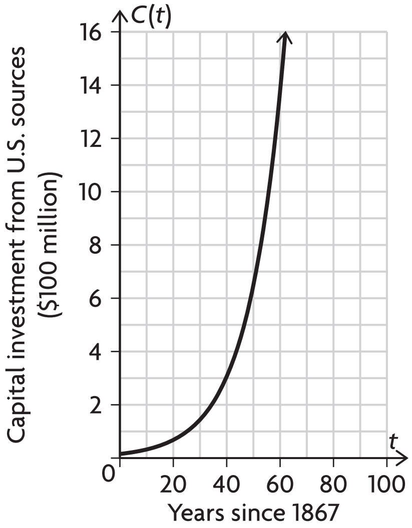
Since $e^{-t} > 0$ for all $t$, $P'(t) = 0$ when

$$4e^{-3t} = 1$$
$$e^{-3t} = \frac{1}{4}$$
$$-3t = \ln(0.25)$$
$$t = \frac{-\ln(0.25)}{3}$$
$$= 0.462.$$

$P(0) = 0$
$P(0.462) \doteq 47.2$
$P(3) \doteq 4.98$
The highest percentage of people spreading the rumour is 47.2% and occurs at the 0.462 h point.

**7.** $C = 0.015 \times 10^9e^{0.075\,33t}, 0 \le t \le 100$

**a.**




**b.** $\frac{dC}{dt} = 0.015 \times 10^9 \times 0.07533e^{0.075\,33t}$
In 1947, $t = 80$ and the growth rate was

$$\frac{dC}{dt} = 0.468\,05 \times 10^9 \text{ dollars/year.}$$

In 1967, $t = 100$ and the growth rate was

$$\frac{dC}{dt} = 2.1115 \times 10^9 \text{ dollars/year.}$$

The ratio of growth rates of 1967 to that of 1947 is

$$\frac{2.1115 \times 10^9}{0.468\,05 \times 10^9} = \frac{4.511}{1}.$$

The growth rate of capital investment grew from 468 million dollars per year in 1947 to 2.112 billion dollars per year in 1967.

**c.** In 1967, the growth rate of investment as a percentage of the amount invested is

$$\frac{2.1115 \times 10^9}{28.0305 \times 10^9} \times 100 = 7.5\%.$$

**d.** In 1977, $t = 110$

$C = 59.537 \times 10^9$ dollars

$$\frac{dC}{dt} = 4.4849 \times 10^9 \text{ dollars/year.}$$

**e.** Statistics Canada data shows the actual amount of U.S. investment in 1977 was $62.5 \times 10^9$ dollars. The error in the model is 3.5%.

**f.** In 2007, $t = 140$.
The expected investment and growth rates are
$C = 570.490 \times 10^9$ dollars and $\frac{dC}{dt} = 42.975 \times 10^9$ dollars/year.

**8. a.** The growth function is $N = 2^{\frac{t}{5}}$.

The number killed is given by $K = e^{\frac{t}{3}}$.
After 60 minutes, $N = 2^{12}$.
Let $T$ be the number of minutes after 60 minutes.
The population of the colony at any time, $T$ after the first 60 minutes is

$$P = N - k$$
$$= 2^{\frac{60+T}{5}} - e^{\frac{T}{3}}, T \geq 0$$
$$\frac{dP}{dt} = 2^{\frac{60+T}{5}}\left(\frac{1}{5}\right)\ln 2 - \frac{1}{3}e^{\frac{T}{3}}$$
$$= 2^{\frac{12+T}{5}}\left(\frac{\ln 2}{5}\right) - \frac{1}{3}e^{\frac{T}{3}}$$
$$= 2^{12} \cdot 2^{\frac{T}{5}}\left(\frac{\ln 2}{5}\right) - \frac{1}{3}e^{\frac{T}{3}}$$

$\frac{dP}{dt} = 0$ when $2^{12}\frac{\ln 2}{5}2^{\frac{T}{5}} = \frac{1}{3}e^{\frac{T}{3}}$ or
$3\frac{\ln 2}{5} \cdot 2^{12}\, 2^{\frac{T}{5}} = e^{\frac{T}{3}}$.
We take the natural logarithm of both sides:

$$\ln\left(3.2^{12}\frac{\ln 2}{5}\right) + \frac{T}{5}\ln 2 = \frac{T}{3}$$
$$7.4404 = T\left(\frac{1}{3} - \frac{\ln 2}{5}\right)$$
$$T = \frac{7.4404}{0.1947} = 38.2 \text{ min.}$$

At $T = 0$, $P = 2^{12} = 4096$.
At $T = 38.2$, $P = 478\,158$.

For $T > 38.2$, $\frac{dP}{dt}$ is always negative.
The maximum number of bacteria in the colony occurs 38.2 min after the drug was introduced.
At this time the population numbers 478 158.

**b.** $P = 0$ when $2^{\frac{60+T}{5}} = e^{\frac{T}{3}}$

$$\frac{60 + T}{5}\ln 2 = \frac{T}{3}$$
$$12\ln 2 = T\left(\frac{1}{3} - \frac{\ln 2}{5}\right)$$
$$T = 42.72$$

The colony will be obliterated 42.72 minutes after the drug was introduced.

**9.** Let $t$ be the number of minutes assigned to study for the first exam and $30 - t$ minutes assigned to study for the second exam. The measure of study effectiveness for the two exams is given by

$$E(t) = E_1(t) + E_2(30 - t), 0 \leq t \leq 30$$
$$= 0.5\left(10 + te^{-\frac{t}{10}}\right) + 0.6\left(9 + (30 - t)e^{-\frac{30-t}{20}}\right)$$
$$E'(t) = 0.5\left(e^{-\frac{t}{10}} - \frac{1}{10}te^{-\frac{t}{10}}\right)$$
$$+ 0.6\left(-e^{-\frac{30-t}{5}} + \frac{1}{20}(30 - t)e^{-\frac{30-t}{20}}\right)$$
$$= 0.05e^{-\frac{t}{10}}(10 - t) + 0.03e^{-\frac{30-t}{5}}$$
$$(-20 + 30 - t)$$
$$= \left(0.05e^{-\frac{t}{10}} + 0.03e^{-\frac{30-t}{5}}\right)(10 - t)$$

$E'(t) = 0$ when $10 - t = 0$
$t = 10$ (The first factor is always a positive number.)
$E(0) = 5 + 5.4 + 18e^{-\frac{3}{2}} = 14.42$
$E(10) = 16.65$
$E(30) = 11.15$
For maximum study effectiveness, 10 h of study should be assigned to the firs exam and 20 h of study for the second exam.

**10.** Use the algorithm for finding extreme values. First, find the derivative $f'(x)$. Then, find any critical points by setting $f'(x) = 0$ and solving for $x$. Also, find the values of $x$ for which $f'(x)$ is undefined. Together these are the critical values. Now, evaluate $f(x)$ for the critical values and the endpoints 2 and $-2$. The highest value will be the absolute maximum on the interval and the lowest value will be the absolute minimum on the interval.

**11. a.** $f'(x) = (x^2)(e^x) + (e^x)(2x)$
$= e^x(x^2 + 2x)$
The function is increasing when $f'(x) > 0$ and decreasing when $f'(x) < 0$. First, find the critical values of $f'(x)$. Solve $e^x = 0$ and $(x^2 + 2x) = 0$
$e^x$ is never equal to zero.
$x^2 + 2x = 0$
$x(x + 2) = 0$.
So, the critical values are 0 and $-2$.

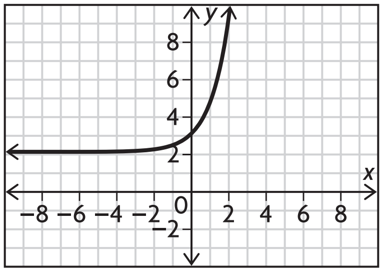
| Interval | $e^x(x^2 + 2x)$ |
|---|---|
| $x < -2$ | + |
| $-2 < x < 0$ | − |
| $0 < x$ | + |

So, $f(x)$ is increasing on the intervals $(-\infty, -2)$ and $(0, \infty)$.
Also, $f(x)$ is decreasing on the interval $(-2, 0)$.

**b.** At $x = 0$, $f'(x)$ switches from decreasing on the left of zero to increasing on the right of zero. So, $x = 0$ is a minimum. Since it is the only critical point that is a minimum, it is the $x$-coordinate of the absolute minimum value of $f(x)$. The absolute minimum value is $f(0) = 0$.

**12. a.** $y' = e^x$

Setting $e^x = 0$ yields no solutions for $x$. $e^x$ is a function that is always increasing. So, there is no maximum or minimum value for $y = e^x + 2$.




**b.** $y' = (x)(e^x) + (e^x)(1)$

$= e^x(x + 1)$

Solve $e^x = 0$ and $(x + 1) = 0$

$e^x$ is never equal to zero.

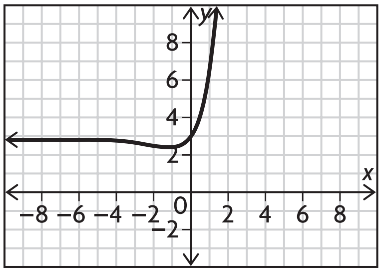
$x + 1 = 0$

$x = -1.$

So there is one critical point: $x = -1$.

| Interval | $e^x(x + 1)$ |
|---|---|
| $x < -1$ | − |
| $x > -1$ | + |

So $y$ is decreasing on the left of $x = -1$ and increasing on the right of $x = -1$. So $x = -1$ is the $x$-coordinate of the minimum of $y$. The minimum value is $-e^{-1} + 3 \doteq 2.63$. There is no maximum value.




**c.** $y' = (2x)(2e^{2x}) + (e^{2x})(2)$

$= 2e^{2x}(2x + 1)$

Solve $2e^{2x} = 0$ and $(2x + 1) = 0$

$2e^{2x}$ is never equal to zero.

$2x + 1 = 0$

$$x = -\frac{1}{2}$$

So there is one critical point: $x = -\frac{1}{2}$.

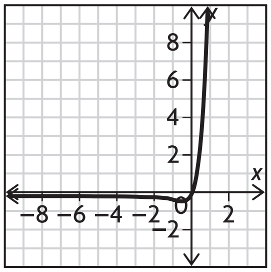
| Interval | $2e^{2x}(2x + 1)$ |
|---|---|
| $x < -\frac{1}{2}$ | − |
| $x > -\frac{1}{2}$ | + |

So $y$ is decreasing on the left of $x = -\frac{1}{2}$ and increasing on the right of $x = -\frac{1}{2}$. So $x = -\frac{1}{2}$ is the $x$-coordinate of the minimum of $y$. The minimum value is

$$2\left(-\frac{1}{2}\right)(e^{2(-\frac{1}{2})})$$

$= -e^{-1}$

$\doteq -0.37$. There is no maximum value.

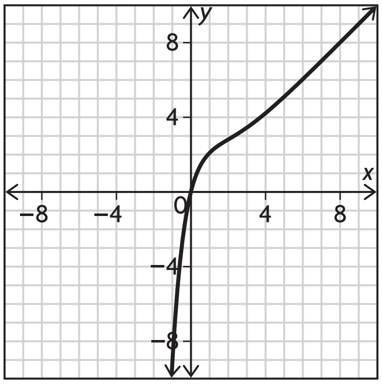



**d.** $y' = (3x)(-e^{-x}) + (e^{-x})(3) + 1$

$= 3e^{-x}(1 - x) + 1$

Solve $3e^{-x}(1 - x) + 1 = 0$.

This gives no real solutions. By looking at the graph of $y = f(x)$, one can see that the function is always increasing. So, there is no maximum or minimum value for $y = 3xe^{-x} + x$.




**13.** $P'(x) = (x)(-xe^{-0.5x^2}) + (e^{-0.5x^2})(1)$

$= e^{-0.5x^2}(-x^2 + 1)$

Solve $e^{-0.5x^2} = 0$ and $(1 - x^2) = 0$.

$e^{-0.5x^2}$ gives no critical points.

$1 - x^2 = 0$

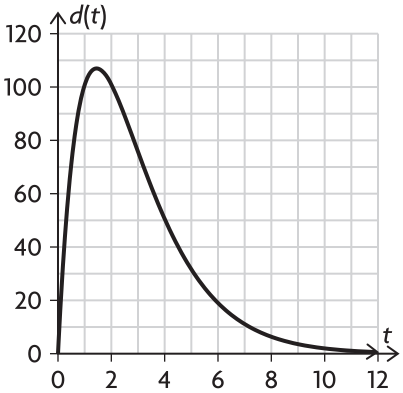
$(1 - x)(1 + x) = 0$

So $x = 1$ and $x = -1$ are the critical points.
So $P(x)$ is decreasing on the left of $x = -1$ and on

| Interval | $e^{-0.5x^2}(-x^2 + 1)$ |
|---|---|
| $x < -1$ | − |
| $-1 < x < 1$ | + |
| $1 < x$ | − |

the right of $x = 1$ and it is increasing between $x = -1$ and $x = 1$. So $x = -1$ is the $x$-coordinate of the minimum of $P(x)$. Also, $x = 1$ is the $x$-coordinate of the maximum of $P(x)$. The minimum value is
$P(-1) = (-1)(e^{-0.5(-1)^2}) = -e^{-0.5} \doteq -0.61.$
The maximum value is
$P(1) = (1)(e^{-0.5(1)^2}) = e^{-0.5} \doteq 0.61.$

**14. a.**




**b.** The speed is increasing when $d'(t) > 0$ and the speed is decreasing when $d'(t) < 0$.

$d'(t) = (200t)(-2^{-t})(\ln 2) + (2^{-t})(200)$
$\quad = 200(2)^{-t}(-t\ln 2 + 1)$

Solve $200(2)^{-t} = 0$ and $-t\ln 2 + 1 = 0$.
$200(2)^{-t}$ gives no critical points.
$-t\ln 2 + 1 = 0$

$$t = \frac{1}{\ln 2} \doteq 1.44$$

So $t = \frac{1}{\ln 2}$ is the critical point.

| Interval | $200(2)^{-t}(-t\ln 2 + 1)$ |
|---|---|
| $t < \frac{1}{\ln 2}$ | + |
| $t > \frac{1}{\ln 2}$ | − |

So the speed of the closing door is increasing when $0 < t < \frac{1}{\ln 2}$ and decreasing when $t > \frac{1}{\ln 2}$.

**c.** There is a maximum at $t = \frac{1}{\ln 2}$ since $d'(t) < 0$ for $t < \frac{1}{\ln 2}$ and $d'(t) > 0$ for $t > \frac{1}{\ln 2}$.

The maximum speed is

$$d\left(\frac{1}{\ln 2}\right) = 200\left(\frac{1}{\ln 2}\right)(2)^{-\frac{1}{\ln 2}} \doteq 106.15 \text{ degrees/s}$$

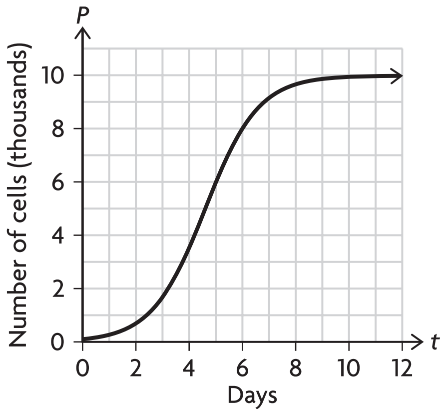
**d.** The door seems to be closed for $t > 10$ s.

**15.** The solution starts in a similar way to that of 9. The effectiveness function is
$E(t) = 0.5\left(10 + te^{-\frac{t}{10}}\right) + 0.6\left(9 + (25 - t)e^{-\frac{25-t}{20}}\right).$
The derivative simplifies to
$E'(t) = 0.05e^{-\frac{t}{10}}(10 - t) + 0.03e^{-\frac{25-t}{20}}(5 - t).$
This expression is very difficult to solve analytically. By calculation on a graphing calculator, we can determine the maximum effectiveness occurs when $t = 8.16$ hours.

**16.** $P = \dfrac{aL}{a + (L - a)e^{-kLt}}$

**a.** We are given $a = 100$, $L = 10\,000$, $k = 0.0001$.

$$P = \frac{10^6}{100 + 9900e^{-t}} = \frac{10^4}{1 + 99e^{-t}}$$
$$= 10^4(1 + 99e^{-t})^{-1}$$




**b.** We need to determine when the derivative of the growth rate $\left(\frac{dP}{dt}\right)$ is zero, i.e., when $\frac{d^2P}{dt^2} = 0$.

$$\frac{dP}{dt} = \frac{-10^4(-99e^{-t})}{(1 + 99e^{-t})^2} = \frac{990000e^{-t}}{(1 + 99e^{-t})^2}$$

$$\frac{d^2P}{dt^2} = \frac{-990000e^{-t}(1 + 99e^{-t})^2 - 990000e^{-t}}{(1 + 99e^{-t})^4} \times \frac{(2)(1 + 99e^{-t})(-99e^{-t})}{(1 + 99e^{-t})^4}$$

$$= \frac{-990000e^{-t}(1 + 99e^{-t}) + 198(990000)e^{-2t}}{(1 + 99e^{-t})^3}$$

$$\frac{d^2P}{dt^2} = 0 \text{ when}$$

$$\begin{aligned} 990000e^{-t}(-1 - 99e^{-t} + 198e^{-t}) &= 0 \\ 99e^{-t} &= 1 \\ e^t &= 99 \\ t &= \ln 99 \\ &\doteq 4.6 \end{aligned}$$

After 4.6 days, the rate of change of the growth rate is zero. At this time the population numbers 5012.

**c.** When $t = 3$, $\frac{dP}{dt} = \frac{990\,000e^{-3}}{(1 + 99e^{-3})^2} \doteq 1402$ cells/day.

When $t = 8$, $\frac{dP}{dt} = \frac{990\,000e^{-8}}{(1 + 99e^{-8})^2} \doteq 311$ cells/day.

The rate of growth is slowing down as the colony is getting closer to its limiting value.

## Mid-Chapter Review, pp. 248–249

**1. a.**
$$\begin{aligned} \frac{dy}{dx} &= \frac{d(5e^{-3x})}{dx} \\ &= (5e^{-3x})(-3x)' \\ &= (5e^{-3x})(-3) \\ &= -15e^{-3x} \end{aligned}$$

**b.**
$$\begin{aligned} \frac{dy}{dx} &= \frac{d\left(7e^{\frac{1}{7}x}\right)}{dx} \\ &= \left(7e^{\frac{1}{7}x}\right)\left(\frac{1}{7}x\right)' \\ &= \left(7e^{\frac{1}{7}x}\right)\left(\frac{1}{7}\right) \\ &= e^{\frac{1}{7}x} \end{aligned}$$

**c.**
$$\begin{aligned} \frac{dy}{dx} &= (x^3)(e^{-2x})' + (x^3)'(e^{-2x}) \\ &= (x^3)((e^{-2x})(-2x)') + (3x^2)(e^{-2x}) \\ &= (x^3)((e^{-2x}))(-2) + 3x^2e^{-2x} \\ &= -2x^3e^{-2x} + 3x^2e^{-2x} \\ &= e^{-2x}(-2x^3 + 3x^2) \end{aligned}$$

**d.**
$$\begin{aligned} \frac{dy}{dx} &= (x-1)^2(e^x)' + ((x-1)^2)'(e^x) \\ &= (x-1)^2(e^x) + (2(x-1))(e^x) \\ &= (x^2 - 2x + 1)(e^x) + (2x - 2)(e^x) \\ &= (e^x)(x^2 - 2x + 1 + 2x - 2) \\ &= (e^x)(x^2 - 1) \end{aligned}$$

**e.**
$$\begin{aligned} \frac{dy}{dx} &= 2(x - e^{-x})(x - e^{-x})' \\ &= 2(x - e^{-x})(1 - (e^{-x})(-x)') \\ &= 2(x - e^{-x})(1 - (e^{-x})(-1)) \\ &= 2(x - e^{-x})(1 + e^{-x}) \\ &= 2(x + xe^{-x} - e^{-x} - e^{-x+-x}) \\ &= 2(x + xe^{-x} - e^{-x} - e^{-2x}) \end{aligned}$$

**f.**
$$\begin{aligned} \frac{dy}{dx} &= \frac{(e^x + e^{-x})(e^x - e^{-x})'}{(e^x + e^{-x})^2} - \frac{(e^x - e^{-x})(e^x + e^{-x})'}{(e^x + e^{-x})^2} \\ &= \frac{(e^x + e^{-x})(e^x - (e^{-x})(-x)')}{(e^x + e^{-x})^2} - \frac{(e^x - e^{-x})(e^x + (e^{-x})(-x)')}{(e^x + e^{-x})^2} \\ &= \frac{(e^x + e^{-x})(e^x - (e^{-x})(-1))}{(e^x + e^{-x})^2} - \frac{(e^x - e^{-x})(e^x + (e^{-x})(-1))}{(e^x + e^{-x})^2} \\ &= \frac{(e^x + e^{-x})(e^x + e^{-x})}{(e^x + e^{-x})^2} - \frac{(e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} \\ &= \frac{e^{2x} + e^0 + e^0 + e^{-2x}}{(e^x + e^{-x})^2} - \frac{(e^{2x} - e^0 - e^0 + e^{-2x})}{(e^x + e^{-x})^2} \\ &= \frac{e^{2x} + e^0 + e^0 + e^{-2x} - e^{2x}}{(e^x + e^{-x})^2} + \frac{e^0 + e^0 - e^{-2x}}{(e^x + e^{-x})^2} \\ &= \frac{4}{(e^x + e^{-x})^2} \end{aligned}$$

**2. a.**
$$\begin{aligned} \frac{dP}{dt} &= 100e^{-5t}(-5t)' \\ &= 100e^{-5t}(-5) \\ &= -500e^{-5t} \end{aligned}$$

**b.** The time is needed for when the sample of the substance is at half of the original amount. So, find $t$ when $P = \frac{1}{2}$.

$$\begin{aligned} P &= 100e^{-5t} \\ \frac{1}{2} &= 100e^{-5t} \\ \frac{1}{200} &= e^{-5t} \\ \ln\frac{1}{200} &= -5t \\ \frac{\ln\frac{1}{200}}{-5} &= t \end{aligned}$$

Now, the question asks for $\frac{dP}{dt} = P'$ when

$$t = \frac{\ln \frac{1}{200}}{-5} \doteq 1.06$$

$$P'\left(\frac{\ln \frac{1}{200}}{-5}\right) = -2.5 \text{ (using a calculator)}$$

**3.** $\frac{dy}{dx} = (-x)(e^x)' + (e^x)(-x)'$

$= (-x)(e^x) + (e^x)(-1)$

$= -xe^x - e^x$

At the point $x = 0$,

$\frac{dy}{dx} = -0e^0 - e^0 = -1.$

At the point $x = 0$,

$y = 2 - 0e^0 = 2$

So, an equation of the tangent to the curve at the point $x = 0$ is

$y - 2 = -1(x - 0)$

$y - 2 = -x$

$y = -x + 2$

$x + y - 2 = 0$

**4. a.** $y' = -3(e^x)'$

$= -3e^x$

$y'' = -3e^x$

**b.** $y' = (x)(e^{2x})' + (e^{2x})(x)'$

$= (x)((e^{2x}) + (2x)') + (e^{2x})(1)$

$= (x)((e^{2x})(2)) + e^{2x}$

$= 2xe^{2x} + e^{2x}$

$y'' = (2x)(e^{2x})' + (e^{2x})(2x)' + e^{2x}(2x)'$

$= (2x)((e^{2x})(2x)') + (e^{2x})(2) + (e^{2x})(2)$

$= (2x)((e^{2x})(2)) + 2e^{2x} + 2e^{2x}$

$= 4xe^{2x} + 4e^{2x}$

**c.** $y' = (e^x)(4 - x)' + (4 - x)(e^x)'$

$= (e^x)(-1) + (4 - x)(e^x)$

$= -e^x + 4e^x - xe^x$

$= 3e^x - xe^x$

$y'' = (3e^x)' - [(x)(e^x)' + (e^x)(x)']$

$= 3e^x - [xe^x + (e^x)(1)]$

$= 3e^x - xe^x - e^x$

$= 2e^x - xe^x$

**5. a.** $\frac{dy}{dx} = (8^{2x+5})(\ln 8)(2x + 5)'$

$= (8^{2x+5})(\ln 8)(2)$

$= 2(\ln 8)(8^{2x+5})$

**b.** $\frac{dy}{dx} = 3.2((10)^{0.2x})(\ln 10)(0.2x)'$

$= 3.2((10)^{0.2x})(\ln 10)(0.2)$

$= 0.64(\ln 10)((10)^{.2x})$

**c.** $f'(x) = (x^2)(2^x)' + (2^x)(x^2)'$

$= (x^2)(2^x)(\ln 2) + (2^x)(2x)$

$= (\ln 2)(x^2 2^x) + 2x2^x$

$= 2^x((\ln 2)(x^2) + 2x)$

**d.** $H'(x) = 300((5)^{3x-1})(\ln 5)(3x - 1)'$

$= 300((5)^{3x-1})(\ln 5)(3)$

$= 900(\ln 5)(5)^{3x-1}$

$= 900(\ln 5)(5)^{3x-1}$

**e.** $q'(x) = (1.9)^x \cdot (\ln 1.9) + 1.9(x)^{1.9-1}$

$= (1.9)^x \cdot (\ln 1.9) + 1.9(x)^{0.9}$

$= (\ln 1.9)(1.9)^x + 1.9x^{0.9}$

**f.** $f'(x) = (x - 2)^2(4^x)' + (4^x)((x - 2)^2)'$

$= (x - 2)^2(4^x)(\ln 4) + (4^x)(2(x - 2))$

$= (\ln 4)(4^x)(x - 2)^2 + (4^x)(2x - 4)$

$= 4^x((\ln 4)(x - 2)^2 + 2x - 4)$

**6. a.** The initial number of rabbits in the forest is given by the time $t = 0$.

$R(0) = 500\left(10 + e^{-\frac{0}{10}}\right)$

$= 500(10 + 1)$

$= 500(11)$

$= 5500$

**b.** The rate of change is the derivative, $\frac{dR}{dt}$.

$R(t) = 5000 + 500\left(e^{-\frac{t}{10}}\right)$

$\frac{dR}{dt} = 0 + 500\left(e^{-\frac{t}{10}}\right)\left(-\frac{t}{10}\right)'$

$= 500\left(e^{-\frac{t}{10}}\right)\left(-\frac{1}{10}\right)$

$= -50\left(e^{-\frac{t}{10}}\right)$

**c.** 1 year = 12 months

The question asks for $\frac{dR}{dt} = R'$ when $t = 12$.

$R'(12) = -50\left(e^{-\frac{12}{10}}\right)$

$\doteq -15.06$

**d.** To find the maximum number of rabbits, optimize the function.

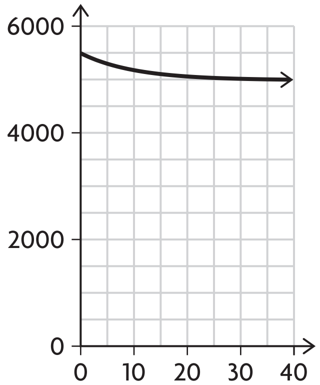
$R'(t) = -50\left(e^{-\frac{t}{10}}\right)$

$0 = -50\left(e^{-\frac{t}{10}}\right)$

$0 = e^{-\frac{t}{10}}$

When solving, the natural log (ln) of both sides must be taken, but (ln 0) does not exist. So there are no solutions to the equation. The function is therefore always decreasing. So, the largest number of rabbits will exist at the earliest time in the interval at time $t = 0$. To check, compare $R(0)$ and $R(36)$. $R(0) = 5500$ and $R(36) \doteq 5013$. So, the largest number of rabbits in the forest during the first 3 years is 5500.

**e.**




The graph is constantly decreasing. The $y$-intercept is at the point (0, 5500). Rabbit populations normally grow exponentially, but this population is shrinking exponentially. Perhaps a large number of rabbit predators such as snakes recently began to appear in the forest. A large number of predators would quickly shrink the rabbit population.

**7.** The highest concentration of the drug can be found by optimizing the given function.

$$C(t) = 10e^{-2t} - 10e^{-3t}$$
$$C'(t) = (10e^{-2t})(-2t)' - (10e^{-3t})(-3t)'$$
$$= (10e^{-2t})(-2) - (10e^{-3t})(-3)$$
$$= -20e^{-2t} + 30e^{-3t}$$

Set the derivative of the function equal to zero and find the critical points.

$$0 = -20e^{-2t} + 30e^{-3t}$$
$$20e^{-2t} = 30e^{-3t}$$
$$\frac{2}{3}e^{-2t} = e^{-3t}$$
$$\frac{2}{3} = \frac{e^{-3t}}{e^{-2t}}$$
$$\frac{2}{3} = (e^{-3t})(e^{2t})$$
$$\frac{2}{3} = e^{-3t+2t}$$
$$\frac{2}{3} = e^{-t}$$
$$\ln\frac{2}{3} = -t$$
$$-\left(\ln\frac{2}{3}\right) = t$$

Therefore, $t = -\left(\ln\frac{2}{3}\right) \doteq 0.41$ is the critical value.
Now, use the algorithm for finding extreme values.

$$C(0) = 10(e^0 - e^0) = 0$$
$$C\left(-\left(\ln\frac{2}{3}\right)\right) \doteq 1.48 \text{ (using a calculator)}$$
$$C(5) = 0.0005$$

So, the function has a maximum when $t = -\left(\ln\frac{2}{3}\right) \doteq 0.41$. Therefore, during the first five hours, the highest concentration occurs at about 0.41 hours.

**8.** $y = ce^{kx}$
$y' = cke^{kx}$
The original function is increasing when its derivative is positive and decreasing when its derivative is negative.
$e^{kx} > 0$ for all $k, x \in \mathbf{R}$.
So, the original function represents growth when $ck > 0$, meaning that $c$ and $k$ must have the same sign. The original function represents decay when $c$ and $k$ have opposite signs.

**9. a.** $A(t) = 5000e^{0.02t}$
$= 5000e^{0.02(0)}$
$= 5000$
The initial population is 5000.

**b.** at $t = 7$
$A(7) = 5000e^{0.02(7)} = 5751$
After a week, the population is 5751.

**c.** at $t = 30$
$A(30) = 5000e^{0.02(30)} = 9111$
After 30 days, the population is 9111.

**10. a.** $P(5) = 760e^{-0.125(5)}$
$\doteq 406.80$ mm Hg

**b.** $P(7) = 760e^{-0.125(7)}$
$\doteq 316.82$ mm Hg

**c.** $P(9) = 760e^{-0.125(9)}$
$\doteq 246.74$ mm Hg

**11.** $A = 100e^{-0.3x}$
$A' = 100e^{-0.3x}(-0.3)$
$= -30e^{-0.3x}$
When 50% of the substance is gone, $y = 50$

$$50 = 100e^{-0.3x}$$
$$0.5 = e^{-0.3x}$$
$$\ln(0.5) = \ln e^{-0.3x}$$
$$\ln(0.5) = -0.3x \ln e$$

$$\frac{\ln 0.5}{\ln e} = -0.3x$$
$$-\frac{\ln 0.5}{0.3 \ln e} = x$$
$$x = 2.31$$
$$A' = -30e^{-0.3x}$$
$$A'(2.31) = -30e^{-0.3(2.31)}$$
$$A' \doteq -15$$

When 50% of the substance is gone, the rate of decay is 15% per year.

**12.** $f(x) = xe^x$

$$f'(x) = xe^x + (1)e^x$$
$$= e^x(x + 1)$$

So $e^x > 0$

$x + 1 > 0$

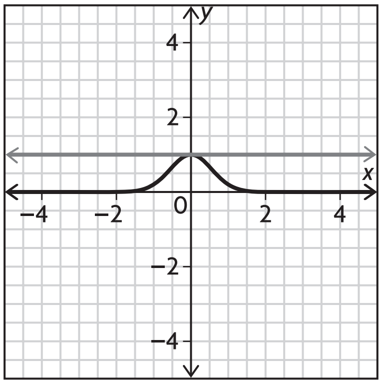
$x > -1$

This means that the function is increasing when $x > -1$.

**13.** $y = 5^{-x^2}$

When $x = 1$,

$$y = \frac{1}{5}$$
$$y' = 5^{-x^2}(-2x)\ln 5$$
$$y' = -\frac{2}{5}\ln 5$$
$$5y - \frac{1}{5} = -\frac{2}{5}\ln 5(x - 1)$$
$$5y - 1 = -2\ln 5(x - 1)$$
$$5y - 1 = (-2\ln 5)x + 2\ln 5$$
$$(2\ln 5)x + 5y = 2\ln 5 + 1$$




**14. a.** $A = P(1 + i)^t$

$$A(t) = 1000(1 + 0.06)^t$$
$$= 1000(1.06)^t$$

**b.** $A'(t) = 1000(1.06)^t(1)\ln(1.06)$

$$= 1000(1.06)^t \ln 1.06$$

**c.** $A'(2) = 1000(1.06)^2 \ln 1.06$

$$= \$65.47$$
$$A'(5) = 1000(1.06)^5 \ln 1.06$$
$$= \$77.98$$
$$A'(10) = 1000(1.06)^{10} \ln 1.06$$
$$= \$104.35$$

**d.** No, the rate is not constant.

**e.** $\frac{A'(2)}{A(2)} = \ln 1.06$

$$\frac{A'(5)}{A(5)} = \ln 1.06$$
$$\frac{A'(10)}{A(10)} = \ln 1.06$$

**f.** All the ratios are equivalent (they equal ln 1.06, which is about 0.058 27), which means that $\frac{A'(t)}{A(t)}$ is constant.

**15.** $y = ce^x$

$$y' = c(e^x) + (0)e^x$$
$$= ce^x$$
$$y = y' = ce^x$$

## 5.4 The Derivatives of *y* = sin *x* and *y* = cos *x*, pp. 256–257

**1. a.** $\frac{dy}{dx} = (\cos 2x) \cdot \frac{d(2x)}{dx}$

$$= 2\cos 2x$$

**b.** $\frac{dy}{dx} = -2(\sin 3x) \cdot \frac{d(3x)}{dx}$

$$= -6\sin 3x$$

**c.** $\frac{dy}{dx} = (\cos(x^3 - 2x + 4)) \cdot \frac{d(x^3 - 2x + 4)}{dx}$

$$= (3x^2 - 2)(\cos(x^3 - 2x + 4))$$

**d.** $\frac{dy}{dx} = -2\sin(-4x) \cdot \frac{d(-4x)}{dx}$

$$= 8\sin(-4x)$$

**e.** $\frac{dy}{dx} = \cos(3x) \cdot \frac{d(3x)}{dx} + \sin(4x) \cdot \frac{d(4x)}{dx}$

$$= 3\cos(3x) + 4\sin(4x)$$

**f.** $\frac{dy}{dx} = 2^x(\ln 2) + 2\cos x + 2\sin x$

**g.** $\frac{dy}{dx} = \cos(e^x) \cdot \frac{d(e^x)}{dx}$

$$= e^x\cos(e^x)$$

**h.** $\frac{dy}{dx} = 3\cos(3x + 2\pi) \cdot \frac{d(3x + 2\pi)}{dx}$

$$= 9\cos(3x + 2\pi)$$

**i.** $\frac{dy}{dx} = 2x - \sin x + 0$
$= 2x - \sin x$

**j.** $\frac{dy}{dx} = \cos\left(\frac{1}{x}\right) \cdot \frac{d\left(\frac{1}{x}\right)}{dx}$
$= -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$

**2. a.** $\frac{dy}{dx} = (2\sin x)(-\sin x) + (\cos x)(2\cos x)$
$= -2\sin^2 x + 2\cos^2 x$
$= 2(\cos^2 x - \sin^2 x)$
$= 2\cos(2x)$

**b.** $y = (x^{-1})(\cos 2x)$
$\frac{dy}{dx} = (x^{-1})(-2\sin 2x) + (\cos 2x)(-x^{-2})$
$= -\frac{2\sin 2x}{x} - \frac{\cos 2x}{x^2}$

**c.** $\frac{dy}{dx} = -\sin(\sin 2x) \cdot \frac{d(\sin 2x)}{dx}$
$= -\sin(\sin 2x) \cdot 2\cos 2x$

**d.** $y = (\sin x)(1 + \cos x)^{-1}$
$\frac{dy}{dx} = (\sin x)(-(1 + \cos x)^{-2} \cdot (-\sin x)$
$+ (1 + \cos x)^{-1}(\cos x)$
$= \frac{-\sin^2 x}{-(1 + \cos x)^2} + \frac{\cos x}{1 + \cos x}$
$= \frac{\sin^2 x}{(1 + \cos x)^2} + \frac{\cos x(1 + \cos x)}{(1 + \cos x)^2}$
$= \frac{\sin^2 x + \cos^2 x + \cos x}{(1 + \cos x)^2}$
$= \frac{1 + \cos x}{(1 + \cos x)^2}$
$= \frac{1}{1 + \cos x}$

**e.** $\frac{dy}{dx} = (e^x)(-\sin x + \cos x) + (\cos x + \sin x)(e^x)$
$= e^x(-\sin x + \cos x + \cos x + \sin x)$
$= e^x(2\cos x)$

**f.** $\frac{dy}{dx} = (2x^3)(\cos x) + (\sin x)(6x^2)$
$-[(3x)(-\sin x) + (\cos x)(3)]$
$= 2x^3\cos x + 6x^2\sin x + 3x\sin x - 3\cos x$

**3. a.** When $x = \frac{\pi}{3}$, $f(x) = f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.
$f'(x) = \cos x$
$f'\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3}$
$= \frac{1}{2}$

So an equation for the tangent at the point $x = \frac{\pi}{3}$ is

$y - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(x - \frac{\pi}{3}\right)$
$2y - \sqrt{3} = x - \frac{\pi}{3}$
$-x + 2y + \left(\frac{\pi}{3} - \sqrt{3}\right) = 0$

**b.** When $x = 0$, $f(x) = f(0) = 0 + \sin(0) = 0$.
$f'(x) = 1 + \cos x$
$f'(0) = 1 + \cos(0)$
$= 1 + 1$
$= 2$

So an equation for the tangent at the point $x = 0$ is
$y - 0 = 2(x - 0)$
$y = 2x$
$-2x + y = 0$

**c.** When $x = \frac{\pi}{4}$, $f(x) = f\left(\frac{\pi}{4}\right) = \cos\left(4 \cdot \frac{\pi}{4}\right)$
$= \cos(\pi)$
$= -1$
$f'(x) = -\sin(4x) \cdot \frac{d(4x)}{dx}$
$= -4\sin(4x)$
$f'\left(\frac{\pi}{4}\right) = -4\sin\left(4 \cdot \frac{\pi}{4}\right)$
$= -4\sin(\pi)$
$= 0$

So an equation for the tangent at the point $x = \frac{\pi}{4}$ is

$y - (-1) = 0\left(x - \frac{\pi}{4}\right)$
$y + 1 = 0$
$y = -1$

**d.** $f(x) = \sin 2x + \cos x$, $x = \frac{\pi}{2}$

The point of contact is $\left(\frac{\pi}{2}, 0\right)$. The slope of the tangent line at any point is $f'(x) = 2\cos 2x - \sin x$. At $\left(\frac{\pi}{2}, 0\right)$, the slope of the tangent line is

$2\cos\pi - \sin\frac{\pi}{2} = -3$.

An equation of the tangent line is $y = -3\left(x - \frac{\pi}{2}\right)$.

**e.** $f(x) = \cos\left(2x + \frac{\pi}{3}\right), x = \frac{\pi}{4}$

The point of tangency is $\left(\frac{\pi}{4}, -\frac{\sqrt{3}}{2}\right)$. The slope of the tangent line at any point is $f'(x) = -2\sin\left(2x + \frac{\pi}{3}\right)$.

At $\left(\frac{\pi}{4}, -\frac{\sqrt{3}}{2}\right)$, the slope of the tangent line is

$-2\sin\left(\frac{5\pi}{6}\right) = -1.$

An equation of the tangent line is

$y + \frac{\sqrt{3}}{2} = -\left(x - \frac{\pi}{4}\right).$

**f.** When $x = \frac{\pi}{2}$, $f(x) = f\left(\frac{\pi}{2}\right) = 2\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)$

$$= 2(1)(0) = 0$$

$$\begin{aligned} f'(x) &= (2\sin x)(-\sin x) + (\cos x)(2\cos x) \\ &= -2\sin^2 x + 2\cos^2 x \\ &= 2(\cos^2 x - \sin^2 x) \\ &= 2\cos(2x) \end{aligned}$$

$$\begin{aligned} f'\left(\frac{\pi}{2}\right) &= 2\cos\left(2 \cdot \frac{\pi}{2}\right) \\ &= 2\cos\pi \\ &= -2 \end{aligned}$$

So an equation for the tangent when $x = \frac{\pi}{2}$ is

$$y - 0 = -2\left(x - \frac{\pi}{2}\right)$$

$$y = -2x + \pi$$

$$2x + y - \pi = 0$$

**4. a.** One could easily find $f'(x)$ and $g'(x)$ to see that they both equal $2(\sin x)(\cos x)$. However, it is easier to notice a fundamental trigonometric identity. It is known that $\sin^2 x + \cos^2 x = 1$. So, $\sin^2 x = 1 - \cos^2 x$.

Therefore, one can notice that $f(x)$ is in fact equal to $g(x)$. So, because $f(x) = g(x)$, $f'(x) = g'(x)$.

**b.** $f'(x)$ and $g'(x)$ are negatives of each other. That is, $f'(x) = 2(\sin x)(\cos x)$ while $g'(x) = -2(\sin x)(\cos x)$.

**5. a.** $v(t) = (\sin(\sqrt{t}))^2$

$$\begin{aligned} v'(t) &= 2\sin(\sqrt{t}) \cdot \frac{d(\sin(\sqrt{t}))}{dt} \\ &= 2\sin(\sqrt{t}) \cdot \cos(\sqrt{t}) \cdot \frac{d(\sqrt{t})}{dt} \\ &= 2\sin(\sqrt{t}) \cdot \cos(\sqrt{t}) \cdot \frac{1}{2}t^{-\frac{1}{2}} \\ &= 2\sin(\sqrt{t}) \cdot \cos(\sqrt{t}) \cdot \frac{1}{2\sqrt{t}} \\ &= \frac{\sin(\sqrt{t})\cos(\sqrt{t})}{\sqrt{t}} \end{aligned}$$

**b.** $v(t) = (1 + \cos t + \sin^2 t)^{\frac{1}{2}}$

$$\begin{aligned} v'(t) &= \frac{1}{2}(1 + \cos t + \sin^2 t)^{-\frac{1}{2}} \\ &\quad \times \frac{d(1 + \cos t + (\sin t)^2)}{dt} \\ &= \frac{-\sin t + 2(\sin t) \cdot \frac{d(\sin t)}{dt}}{2\sqrt{1 + \cos t + \sin^2 t}} \\ &= \frac{-\sin t + 2(\sin t)(\cos t)}{2\sqrt{1 + \cos t + \sin^2 t}} \end{aligned}$$

**c.** $h(x) = \sin x \sin 2x \sin 3x$

So, treat $\sin x \sin 2x$ as one function, say $f(x)$ and treat $\sin 3x$ as another function, say $g(x)$.

Then, the product rule may be used with the chain rule:

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= (\sin x \sin 2x)(3\cos 3x) \\ &\quad + (\sin 3x)[(\sin x)(2\cos 2x) \\ &\quad + (\sin 2x)(\cos x)] \\ &= 3\sin x \sin 2x \cos 3x \\ &\quad + 2\sin x \sin 3x \cos 2x \\ &\quad + \sin 2x \sin 3x \cos x \end{aligned}$$

**d.** 
$$\begin{aligned} m'(x) &= 3(x^2 + \cos^2 x)^2 \cdot \frac{d(x^2 + (\cos x)^2)}{dx} \\ &= 3(x^2 + \cos^2 x)^2 \cdot (2x + 2(\cos x)(-\sin x)) \\ &= 3(x^2 + \cos^2 x)^2 \cdot (2x - 2\sin x \cos x) \end{aligned}$$

**6.** By the algorithm for finding extreme values, the maximum and minimum values occur at points on the graph where $f'(x) = 0$, or at an endpoint of the interval.

**a.** $\frac{dy}{dx} = -\sin x + \cos x$

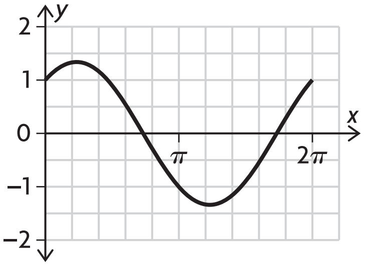
Set $\frac{dy}{dx} = 0$ and solve for $x$ to find any critical points.

$$\begin{aligned} \cos x - \sin x &= 0 \\ \cos x &= \sin x \\ 1 &= \frac{\sin x}{\cos x} \\ 1 &= \tan x \\ x &= \frac{\pi}{4}, \frac{5\pi}{4} \end{aligned}$$

Evaluate $f(x)$ at the critical numbers, including the endpoints of the interval.

| $x$ | 0 | $\frac{\pi}{4}$ | $\frac{5\pi}{4}$ | $2\pi$ |
|---|---|---|---|---|
| $f(x) = \cos x + \sin x$ | 1 | $\sqrt{2}$ | $-\sqrt{2}$ | 1 |

So, the absolute maximum value on the interval is $\sqrt{2}$ when $x = \frac{\pi}{4}$ and the absolute minimum value on the interval is $-\sqrt{2}$ when $x = \frac{5\pi}{4}$.




**b.** $\frac{dy}{dx} = 1 - 2\sin x$

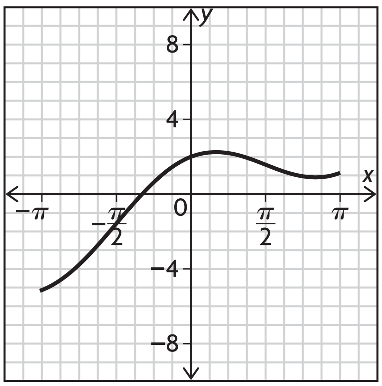
Set $\frac{dy}{dx} = 0$ and solve for $x$ to find any critical points.

$$1 - 2\sin x = 0$$
$$1 = 2\sin x$$
$$\frac{1}{2} = \sin x$$
$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

Evaluate $f(x)$ at the critical numbers, including the endpoints of the interval.

| $x$ | $-\pi$ | $-\frac{\pi}{6}$ | $\frac{\pi}{6}$ | $\pi$ |
|---|---|---|---|---|
| $f(x) = x + 2\cos x$ | $-\pi - 2$ $\doteq -5.14$ | $-\frac{\pi}{6} + \sqrt{3}$ $\doteq 1.21$ | $\frac{\pi}{6} + \sqrt{3}$ $\doteq 2.26$ | $\pi - 2$ $\doteq 1.14$ |

So, the absolute maximum value on the interval is 2.26 when $x = \frac{\pi}{6}$ and the absolute minimum value on the interval is $-5.14$ when $x = -\pi$.




**c.** $\frac{dy}{dx} = \cos x + \sin x$

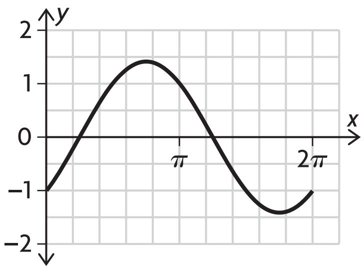
Set $\frac{dy}{dx} = 0$ and solve for $x$ to find any critical points.

$$\cos x + \sin x = 0$$
$$\sin x = -\cos x$$
$$\frac{\sin x}{\cos x} = -1$$
$$\tan x = -1$$
$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

Evaluate $f(x)$ at the critical numbers, including the endpoints of the interval.

| $x$ | 0 | $\frac{3\pi}{4}$ | $\frac{7\pi}{4}$ | 2 |
|---|---|---|---|---|
| $f(x) = \sin x - \cos x$ | $-1$ | $\sqrt{2}$ | $-\sqrt{2}$ | $-1$ |

So, the absolute maximum value on the interval is $\sqrt{2}$ when $x = \frac{3\pi}{4}$ and the absolute minimum value on the interval is $-\sqrt{2}$ when $x = \frac{7\pi}{4}$.




**d.** $\frac{dy}{dx} = 3\cos x - 4\sin x$

Set $\frac{dy}{dx} = 0$ and solve for $x$ to find any critical points.

$$3\cos x - 4\sin x = 0$$
$$3\cos x = 4\sin x$$
$$\frac{3}{4} = \frac{\sin x}{\cos x}$$
$$\frac{3}{4} = \tan x$$
$$\tan^{-1}\left(\frac{3}{4}\right) = \tan^{-1}(\tan x)$$

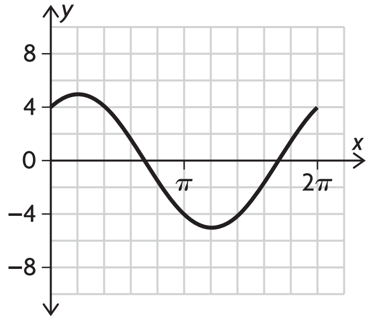
Using a calculator, $x \doteq 0.6435$.
This is a critical value, but there is also one more in the interval $0 \le x \le 2\pi$. The period of $\tan x$ is $\pi$, so adding $\pi$ to the one solution will give another solution in the interval.

$$x = 0.6435 + \pi \doteq 3.7851$$

Evaluate $f(x)$ at the critical numbers, including the endpoints of the interval.

| $x$ | 0 | 0.64 | 3.79 | $2\pi$ |
|---|---|---|---|---|
| $f(x) = 3 \sin x + 4 \cos x$ | 4 | 5 | $-5$ | 4 |

So, the absolute maximum value on the interval is 5 when $x \doteq 0.64$ and the absolute minimum value on the interval is $-5$ when $x \doteq 3.79$.




**7. a.** The particle will change direction when the velocity, $s'(t)$, changes from positive to negative.
$s'(t) = 16 \cos 2t$
Set $s'(t) = 0$ and solve for $t$ to find any critical points.

$$0 = 16 \cos 2t$$
$$0 = \cos 2t$$
$$\frac{\pi}{2}, \frac{3\pi}{2} = 2t$$
$$\frac{\pi}{4}, \frac{3\pi}{4} = t$$

Also, there is no given interval so it will be beneficial to locate all solutions.

Therefore, $t = \frac{\pi}{4} + \pi k, \frac{3\pi}{4} + \pi k$ for some positive integer $k$ constitutes all solutions.

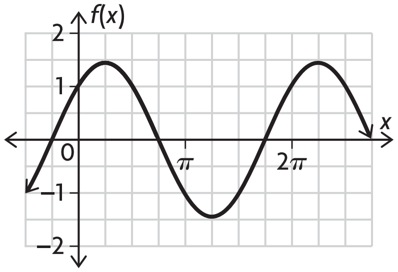
One can create a table and notice that on each side of any value of $t$, the function is increasing on one side and decreasing on the other. So, each $t$ value is either a maximum or a minimum.

| $t$ | $\frac{\pi}{4}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{4}$ | $\frac{7\pi}{4}$ |
|---|---|---|---|---|
| $s(t) = 8 \sin 2t$ | 8 | $-8$ | 8 | $-8$ |

The table continues in this pattern for all critical values $t$. So, the particle changes direction at all critical values. That is, it changes direction for $t = \frac{\pi}{4} + \pi k, \frac{3\pi}{4} + \pi k$ for positive integers $k$.

**b.** From the table or a graph, one can see that the particle's maximum velocity is 8 at the time $t = \frac{\pi}{4} + \pi k$.

**8. a.**




**b.** The tangent to the curve $f(x)$ is horizontal at the point(s) where $f'(x)$ is zero.
$f'(x) = -\sin x + \cos x$
Set $f'(x) = 0$ and solve for $x$ to find any critical points.

$$\cos x - \sin x = 0$$
$$\cos x = \sin x$$
$$1 = \frac{\sin x}{\cos x}$$
$$1 = \tan x$$

$x = \frac{\pi}{4}$ (*Note:* The solution $x = \frac{5\pi}{4}$ is not in the interval $0 \le x \le \pi$ so it is not included.) When $x = \frac{\pi}{4}$, $f(x) = f\left(\frac{\pi}{4}\right) = \sqrt{2}$.
So, the coordinates of the point where the tangent to the curve of $f(x)$ is horizontal is $\left(\frac{\pi}{4}, \sqrt{2}\right)$.

**9.** $\csc x = \frac{1}{\sin x} = (\sin x)^{-1}$

$\sec x = \frac{1}{\cos x} = (\cos x)^{-1}$

Now, the power rule can be used to compute the derivates of $\csc x$ and $\sec x$.

$$((\sin x)^{-1})' = -(\sin x)^{-2} \cdot \frac{d(\sin x)}{dx}$$
$$= -(\sin x)^{-2} \cdot \cos x$$
$$= -\frac{\cos x}{(\sin x)^2}$$
$$((\sin x)^{-1})' = -(\sin x)^{-2} \cdot \frac{d(\sin x)}{dx}$$
$$= -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}$$
$$= -\csc x \cot x$$
$$((\cos x)^{-1})' = -(\cos x)^{-2} \cdot \frac{d(\cos x)}{dx}$$
$$= -(\cos x)^{-2} \cdot (-\sin x)$$
$$= \frac{\sin x}{(\cos x)^2}$$
$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$
$$= \sec x \tan x$$

**10.** $\frac{dy}{dx} = -2 \sin 2x$

At the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$,

$$\frac{dy}{dx} = -2 \sin\left(2 \cdot \frac{\pi}{6}\right)$$
$$= -2 \sin\left(\frac{\pi}{3}\right)$$
$$= -2\left(\frac{\sqrt{3}}{2}\right)$$
$$= -\sqrt{3}$$

Therefore, at the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$, the slope of the tangent to the curve $y = \cos 2x$ is $-\sqrt{3}$.

**11. a.** The particle will change direction when the velocity, $s'(t)$ changes from positive to negative.
$s'(t) = 16 \cos 4t$
Set $s'(t) = 0$ and solve for $t$ to find any critical points.
$$0 = 16 \cos 4t$$
$$0 = \cos 4t$$
$$\frac{\pi}{2}, \frac{3\pi}{2} = 4t$$
$$\frac{\pi}{8}, \frac{3\pi}{8} = t$$
Also, there is no given interval so it will be beneficial to locate all solutions.

Therefore, $t = \frac{\pi}{8} + \pi k, \frac{3\pi}{8} + \pi k$ for some positive integer $k$ constitutes all solutions.

One can create a table and notice that on each side of any value of $t$, the function is increasing on one side and decreasing on the other. So, each $t$ value is either a maximum or a minimum.

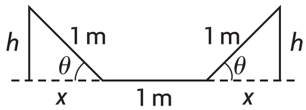
| $t$ | $\frac{\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{7\pi}{8}$ |
|---|---|---|---|---|
| $s(t) = 4 \sin 4t$ | 4 | $-4$ | 4 | $-4$ |

The table continues in this pattern for all critical values $t$. So, the particle changes direction at all critical values. That is, it changes direction for $t = \frac{\pi}{4} + \pi k, \frac{3\pi}{4} + \pi k$ for positive integers $k$.

**b.** From the table or a graph, one can see that the particle's maximum velocity is 4 at the time $t = \frac{\pi}{4} + \pi k$.

**c.** At $t = 0$, $s = 0$, so the minimum distance from the origin is 0. The maximum value of the sine function is 1, so the maximum distance from the origin is $4(1)$ or 4.

**12.**



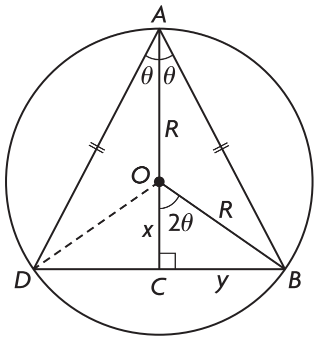

Label the base of a triangle $x$ and the height $h$. So $\cos\theta = \frac{x}{1} = x$ and $\sin\theta = \frac{h}{1} = h$.
Therefore, $x = \cos\theta$ and $h = \sin\theta$.
The irrigation channel forms a trapezoid and the area of a trapezoid is $\frac{(b_1 + b_2)h}{2}$ where $b_1$ and $b_2$ are the bottom and top bases of the trapezoid and $h$ is the height.
$b_1 = 1$
$b_2 = x + 1 + x = \cos\theta + 1 + \cos\theta = 2\cos\theta + 1$
$h = \sin\theta$
Therefore, the area equation is given by
$$A = \frac{(2\cos\theta + 1 + 1)\sin\theta}{2}$$
$$= \frac{(2\cos\theta + 2)\sin\theta}{2}$$
$$= \frac{2\cos\theta\sin\theta + 2\sin\theta}{2}$$
$$= \sin\theta\cos\theta + \sin\theta$$
To maximize the cross-sectional area, differentiate:
$$A' = (\sin\theta)(-\sin\theta) + (\cos\theta)(\cos\theta) + \cos\theta$$
$$= -\sin^2\theta + \cos^2\theta + \cos\theta$$
Using the trig identity $\sin^2\theta + \cos^2\theta = 1$, use the fact that $\sin^2\theta = 1 - \cos^2\theta$.
$$A' = -(1 - \cos^2\theta) + \cos^2\theta + \cos\theta$$
$$= -1 + \cos^2\theta + \cos^2\theta + \cos\theta$$
$$= 2\cos^2\theta + \cos\theta - 1$$
Set $A' = 0$ to find the critical points.
$0 = 2\cos^2\theta + \cos\theta - 1$
$0 = (2\cos\theta - 1)(\cos\theta + 1)$
Solve the two expressions for $\theta$.
$$2\cos\theta = 1$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}$$
Also, $\cos\theta = -1$
$$\theta = \pi$$
(*Note:* The question only seeks an answer around $0 \le \theta \le \frac{\pi}{2}$. So, there is no need to find all solutions by adding $k\pi$ for all integer values of $k$.)
The area, $A$, when $\theta = \pi$ is 0 so that answer is disregarded for this problem.

When $\theta = \frac{\pi}{3}$,

$$A = \sin\frac{\pi}{3}\cos\frac{\pi}{3} + \sin\frac{\pi}{3}$$
$$= \left(\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\right) + \frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{3}}{4} + \frac{2\sqrt{3}}{4}$$
$$= \frac{3\sqrt{3}}{4}$$

The area is maximized by the angle $\theta = \frac{\pi}{3}$.

**13.** Let O be the centre of the circle with line segments drawn and labeled, as shown.




In $\triangle OCB$, $\angle COB = 2\theta$.

Thus, $\frac{y}{R} = \sin 2\theta$ and $\frac{x}{R} = \cos 2\theta$,
so $y = R\sin 2\theta$ and $x = R\cos 2\theta$.

The area $A$ of $\triangle ABD$ is

$$A = \frac{1}{2}|DB||AC|$$
$$= y(R + x)$$
$$= R\sin 2\theta(R + R\cos 2\theta)$$
$$= R^2(\sin 2\theta + \sin 2\theta\cos 2\theta), \text{ where } 0 < 2\theta < \pi$$

$$\frac{dA}{d\theta} = R^2(2\cos 2\theta + 2\cos 2\theta\cos 2\theta + \sin 2\theta(-2\sin 2\theta)).$$

We solve $\frac{dA}{d\theta} = 0$:

$$2\cos^2 2\theta - 2\sin^2 2\theta + 2\cos 2\theta = 0$$
$$2\cos^2 2\theta + \cos 2\theta - 1 = 0$$
$$(2\cos 2\theta - 1)(\cos 2\theta + 1) = 0$$
$$\cos 2\theta = \frac{1}{2} \text{ or } \cos 2\theta = -1$$
$$2\theta = \frac{\pi}{3} \text{ or } 2\theta = \pi \text{ (not in domain).}$$

As $2\theta \to 0$, $A \to 0$ and as $2\theta \to \pi$, $A \to 0$. The maximum area of the triangle is $\frac{3\sqrt{3}}{4}R^2$ when $2\theta = \frac{\pi}{3}$, i.e., $\theta = \frac{\pi}{6}$.

**14.** First find $y''$.

$$y = A\cos kt + B\sin kt$$
$$y' = -kA\sin kt + kB\cos kt$$
$$y'' = -k^2A\cos kt - k^2B\sin kt$$

So, $y'' + k^2y$

$$= -k^2A\cos kt - k^2B\sin kt + k^2(A\cos kt + B\sin kt)$$
$$= -k^2A\cos kt - k^2B\sin kt + k^2A\cos kt + k^2B\sin kt$$
$$= 0$$

Therefore, $y'' + k^2y = 0$.

## 5.5 The Derivative of $y = \tan x$, p. 260

**1. a.** $\frac{dy}{dx} = \sec^2 3x\left(\frac{d}{dx}3x\right)$
$= 3\sec^2 3x$

**b.** $\frac{dy}{dx} = 2\sec^2 x - \sec 2x\left(\frac{d}{dx}2x\right)$
$= 2\sec^2 x - 2\sec 2x$

**c.** $\frac{dy}{dx} = 2\tan(x^3)\left(\frac{d}{dx}\tan(x^3)\right)$

$\frac{dy}{dx} = 2\tan(x^3)\left(\frac{d}{dx}\tan(x^3)\right)$
$= 2\tan(x^3)\sec^2(x^3)\left(\frac{d}{dx}x^3\right)$
$= 6x^2\tan(x^3)\sec^2(x^3)$

**d.** $\frac{dy}{dx} = \frac{2x\tan\pi x - x^2\sec^2\pi x\left(\frac{d}{dx}\pi x\right)}{\tan^2\pi x}$
$= \frac{2x\tan\pi x - \pi x^2\sec^2\pi x}{\tan^2\pi x}$
$= \frac{x(2\tan\pi x - \pi x\sec^2\pi x)}{\tan^2\pi x}$

**e.** $\frac{dy}{dx} = \sec^2(x^2)\left(\frac{d}{dx}x^2\right) - 2\tan x\left(\frac{d}{dx}\right)(\tan x)$
$= 2x\sec^2(x^2) - 2\tan x\sec^2 x$

**f.** $\frac{dy}{dx} = \tan 5x(3\cos 5x)\left(\frac{d}{dx}5x\right) + 3\sin 5x\sec^2 5x\left(\frac{d}{dx}5x\right)$
$= 15\tan 5x\cos 5x + 15\sin 5x\sec^2 5x$
$= 15(\tan 5x\cos 5x + \sin 5x\sec^2 5x)$

**2. a.** The general equation for the line tangent to the function $f(x)$ at the point $(a, b)$ is
$y - b = f'(x)(x - a)$.

$f(x) = \tan x$
$f'(x) = \sec^2 x$

$$f\left(\frac{\pi}{4}\right) = 0$$
$$f'\left(\frac{\pi}{4}\right) = 2$$

The equation for the line tangent to the function $f(x)$ at $x = \frac{\pi}{4}$ is $y = 2\left(x - \frac{\pi}{4}\right)$.

**b.** The general equation for the line tangent to the function $f(x)$ at the point $(a, b)$ is $y - b = f'(x)(x - a)$.

$$f(x) = 6\tan x - \tan 2x$$
$$f'(x) = 6\sec^2 x - \sec^2 2x\left(\frac{d}{dx}2x\right)$$
$$f'(x) = 6\sec^2 x - 2\sec^2 2x$$
$$f(0) = 0$$
$$f'(0) = -2$$

The equation for the line tangent to the function $f(x)$ at $x = 0$ is $y = -2x$.

**3. a.** $\frac{dy}{dx} = \sec^2 x(\sin x)\left(\frac{d}{dx}\sin x\right)$
$= \cos x \sec^2(\sin x)$

**b.** $\frac{dy}{dx} = -2[\tan(x^2 - 1)]^{-3}\left(\frac{d}{dx}\tan(x^2 - 1)\right)$
$= -2[\tan(x^2 - 1)]^{-3}\sec^2(x^2 - 1) \times \left(\frac{d}{dx}(x^2 - 1)\right)$
$= -4x[\tan(x^2 - 1)]^{-3}\sec^2(x^2 - 1)$

**c.** $\frac{dy}{dx} = 2\tan(\cos x)\left(\frac{d}{dx}\tan(\cos x)\right)$
$= 2\tan(\cos x)\sec^2(\cos x)\left(\frac{d}{dx}\cos x\right)$
$= -2\tan(\cos x)\sec^2(\cos x)\sin x$

**d.** $\frac{dy}{dx} = 2(\tan x + \cos x)\left(\frac{d}{dx}\tan x + \cos x\right)$
$= 2(\tan x + \cos x)(\sec^2 x - \sin x)$

**e.** $\frac{dy}{dx} = \tan x(3\sin^2 x)\left(\frac{d}{dx}\sin x\right) + \sin^3 x\sec^2 x$
$= 3\tan x\sin^2 x\cos x + \sin^3 x\sec^2 x$
$= \sin^2 x(3\tan x\cos x + \sin x\sec^2 x)$

**f.** $\frac{dy}{dx} = e^{\tan\sqrt{x}}\left(\frac{d}{dx}\tan\sqrt{x}\right)$
$= e^{\tan\sqrt{x}}(\sec^2\sqrt{x})\left(\frac{d}{dx}\sqrt{x}\right)$
$= \frac{1}{2\sqrt{x}}e^{\tan\sqrt{x}}\sec^2\sqrt{x}$

**4. a.** $\frac{dy}{dx} = \tan x\cos x + \sin x\sec^2 x$
$= \frac{\sin x}{\cos x}\cdot\cos x + \sin x\cdot\frac{1}{\cos^2 x}$
$= \sin x + \frac{\sin x}{\cos^2 x}$

$\frac{d^2y}{dx^2} = \cos x + \frac{\cos^3 x}{\cos^4 x} - \frac{\sin x(2\cos x)\left(\frac{d}{dx}\cos x\right)}{\cos^4 x}$
$= \cos x + \frac{\cos^3 x + 2\sin^2 x\cos x}{\cos^4 x}$
$= \cos x + \frac{1}{\cos x} + \frac{2\sin^2 x}{\cos^3 x}$
$= \cos x + \sec x + \frac{2\sin^2 x}{\cos^3 x}$

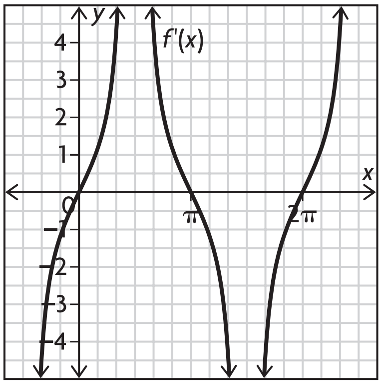
**b.** $\frac{dy}{dx} = 2\tan x\left(\frac{d}{dx}\tan x\right)$
$= 2\tan x\sec^2 x$
$= \frac{2\sin x}{\cos x}\cdot\frac{1}{\cos^2 x}$
$= \frac{2\sin x}{\cos^3 x}$

$\frac{d^2y}{dx^2} = \frac{2\cos^4 x - 6\sin x\cos^2 x\left(\frac{d}{dx}\cos x\right)}{\cos^6 x}$
$= \frac{2\cos^4 x + 6\sin^2 x\cos^2 x}{\cos^6 x}$
$= \frac{2}{\cos^2 x} + \frac{6\sin^2 x}{\cos^2 x}\cdot\frac{1}{\cos^2 x}$
$= 2\sec^2 x + 6\tan^2 x\sec^2 x$
$= 2\sec^2 x(1 + 3\tan^2 x)$

**5.** The slope of $f(x) = \sin x\tan x$ equals zero when the derivative equals zero.

$$f(x) = \sin x\tan x$$
$f'(x) = \sin x(\sec^2 x) + \tan x(\cos x)$
$= \sin x(\sec^2 x) + \frac{\sin x}{\cos x}(\cos x)$
$= \sin x(\sec^2 x) + \sin x$
$= \sin x(\sec^2 x + 1)$

$\sec^2 x + 1$ is always positive, so the derivative is 0 only when $\sin x = 0$. So, $f'(x)$ equals 0 when $x = 0$, $x = \pi$, and $x = 2\pi$. The solutions can be verified by examining the graph of the derivative function shown below.




**6.** The local maximum point occurs when the derivative equals zero.

$$\frac{dy}{dx} = 2 - \sec^2 x$$
$$2 - \sec^2 x = 0$$
$$\sec^2 x = 2$$
$$\sec x = \pm\sqrt{2}$$
$$x = \pm\frac{\pi}{4}$$

$\frac{dy}{dx} = 0$ when $x = \pm\frac{\pi}{4}$, so the local maximum point occurs when $x = \pm\frac{\pi}{4}$. Solve for $y$ when $x = \frac{\pi}{4}$.

$$y = 2\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{4}\right)$$
$$y = \frac{\pi}{2} - 1$$
$$y = 0.57$$

Solve for $y$ when $x = -\frac{\pi}{4}$.

$$y = 2\left(-\frac{\pi}{4}\right) - \tan\left(-\frac{\pi}{4}\right)$$
$$y = -\frac{\pi}{2} + 1$$
$$y = -0.57$$

The local maximum occurs at the point $\left(\frac{\pi}{4}, 0.57\right)$.

**7.** $y = \sec x + \tan x$

$$= \frac{1}{\cos x} + \frac{\sin x}{\cos x}$$
$$= \frac{1 + \sin x}{\cos x}$$
$$\frac{dy}{dx} = \frac{\cos^2 x - (1 + \sin x)(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x - (-\sin x - \sin^2 x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1 + \sin x}{\cos^2 x}$$

The denominator is never negative. $1 + \sin x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, since $\sin x$ reaches its minimum of $-1$ at $x = \frac{\pi}{2}$. Since the derivative of the original function is always positive in the specified interval, the function is always increasing in that interval.

**8.** When $x = \frac{\pi}{4}$, $y = 2\tan\left(\frac{\pi}{4}\right)$

$$= 2$$
$$y' = 2\sec^2 x$$

When $x = \frac{\pi}{4}$, $y' = 2\left(\sec\frac{\pi}{4}\right)^2$

$$= 2(\sqrt{2})^2$$
$$= 4$$

So an equation for the tangent at the point $x = \frac{\pi}{4}$ is

$$y - 2 = 4\left(x - \frac{\pi}{4}\right)$$
$$y - 2 = 4x - \pi$$
$$-4x + y - (2 - \pi) = 0$$

**9.** Write $\tan x = \frac{\sin x}{\cos x}$ and use the quotient rule to derive the derivative of the tangent function.

**10.** $y = \cot x$

$$y = \frac{1}{\tan x}$$
$$\frac{dy}{dx} = \frac{\tan x(0) - (1)\sec^2 x}{\tan^2 x}$$
$$= \frac{-\sec^2 x}{\tan^2 x}$$
$$= \frac{\frac{-1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}}$$
$$= \frac{-1}{\sin^2 x}$$
$$= -\csc^2 x$$

**11.** Using the fact from question 10 that the derivative of $\cot x$ is $-\csc^2 x$,

$$f'(x) = -4\csc^2 x$$
$$= -4(\csc x)^2$$
$$f''(x) = -8(\csc x) \cdot \frac{d(\csc x)}{dx}$$
$$= -8(\csc x) \cdot (-\csc x \cot x)$$
$$= 8\csc^2 x \cot x$$

## Review Exercise, pp. 263–265

**1. a.** $y' = 0 - e^x$
$= -e^x$

**b.** $y' = 2 + 3e^x$

**c.** $y' = e^{2x+3} \cdot \dfrac{d(2x + 3)}{dx}$
$= 2e^{2x+3}$

**d.** $y' = e^{-3x^2+5x} \cdot \dfrac{d(-3x^2 + 5x)}{dx}$
$= (-6x + 5)e^{-3x^2+5x}$

**e.** $y' = (x)(e^x) + (e^x)(1)$
$= e^x(x + 1)$

**f.** $s' = \dfrac{(e^t + 1)(e^t) - (e^t - 1)(e^t)}{(e^t + 1)^2}$
$= \dfrac{e^{2t} + e^t - (e^{2t} - e^t)}{(e^t + 1)^2}$
$= \dfrac{2e^t}{(e^t + 1)^2}$

**2. a.** $\dfrac{dy}{dx} = 10^x \ln 10$

**b.** $\dfrac{dy}{dx} = 4^{3x^2} \cdot \ln 4 \cdot \dfrac{d(3x^2)}{dx}$
$= 6x(4^{3x^2})\ln 4$

**c.** $\dfrac{dy}{dx} = (5x)(5^x \ln 5) + (5^x)(5)$
$= 5 \cdot 5^x(x \ln 5 + 1)$

**d.** $\dfrac{dy}{dx} = (x^4)(2^x \ln 2) + (2^x)(4x^3)$
$= x^3 \cdot 2^x(x \ln 2 + 4)$

**e.** $y = (4x)(4^{-x})$
$\dfrac{dy}{dx} = (4x)(-4^{-x} \ln 4) + (4^{-x})(4)$
$= 4 \cdot 4^{-x}(-x \ln 4 + 1)$
$= \dfrac{4 - 4x \ln 4}{4^x}$

**f.** $y = (5^{\sqrt{x}})(x^{-1})$
$\dfrac{dy}{dx} = (5^{\sqrt{x}})(-x^{-2}) + (x^{-1})\left(5^{\sqrt{x}} \cdot \ln 5 \cdot \dfrac{d(\sqrt{x})}{dx}\right)$
$= (5^{\sqrt{x}})\left(-\dfrac{1}{x^2}\right) + (x^{-1})\left(5^{\sqrt{x}} \cdot \ln 5 \cdot \dfrac{1}{2\sqrt{x}}\right)$
$= 5^{\sqrt{x}}\left(-\dfrac{1}{x^2} + \dfrac{\ln 5}{2x\sqrt{x}}\right)$

**3. a.** $\dfrac{dy}{dx} = 3 \cos(2x) \cdot \dfrac{d(2x)}{dx} + 4 \sin(2x) \cdot \dfrac{d(2x)}{dx}$
$= 6 \cos(2x) + 8 \sin(2x)$

**b.** $\dfrac{dy}{dx} = \sec^2(3x) \cdot \dfrac{d(3x)}{dx}$
$= 3 \sec^2(3x)$

**c.** $y = (2 - \cos x)^{-1}$
$\dfrac{dy}{dx} = -(2 - \cos x)^{-2} \cdot \dfrac{d(2 - \cos x)}{dx}$
$= -\dfrac{\sin x}{(2 - \cos x)^2}$

**d.** $\dfrac{dy}{dx} = (x)\left(\sec^2(2x) \cdot \dfrac{d(2x)}{dx}\right) + (\tan(2x))(1)$
$= 2x \sec^2(2x) + \tan 2x$

**e.** $\dfrac{dy}{dx} = (\sin 2x)\left(e^{3x} \cdot \dfrac{d(3x)}{dx}\right)$
$+ (e^{3x})\left(\cos 2x \cdot \dfrac{d(2x)}{dx}\right)$
$= 3e^{3x} \sin 2x + 2e^{3x} \cos 2x$
$= e^{3x}(3 \sin 2x + 2 \cos 2x)$

**f.** $y = (\cos(2x))^2$
$\dfrac{dy}{dx} = 2(\cos(2x)) \cdot \dfrac{d(\cos(2x))}{dx}$
$= 2(\cos(2x)) \cdot -\sin(2x) \cdot \dfrac{d(2x)}{dx}$
$= -4 \cos(2x) \sin(2x)$

**4. a.** $f(x) = e^x \cdot x^{-1}$
$f'(x) = (e^x)(-x^{-2}) + (x^{-1})(e^x)$
$= e^x\left(-\dfrac{1}{x^2} + \dfrac{1}{x}\right)$
$= e^x\left(\dfrac{-x + x^2}{x^3}\right)$

Now, set $f'(x) = 0$ and solve for $x$.
$0 = e^x\left(\dfrac{-x + x^2}{x^3}\right)$
Solve $e^x = 0$ and $\dfrac{x^2 - x}{x^3} = 0$.
$e^x$ is never zero.
$\dfrac{x^2 - x}{x^3} = 0$
$x^2 - x = 0$
$x(x - 1) = 0$
So $x = 0$ or $x = 1$.
(Note, however, that $x$ cannot be zero because this would cause division by zero in the original function.)
So $x = 1$.

**b.** The function has a horizontal tangent at $(1, e)$.

**5. a.** $f'(x) = (x)\left(e^{-2x} \cdot \frac{d(-2x)}{dx}\right) + (e^{-2x})(1)$

$= -2xe^{-2x} + e^{-2x}$

$= e^{-2x}(-2x + 1)$

$f'\left(\frac{1}{2}\right) = e^{-2\frac{1}{2}}\left(-2 \cdot \frac{1}{2} + 1\right)$

$= e^{-1}(-1 + 1)$

$= 0$

**b.** This means that the slope of the tangent to $f(x)$ at the point with $x$-coordinate $\frac{1}{2}$ is 0.

**6. a.** $y' = (x)(e^x) + (e^x)(1) - e^x$

$= xe^x$

$y'' = (x)(e^x) + (e^x)(1)$

$= xe^x + e^x$

$= e^x(x + 1)$

**b.** $y' = (x)(10e^{10x}) + (e^{10x})(1)$

$= 10xe^{10x} + e^{10x}$

$y'' = (10x)(10e^{10x}) + (e^{10x})(10) + 10e^{10x}$

$= 100xe^{10x} + 10e^{10x} + 10e^{10x}$

$= 100xe^{10x} + 20e^{10x}$

$= 20e^{10x}(5x + 1)$

**7.** $y = \frac{e^{2x} - 1}{e^{2x} + 1}$

$\frac{dy}{dx} = \frac{2e^{2x}(e^{2x} + 1) - (e^{2x} - 1)(2e^{2x})}{(e^{2x} + 1)^2}$

$= \frac{2e^{4x} + 2e^{2x} - 2e^{4x} + 2e^{2x}}{(e^{2x} + 1)^2}$

$= \frac{4e^{2x}}{(e^{2x} + 1)^2}$

Now, $1 - y^2 = 1 - \frac{e^{4x} - 2e^{2x} + 1}{(e^{2x} + 1)^2}$

$= \frac{e^{4x} + 2e^{2x} + 1 - e^{4x} + 2e^{2x} - 1}{(e^{2x} + 1)^2}$

$= \frac{4e^{2x}}{(3^{2x} + 1)^2} = \frac{dy}{dx}$

**8.** The slope of the required tangent line is 3.
The slope at any point on the curve is given by
$\frac{dy}{dx} = 1 + e^{-x}$.
To find the point(s) on the curve where the tangent has slope 3, we solve:

$1 + e^{-x} = 3$

$e^{-x} = 2$

$-x = \ln 2$

$x = -\ln 2.$

The point of contact of the tangent is
$(-\ln 2, -\ln 2 - 2)$.

The equation of the tangent line is
$y + \ln 2 + 2 = 3(x + \ln 2)$ or
$3x - y + 2\ln 2 - 2 = 0$.

**9.** When $x = \frac{\pi}{2}$,

$y = f(x) = f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2}(1) = \frac{\pi}{2}$

$y' = f'(x) = (x)(\cos x) + (\sin x)(1)$

$= x\cos x + \sin x$

$f'\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}$

$= \frac{\pi}{2}(0) + 1$

$= 1$

So an equation for the tangent at the point $x = \frac{\pi}{2}$ is

$y - \frac{\pi}{2} = 1\left(x - \frac{\pi}{2}\right)$

$y - \frac{\pi}{2} = x - \frac{\pi}{2}$

$y = x$

$-x + y = 0$

**10.** If $s(t) = \frac{\sin t}{3 + \cos 2t}$ is the function describing an object's position at time $t$, then $v(t) = s'(t)$ is the function describing the object's velocity at time $t$. So

$v(t) = s'(t)$

$= \frac{(3 + \cos 2t)(\cos t) - (\sin t)(-2\sin 2t)}{(3 + \cos 2t)^2}$

$s'\left(\frac{\pi}{4}\right) = \frac{\left(3 + \cos 2 \cdot \frac{\pi}{4}\right)\left(\cos\frac{\pi}{4}\right)}{\left(3 + \cos 2 \cdot \frac{\pi}{4}\right)^2} - \frac{\left(\sin\frac{\pi}{4}\right)\left(-2\sin 2 \cdot \frac{\pi}{4}\right)}{\left(3 + \cos 2 \cdot \frac{\pi}{4}\right)^2}$

$= \frac{\left(3 + \cos\frac{\pi}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(-2\sin\frac{\pi}{2}\right)}{\left(3 + \cos\frac{\pi}{2}\right)^2}$

$= \frac{(3 + 0)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)(-2 \cdot 1)}{(3 + 0)^2}$

$= \frac{\frac{3\sqrt{2}}{2} + \sqrt{2}}{9}$

$= \frac{3\sqrt{2} + 2\sqrt{2}}{2} \cdot \frac{1}{9}$

$= \frac{5\sqrt{2}}{18}$

So, the object's velocity at time $t = \frac{\pi}{4}$ is $\frac{5\sqrt{2}}{18} \doteq 0.3928$ metres per unit of time.

**11. a.** The question asks for the time $t$ when $N'(t) = 0$.

$$N(t) = 60\,000 + 2000te^{-\frac{t}{20}}$$

$$N'(t) = 0 + (2000t)\left(-\frac{1}{20}e^{-\frac{t}{20}}\right) + (e^{-\frac{t}{20}})(2000)$$
$$= -100te^{-\frac{t}{20}} + 2000e^{-\frac{t}{20}}$$
$$= 100e^{-\frac{t}{20}}(-t + 20)$$

Set $N'(t) = 0$ and solve for $t$.

$$0 = 100e^{-\frac{t}{20}}(-t + 20)$$

$100e^{-\frac{t}{20}}$ is never equal to zero.

$$-t + 20 = 0$$
$$20 = t$$

Therefore, the rate of change of the number of bacteria is equal to zero when time $t = 20$.

**b.** The question asks for $\frac{dM}{dt} = M'(t)$ when $t = 10$. That is, it asks for $M'(10)$.

$$M(t) = (N + 1000)^{\frac{1}{3}}$$

$$M'(t) = \frac{1}{3}(N + 1000)^{-\frac{2}{3}} \cdot \frac{d(N + 1000)}{dt}$$
$$= \frac{1}{3(N + 1000)^{\frac{2}{3}}} \cdot \frac{dN}{dt}$$

From part a., $\frac{dN}{dt} = N'(t) = 100e^{-\frac{t}{20}}(-t + 20)$ and $N(t) = 60000 + 2000te^{-\frac{t}{20}}$

So $M'(t) = \frac{100e^{-\frac{1}{20}}(-t + 20)}{3(N + 1000)^{\frac{2}{3}}}$

First calculate $N(10)$.

$$N(10) = 60000 + 2000(10)e^{-\frac{10}{20}}$$
$$= 60000 + 20000e^{-\frac{1}{2}}$$
$$\doteq 72131$$

$$\text{So } M'(10) = \frac{100e^{-\frac{10}{20}}(-10 + 20)}{3(N(10) + 1000)^{\frac{2}{3}}}$$
$$= \frac{100e^{-\frac{1}{2}}(10)}{3(72131 + 1000)^{\frac{2}{3}}}$$
$$\doteq \frac{606.53}{5246.33}$$
$$\doteq 0.1156$$

So, after 10 days, about 0.1156 mice are infected per day. Essentially, almost 0 mice are infected per day when $t = 10$.

**12. a.** $c_1(t) = te^{-t}; c_1(0) = 0$

$$c_1'(t) = e^{-t} - te^{-t}$$
$$= e^{-t}(1 - t)$$

Since $e^{-t} > 0$ for all $t$, $c_1'(t) = 0$ when $t = 1$. Since $c_1'(t) > 0$ for $0 \leq t < 1$, and $c_1'(t) < 0$ for all $t > 1$, $c_1(t)$ has a maximum value of $\frac{1}{e} \doteq 0.368$ at $t = 1$ h.

$$c_2(t) = t^2e^{-t}; c_2(0) = 0$$
$$c_2'(t) = 2te^{-t} - t^2e^{-t}$$
$$= te^{-t}(2 - t)$$

$c_2'(t) = 0$ when $t = 0$ or $t = 2$.

Since $c_2'(t) > 0$ for $0 < t < 2$ and $c_2'(t) < 0$ for all $t > 2$, $c_2(t)$ has a maximum value of $\frac{4}{e^2} \doteq 0.541$ at $t = 2$ h. The larger concentration occurs for medicine $c_2$.

**b.** $c_1(0.5) = 0.303$

$c_2(0.5) = 0.152$

In the first half-hour, the concentration of $c_1$ increases from 0 to 0.303, and that of $c_2$ increases from 0 to 0.152. Thus, $c_1$ has the larger concentration over this interval.

**13. a.** $y = (2 + 3e^{-x})^3$

$$y' = 3(2 + 3e^{-x})^2[0 + 3e^{-x}(-1)]$$
$$= 3(2 + 3e^{-x})^2(-3e^{-x})$$
$$= -9e^{-x}(2 + 3e^{-x})^2$$

**b.** $y = x^e$

$$y' = ex^{e-1}$$

**c.** $y = e^{e^x}$

$$y' = e^{e^x}(e^x)(1)$$
$$= e^{x + e^x}$$

**d.** $y = (1 - e^{5x})^5$

$$y' = 5(1 - e^{5x})^4[0 - e^{5x}(5)]$$
$$= -25e^{5x}(1 - e^{5x})^4$$

**14. a.** $y = 5^x$

$$y' = 5^x \ln 5$$

**b.** $y = (0.47)^x$

$$y' = (0.47)^x \ln(0.47)$$

**c.** $y = (52)^{2x}$

$$y' = (52)^{2x}(2) \ln 52$$
$$= 2(52)^{2x} \ln 52$$

**d.** $y = 5(2)^x$

$$y' = 5(2)^x \ln 2$$

**e.** $y = 4e^x$

$$y' = 4e^x(1) \ln e$$
$$= 4e^x$$

**f.** $y = -2(10)^{3x}$

$$y' = -2(3)10^{3x} \ln 10$$
$$= -6(10)^{3x} \ln 10$$

**15. a.** $y' = \cos 2^x \cdot \frac{d(2^x)}{dx}$

$$= 2^x \ln 2 \cos 2^x$$

**b.** $y' = (x^2)(\cos x) + (\sin x)(2x)$
$= x^2 \cos x + 2x \sin x$

**c.** $y' = \cos\left(\frac{\pi}{2} - x\right) \cdot \frac{d(\frac{\pi}{2} - x)}{dx}$
$= -\cos\left(\frac{\pi}{2} - x\right)$

**d.** $y' = (\cos x)(\cos x) + (\sin x)(-\sin x)$
$= \cos^2 x - \sin^2 x$

**e.** $y = (\cos x)^2$
$y' = 2(\cos x) \cdot \frac{d(\cos x)}{dx}$
$= -2 \cos x \sin x$

**f.** $y = \cos x\ (\sin x)^2$
$y' = (\cos x)(2\ (\sin x)(\cos x)) + (\sin x)^2(-\sin x)$
$= 2 \sin x \cos^2 x - \sin^3 x$

**16.** Compute $\frac{dy}{dx}$ when $x = \frac{\pi}{2}$ to find the slope of the line at the given point.
$y' = -\sin x$
So, at the point $x = \frac{\pi}{2}$, $y' = f'(x)$ is
$f'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1.$
Therefore, an equation of the line tangent to the curve at the given point is

$$y - 0 = -1\left(x - \frac{\pi}{2}\right)$$

$$y = -x + \frac{\pi}{2}$$

$$x + y - \frac{\pi}{2} = 0$$

**17.** The velocity of the object at any time $t$ is $v = \frac{ds}{dt}$.
Thus, $v = 8\ (\cos(10\pi t))(10\pi)$
$= 80\pi \cos(10\pi t).$
The acceleration at any time $t$ is $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.
Hence, $a = 80\pi(-\sin(10\pi t))(10\pi) = -800\pi^2 \sin(10\pi t)$.
Now, $\frac{d^2s}{dt^2} + 100\pi^2 s$
$= -800\pi^2 \sin(10\pi t)$
$+ 100\pi^2(8 \sin(10\pi t)) = 0.$

**18.** Since $s = 5 \cos\left(2t + \frac{\pi}{4}\right)$,

$$v = \frac{ds}{dt} = 5\left(-\sin\left(2t + \frac{\pi}{4}\right)\right)$$

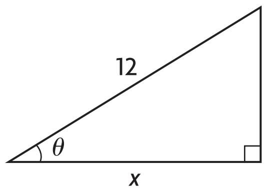
$$= -10 \sin\left(2t + \frac{\pi}{4}\right)$$

and $a = \frac{dv}{dt} = -10\left(\cos\left(2t + \frac{\pi}{4}\right)\right)$

$$= -20 \cos\left(2t + \frac{\pi}{4}\right).$$

The maximum values of the displacement, velocity, and acceleration are 5, 10, and 20, respectively.

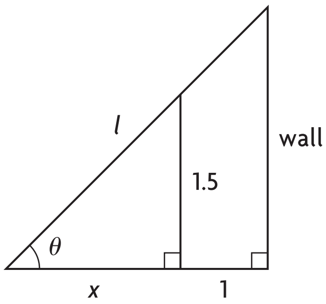
**19.** Let the base angle be $\theta$, $0 < \theta < \frac{\pi}{2}$, and let the sides of the triangle have lengths $x$ and $y$, as shown. Let the perimeter of the triangle be $P$ cm.




Now, $\frac{x}{12} = \cos\theta$ and $\frac{y}{12} = \sin\theta$
so $x = 12 \cos\theta$ and $y = 12 \sin\theta$.
Therefore, $P = 12 + 12\cos\theta + 12\sin\theta$ and
$\frac{dP}{d\theta} = -12\sin\theta + 12\cos\theta$.
For critical values, $-12\sin\theta + 12\cos\theta = 0$
$\sin\theta = \cos\theta$
$\tan\theta = 1$

$\theta = \frac{\pi}{4}$, since $0 < \theta < \frac{\pi}{2}$.
When $\theta = \frac{\pi}{4}$, $P = 12 + \frac{12}{\sqrt{2}} + \frac{12}{\sqrt{2}}$
$= 12 + \frac{24}{\sqrt{2}}$
$= 12 + 12\sqrt{2}.$
As $\theta \to 0^+$, $\cos\theta \to 1$, $\sin\theta \to 0$, and
$P \to 12 + 12 + 0 = 24.$
As $\theta \to \frac{\pi}{2}$, $\cos\theta \to 0$, $\sin\theta \to 1$ and
$P \to 12 + 0 + 12 = 24.$
Therefore, the maximum value of the perimeter is $12 + 12\sqrt{2}$ cm, and occurs when the other two angles are each $\frac{\pi}{4}$ rad, or 45°.

**20.** Let $l$ be the length of the ladder, $\theta$ be the angle between the foot of the ladder and the ground, and $x$ be the distance of the foot of the ladder from the fence, as shown.
Thus, $\frac{x + 1}{l} = \cos\theta$ and $\frac{1.5}{x} = \tan\theta$
$x + 1 = l\cos\theta$ where $x = \frac{1.5}{\tan\theta}$.




Replacing $x$, $\frac{1.5}{\tan\theta} + 1 = l\cos\theta$

$$l = \frac{1.5}{\sin\theta} + \frac{1}{\cos\theta},\ 0 < \theta < \frac{\pi}{2}$$

$$\frac{dl}{d\theta} = -\frac{1.5\cos\theta}{\sin^2\theta} + \frac{\sin\theta}{\cos^2\theta}$$

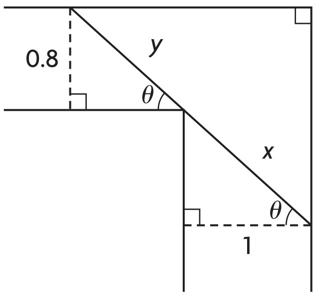
$$= \frac{-1.5\cos^3\theta + \sin^3\theta}{\sin^2\theta\cos^2\theta}.$$

Solving $\frac{dl}{d\theta} = 0$ yields:

$$\sin^3\theta - 1.5\cos^3\theta = 0$$
$$\tan^3\theta = 1.5$$
$$\tan\theta = \sqrt[3]{1.5}$$
$$\theta \doteq 0.46365.$$

The length of the ladder corresponding to this value of $\theta$ is $l \doteq 4.5$ m. As $\theta \to 0^+$ and $\frac{\pi^-}{2}$, $l$ increases without bound. Therefore, the shortest ladder that goes over the fence and reaches the wall has a length of 4.5 m.

**21.** The longest pole that can fit around the corner is determined by the minimum value of $x + y$. Thus, we need to find the minimum value of $l = x + y$.




From the diagram, $\frac{0.8}{y} = \sin\theta$ and $\frac{1}{x} = \cos\theta$.

Thus, $l = \frac{1}{\cos\theta} + \frac{0.8}{\sin\theta},\ 0 \le \theta \le \frac{\pi}{2}$:

$$\frac{dl}{d\theta} = \frac{1\sin\theta}{\cos^2\theta} - \frac{0.8\cos\theta}{\sin^2\theta}$$

$$= \frac{0.8\sin^3\theta - \cos^3\theta}{\cos^2\theta\sin^2\theta}.$$

Solving $\frac{dl}{d\theta} = 0$ yields:

$$0.8\sin^3\theta - \cos^3\theta = 0$$
$$\tan^3\theta = 1.25$$
$$\tan\theta = \sqrt[3]{1.25}$$
$$\tan\theta \doteq 1.077$$
$$\theta \doteq 0.822.$$

Now, $l = \frac{0.8}{\cos(0.822)} + \frac{1}{\sin(0.822)} \doteq 2.5$.

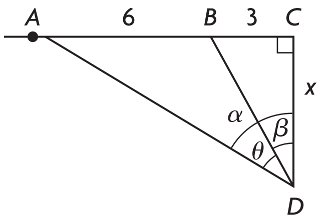
When $\theta = 0$, the longest possible pole would have a length of 0.8 m. When $\theta = \frac{\pi}{2}$, the longest possible pole would have a length of 1 m. Therefore, the longest pole that can be carried horizontally around the corner is one of length 2.5 m.

**22.** We want to find the value of $x$ that maximizes $\theta$.

Let $\angle ADC = \alpha$ and $\angle BDC = \beta$.

Thus, $\theta = \alpha - \beta$:

$$\tan\theta = \tan(\alpha - \beta)$$
$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$




From the diagram, $\tan\alpha = \frac{9}{x}$ and $\tan\beta = \frac{3}{x}$.

Hence, $\tan\theta = \dfrac{\frac{9}{x} - \frac{3}{x}}{1 + \frac{27}{x^2}}$

$$= \frac{9x - 3x}{x^2 + 27}$$
$$= \frac{6x}{x^2 + 27}.$$

We differentiate implicitly with respect to $x$:

$$\sec^2\theta\frac{d\theta}{dx} = \frac{6(x^2 + 27) - 6x(2x)}{(x^2 + 27)^2}$$

$$\frac{d\theta}{dx} = \frac{162 - 6x^2}{\sec^2\theta(x^2 + 27)^2}$$

Solving $\frac{d\theta}{dx} = 0$ yields:

$$162 - 6x^2 = 0$$
$$x^2 = 27$$
$$x = 3\sqrt{3}.$$

**23. a.** $f(x) = 4(\sin(x-2))^2$

$f'(x) = 8\sin(x-2)\cos(x-2)$

$f''(x) = (8\sin(x-2))(-\sin(x-2)) + (\cos(x-2))(8\cos(x-2))$

$= -8\sin^2(x-2) + 8\cos^2(x-2)$

**b.** $f(x) = (2\cos x)(\sec x)^2$

$f'(x) = (2\cos x)(2\sec x \cdot \sec x\tan x) + (\sec x)^2(-2\sin x)$

$= (4\cos x)(\sec^2 x\tan x) - 2\sin x(\sec x)^2$

Using the product rule multiple times,

$f''(x) = (4\cos x)\left[\sec^2 x \cdot \sec^2 x + \tan x(2\sec x \cdot \sec x\tan x)\right]$

$+ (\sec^2 x\tan x)(-4\sin x)$

$+ (-2\sin x)(2\sec x \cdot \sec x\tan x)$

$+ (\sec x)^2(-2\cos x)$

$= 4\cos x\sec^4 x + 8\cos x\tan^2 x\sec^2 x - 4\sin x\tan x\sec^2 x - 4\sin x\tan x\sec^2 x - 2\cos x\sec^2 x$

$= 4\cos x\sec^4 x + 8\cos x\tan^2 x\sec^2 x - 8\sin x\tan x\sec^2 x - 2\cos x\sec^2 x$

## Chapter 5 Test, p. 266

**1. a.** $y = e^{-2x^2}$

$$\frac{dy}{dx} = -4xe^{-2x^2}$$

**b.** $y = 3^{x^2+3x}$

$$\frac{dy}{dx} = 3^{x^2+3x} \cdot \ln 3 \cdot (2x+3)$$

**c.** $y = \frac{e^{3x} + e^{-3x}}{2}$

$$\frac{dy}{dx} = \frac{1}{2}[3e^{3x} - 3e^{-3x}]$$

$$= \frac{3}{2}[e^{3x} - e^{-3x}]$$

**d.** $y = 2\sin x - 3\cos 5x$

$$\frac{dy}{dx} = 2\cos x - 3(-\sin 5x)(5)$$

$$= 2\cos x + 15\sin 5x$$

**e.** $y = \sin^3(x^2)$

$$\frac{dy}{dx} = 3\sin^2(x^2)(\cos(x^2)(2x))$$

$$= 6x\sin^2(x^2)\cos(x^2)$$

**f.** $y = \tan\sqrt{1-x}$

$$\frac{dy}{dx} = \sec^2\sqrt{1-x}\left(\frac{1}{2} \times \frac{1}{\sqrt{1-x}}\right)(-1)$$

$$= -\frac{\sec^2\sqrt{1-x}}{2\sqrt{1-x}}$$

**2.** The given line is $-6x + y = 2$ or $y = 6x + 2$, so the slope is 6.

$y = 2e^{3x}$

$$\frac{dy}{dx} = 2e^{3x}(3)$$

$$= 6e^{3x}$$

In order for the tangent line to be parallel to the given line, the derivative has to equal 6 at the tangent point.

$6e^{3x} = 6$

$e^{3x} = 1$

$x = 0$

When $x = 0$, $y = 2$.

The equation of the tangent line is $y - 2 = 6(x - 0)$ or $-6x + y = 2$. The tangent line is the given line.

**3.** $y = e^x + \sin x$

$$\frac{dy}{dx} = e^x + \cos x$$

When $x = 0$, $\frac{dy}{dx} = 1 + 1$ or 2, so the slope of the tangent line at $(0, 1)$ is 2.

The equation of the tangent line at $(0, 1)$ is $y - 1 = 2(x - 0)$ or $-2x + y = 1$.

**4.** $v(t) = 10e^{-kt}$

**a.** $a(t) = v'(t) = -10ke^{-kt}$

$= -k(10e^{-kt})$

$= -kv(t)$

Thus, the acceleration is a constant multiple of the velocity. As the velocity of the particle decreases, the acceleration increases by a factor of $k$.

**b.** At time $t = 0$, $v = 10$ cm/s.

**c.** When $v = 5$, we have $10e^{-kt} = 5$

$$e^{-kt} = \frac{1}{2}$$

$$-kt = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$t = \frac{\ln 2}{k}.$$

After $\frac{\ln 2}{k}$ s have elapsed, the velocity of the particle is 5 cm/s. The acceleration of the particle is $-5k$ at this time.

**5. a.** $f(x) = (\cos x)^2$

$$f'(x) = 2(\cos x) \cdot \frac{d(\cos x)}{dx}$$

$= 2(\cos x) \cdot (-\sin x)$

$= -2\sin x\cos x$

$f''(x) = (-2\sin x)(-\sin x) + (\cos x)(-2\cos x)$

$= 2\sin^2 x - 2\cos^2 x$

$= 2(\sin^2 x - \cos^2 x)$

**b.** $f(x) = \cos x \cot x$

$f'(x) = (\cos x)(-\csc^2 x) + (\cot x)(-\sin x)$

$= -\cos x \cdot \dfrac{1}{\sin^2 x} - \dfrac{\cos x}{\sin x} \cdot \sin x$

$= -\dfrac{\cos x}{\sin x} \cdot \dfrac{1}{\sin x} - \cos x$

$= -\cot x \csc x - \cos x$

$f''(x) =$

$(-\cot x)(-\csc x \cot x) + (\csc x)(\csc^2 x) + \sin x$

$= \csc x \cot^2 x + \csc^3 x + \sin x$

**6.** $f(x) = (\sin x)^2$

To find the absolute extreme values, first find the derivative, set it equal to zero, and solve for $x$.

$f'(x) = 2\,(\sin x) \cdot \dfrac{d(\sin x)}{dx}$

$= 2 \sin x \cos x$

$= \sin 2x$

Now set $f'(x) = 0$ and solve for $x$.

$0 = \sin 2x$

$2x = 0, \pi, 2\pi$

$x = 0, \dfrac{\pi}{2}, \pi$ in the interval $0 \le x \le \pi$.

Evaluate $f(x)$ at the critical numbers, including the endpoints of the interval.

| $x$ | 0 | $\frac{\pi}{2}$ | $\pi$ |
|---|---|---|---|
| $f(x) = (\sin^2 x)$ | 0 | 1 | 0 |

So, the absolute maximum value on the interval is 1 when $x = \dfrac{\pi}{2}$ and the absolute minimum value on the interval is 0 when $x = 0$ and $x = \pi$.

**7.** $y = f(x) = 5^x$

Find the derivative, $f'(x)$, and evaluate the derivative at $x = 2$ to find the slope of the tangent when $x = 2$.

$f'(x) = 5^x \ln 5$

$f'(2) = 5^2 \ln 5$

$= 25 \ln 5$

$\doteq 40.24$

**8.** $y = xe^x + 3e^x$

To find the maximum and minimum values, first find the derivative, set it equal to zero, and solve for $x$.

$y' = (x)(e^x) + (e^x)(1) + 3e^x$

$= xe^x + e^x + 3e^x$

$= xe^x + 4e^x$

$= e^x(x + 4)$

Now set $y' = 0$ and solve for $x$.

$0 = e^x(x + 4)$

$e^x$ is never equal to zero.

$(x + 4) = 0$

So $x = -4$.

Therefore, the critical value is $-4$.

| Interval | $e^x(x + 4)$ |
|---|---|
| $x < -4$ | − |
| $-4 < x$ | + |

So $f(x)$ is decreasing on the left of $x = -4$ and increasing on the right of $x = -4$. Therefore, the function has a minimum value at $\left(-4, -\dfrac{1}{e^4}\right)$. There is no maximum value.

**9.** $f(x) = 2 \cos x - \sin 2x$

So, $f(x) = 2 \cos x - 2 \sin x \cos x$.

**a.** $f'(x) = -2 \sin x - (2 \sin x)(-\sin x) - (\cos x)(2 \cos x)$

$= -2 \sin x + 2 \sin^2 x - 2 \cos^2 x$

Set $f'(x) = 0$ to solve for the critical values.

$-2 \sin x + 2 \sin^2 x - 2 \cos^2 x = 0$

$-2 \sin x + 2 \sin^2 x - 2(1 - \sin^2 x) = 0$

$-2 \sin x + 2 \sin^2 x - 2 + 2 \sin^2 x = 0$

$4 \sin^2 x - 2 \sin x - 2 = 0$

$(2 \sin x + 1)(2 \sin x - 2) = 0$

$2 \sin x + 1 = 0$ and $2 \sin x - 2 = 0$

So, $\sin x = -\frac{1}{2}$.

In the given interval, this occurs when $x = -\dfrac{\pi}{6}, -\dfrac{5\pi}{6}$.

Also, $\sin x = 1$.

In the given interval, this occurs when $x = \dfrac{\pi}{2}$.

Therefore, on the given interval, the critical numbers for $f(x)$ are $x = -\dfrac{\pi}{6}, -\dfrac{5\pi}{6}, \dfrac{\pi}{2}$.

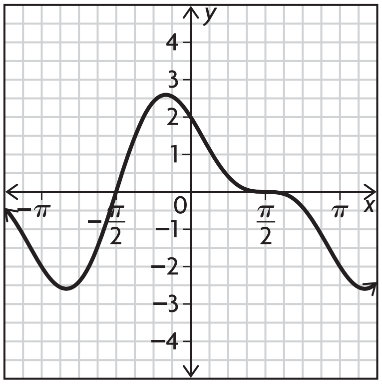
**b.** To determine the intervals where $f(x)$ is increasing and where $f(x)$ is decreasing, find the slope of $f(x)$ in the intervals between the endpoints and the critical numbers. To do this, it helps to make a table.

| $x$ | slope of $f(x)$ |
|---|---|
| $-\pi \le x < -\frac{5\pi}{6}$ | − |
| $-\frac{5\pi}{6} < x < -\frac{\pi}{6}$ | + |
| $-\frac{\pi}{6} < x < \frac{\pi}{2}$ | − |
| $\frac{\pi}{2} < x \le \pi$ | − |

So, $f(x)$ is increasing on the interval $-\dfrac{5\pi}{6} < x < -\dfrac{\pi}{6}$ and $f(x)$ is decreasing on the intervals $-\pi \le x < -\dfrac{5\pi}{6}$ and $-\dfrac{\pi}{6} < x < \pi$.

**c.** From the table in part b., it can be seen that there is a local maximum at the point where $x = -\frac{\pi}{6}$ and there is a local minimum at the point where $x = -\frac{5\pi}{6}$.

**d.**




## Cumulative Review of Calculus

**1. a.** $f(x) = 3x^2 + 4x - 5$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
$$= \lim_{h \to 0} \frac{f(2+h) - 15}{h}$$
$$= \lim_{h \to 0} \frac{3(2+h)^2 + 4(2+h) - 5 - 15}{h}$$
$$= \lim_{h \to 0} \frac{12 + 12h + 3h^2 + 8 + 4h - 20}{h}$$
$$= \lim_{h \to 0} \frac{3h^2 + 16h}{h}$$
$$= \lim_{h \to 0} 3h + 16$$
$$= 16$$

**b.** $f(x) = \frac{2}{x-1}$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
$$= \lim_{h \to 0} \frac{f(2+h) - 2}{h}$$
$$= \lim_{h \to 0} \frac{\frac{2}{2+h-1} - 2}{h}$$
$$= \lim_{h \to 0} \frac{\frac{2}{1+h} - \frac{2(1+h)}{1+h}}{h}$$
$$= \lim_{h \to 0} \frac{2 - 2(1+h)}{h(1+h)}$$
$$= \lim_{h \to 0} \frac{-2h}{h(1+h)}$$
$$= \lim_{h \to 0} \frac{-2}{1+h}$$
$$= -2$$

**c.** $f(x) = \sqrt{x+3}$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
$$= \lim_{h \to 0} \frac{f(6+h) - 3}{h}$$
$$= \lim_{h \to 0} \frac{\sqrt{h+9} - 3}{h}$$
$$= \lim_{h \to 0} \frac{(\sqrt{h+9} - 3)(\sqrt{h+9} + 3)}{h(\sqrt{h+9} + 3)}$$
$$= \lim_{h \to 0} \frac{h + 9 - 9}{h(\sqrt{h+9} + 3)}$$
$$= \lim_{h \to 0} \frac{h}{h(\sqrt{h+9} + 3)}$$
$$= \lim_{h \to 0} \frac{1}{(\sqrt{h+9} + 3)}$$
$$= \frac{1}{6}$$

**d.** $f(x) = 2^{5x}$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
$$= \lim_{h \to 0} \frac{2^{5(1+h)} - 32}{h}$$
$$= \lim_{h \to 0} \frac{2^5 \cdot 2^{5h} - 32}{h}$$
$$= \lim_{h \to 0} \frac{32(2^{5h} - 1)}{h}$$
$$= 32 \lim_{h \to 0} \frac{5(2^{5h} - 1)}{5h}$$
$$= 160 \lim_{h \to 0} \frac{(2^{5h} - 1)}{5h}$$
$$= 160 \ln 2$$

**2. a.** average velocity $= \frac{\text{change in distance}}{\text{change in time}}$

$$= \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$
$$= \frac{[2(4)^2 + 3(4) + 1] - [(2(1)^2 + 3(1) + 1)]}{4 - 1}$$
$$= \frac{45 - 6}{3}$$
$$= 13 \text{ m/s}$$

**b.** instantaneous velocity = slope of the tangent

$$m = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

$$= \lim_{h\to 0} \frac{s(3+h) - s(3)}{h}$$

$$= \lim_{h\to 0} \left[\frac{2(3+h)^2 + 3(3+h) + 1}{h} - \frac{(2(3)^2 + 3(3) + 1)}{h}\right]$$

$$= \lim_{h\to 0} \frac{18 + 12h + 2h^2 + 9 + 3h + 1 - 28}{h}$$

$$= \lim_{h\to 0} \frac{15h + 2h^2}{h}$$

$$= \lim_{h\to 0} (15 + 2h)$$

$$= 15 \text{ m/s}$$

**3.** $$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$\lim_{h\to 0} \frac{(4+h)^3 - 64}{h} = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h}$$

$$(4+h)^3 - 64 = f(4+h) - f(4)$$

Therefore, $f(x) = x^3$.

**4. a.** Average rate of change in distance with respect to time is average velocity, so

$$\text{average velocity} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

$$= \frac{s(3) - s(1)}{3 - 1}$$

$$= \frac{4.9(3)^2 - 4.9(1)}{3 - 1}$$

$$= 19.6 \text{ m/s}$$

**b.** Instantaneous rate of change in distance with respect to time = slope of the tangent.

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{4.9(2+h)^2 - 4.9(2)^2}{h}$$

$$= \lim_{h\to 0} \frac{19.6 + 19.6h + 4.9h^2 - 19.6}{h}$$

$$= \lim_{h\to 0} \frac{19.6h + 4.9h^2}{h}$$

$$= \lim_{h\to 0} 19.6 + 4.9h$$

$$= 19.6 \text{ m/s}$$

**c.** First, we need to determine $t$ for the given distance:

$146.9 = 4.9t^2$

$29.98 = t^2$

$5.475 = t$

Now use the slope of the tangent to determine the instantaneous velocity for $t = 5.475$:

$$m = \lim_{h\to 0} \frac{f(5.475+h) - f(5.475)}{h}$$

$$= \lim_{h\to 0} \frac{4.9(5.475+h)^2 - 4.9(5.475)^2}{h}$$

$$= \lim_{h\to 0} \frac{146.9 + 53.655h + 4.9h^2 - 146.9}{h}$$

$$= \lim_{h\to 0} \frac{53.655h + 4.9h^2}{h}$$

$$= \lim_{h\to 0} [53.655 + 4.9h]$$

$$= 53.655 \text{ m/s}$$

**5. a.** Average rate of population change

$$= \frac{p(t_2) - p(t_1)}{t_2 - t_1}$$

$$= \frac{2(8)^2 + 3(8) + 1 - (2(0) + 3(0) + 1)}{8 - 0}$$

$$= \frac{128 + 24 + 1 - 1}{8 - 0}$$

$= 19$ thousand fish/year

**b.** Instantaneous rate of population change

$$= \lim_{h\to 0} \frac{p(t+h) - p(t)}{h}$$

$$= \lim_{h\to 0} \frac{p(5+h) - p(5)}{h}$$

$$= \lim_{h\to 0} \left[\frac{2(5+h)^2 + 3(5+h) + 1}{h} - \frac{(2(5)^2 + 3(5) + 1)}{h}\right]$$

$$= \lim_{h\to 0} \frac{50 + 20h + 2h^2 + 15 + 3h + 1 - 66}{h}$$

$$= \lim_{h\to 0} \frac{2h^2 + 23h}{h}$$

$$= \lim_{h\to 0} 2h + 23$$

$= 23$ thousand fish/year

**6. a. i.** $f(2) = 3$

**ii.** $\lim_{x\to 2^-} f(x) = 1$

**iii.** $\lim_{x\to 2^+} f(x) = 3$

**iv.** $\lim_{x\to 6} f(x) = 2$

**b.** No, $\lim_{x\to 4} f(x)$ does not exist. In order for the limit to exist, $\lim_{x\to 4^-} f(x)$ and $\lim_{x\to 4^+} f(x)$ must exist and they must be equal. In this case, $\lim_{x\to 4^-} f(x) = \infty$, but $\lim_{x\to 4^+} f(x) = -\infty$, so $\lim_{x\to 4} f(x)$ does not exist.

**7.** $f(x)$ is discontinuous at $x = 2$. $\lim_{x\to 2^-} f(x) = 5$, but $\lim_{x\to 2^+} f(x) = 3$.

**8. a.** $\lim_{x\to 0} \frac{2x^2 + 1}{x - 5} = \frac{2(0)^2 + 1}{0 - 5}$

$= -\frac{1}{5}$

**b.** $\lim_{x\to 3} \frac{x - 3}{\sqrt{x + 6} - 3}$

$= \lim_{x\to 3} \frac{(x - 3)(\sqrt{x + 6} + 3)}{(\sqrt{x + 6} - 3)(\sqrt{x + 6} + 3)}$

$= \lim_{x\to 3} \frac{(x - 3)(\sqrt{x + 6} + 3)}{x + 6 - 9}$

$= \lim_{x\to 3} \frac{(x - 3)(\sqrt{x + 6} + 3)}{x - 3}$

$= \lim_{x\to 3} \sqrt{x + 6} + 3$

$= 6$

**c.** $\lim_{x\to -3} \frac{\frac{1}{x} + \frac{1}{3}}{x + 3}$

$= \lim_{x\to -3} \frac{\frac{x + 3}{3x}}{x + 3}$

$= \lim_{x\to -3} \frac{x + 3}{3x(x + 3)}$

$= \lim_{x\to -3} \frac{1}{3x}$

$= -\frac{1}{9}$

**d.** $\lim_{x\to 2} \frac{x^2 - 4}{x^2 - x - 2}$

$= \lim_{x\to 2} \frac{(x + 2)(x - 2)}{(x + 1)(x - 2)}$

$= \lim_{x\to 2} \frac{x + 2}{x + 1}$

$= \frac{4}{3}$

**e.** $\lim_{x\to 2} \frac{x - 2}{x^3 - 8}$

$= \lim_{x\to 2} \frac{x - 2}{(x - 2)(x^2 + 2x + 4)}$

$= \lim_{x\to 2} \frac{1}{x^2 + 2x + 4}$

$= \frac{1}{12}$

**f.** $\lim_{x\to 0} \frac{\sqrt{x + 4} - \sqrt{4 - x}}{x}$

$= \lim_{x\to 0} \frac{(\sqrt{x + 4} - \sqrt{4 - x})(\sqrt{x + 4} + \sqrt{4 - x})}{x(\sqrt{x + 4} + \sqrt{4 - x})}$

$= \lim_{x\to 0} \frac{x + 4 - (4 - x)}{x(\sqrt{x + 4} + \sqrt{4 - x})}$

$= \lim_{x\to 0} \frac{2x}{x(\sqrt{x + 4} + \sqrt{4 - x})}$

$= \lim_{x\to 0} \frac{2}{(\sqrt{x + 4} + \sqrt{4 - x})}$

$= \frac{1}{2}$

**9. a.** $f(x) = 3x^2 + x + 1$

$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$

$= \lim_{h\to 0} \left[\frac{3(x + h)^2 + (x + h) + 1}{h} - \frac{(3x^2 + x + 1)}{h}\right]$

$= \lim_{h\to 0} \left[\frac{3x^2 + 6hx + 6h^2 + x + h}{h} + \frac{1 - 3x^2 - x - 1}{h}\right]$

$= \lim_{h\to 0} \frac{6hx + 6h^2 + h}{h}$

$= \lim_{h\to 0} 6x + 6h + 1$

$= 6x + 1$

**b.** $f(x) = \frac{1}{x}$

$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{\frac{1}{x + h} - \frac{1}{x}}{h}$

$= \lim_{h\to 0} \frac{x - (x + h)}{h(x)(x + h)}$

$= \lim_{h\to 0} \frac{-h}{h(x)(x + h)}$

$= \lim_{h\to 0} \frac{-1}{x(x + h)}$

$= -\frac{1}{x^2}$

**10. a.** To determine the derivative, use the power rule:

$y = x^3 - 4x^2 + 5x + 2$

$\frac{dy}{dx} = 3x^2 - 8x + 5$

**b.** To determine the derivative, use the chain rule:

$$y = \sqrt{2x^3 + 1}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{2x^3 + 1}}(6x^2)$$

$$= \frac{3x^2}{\sqrt{2x^3 + 1}}$$

**c.** To determine the derivative, use the quotient rule:

$$y = \frac{2x}{x + 3}$$

$$\frac{dy}{dx} = \frac{2(x + 3) - 2x}{(x + 3)^2}$$

$$= \frac{6}{(x + 3)^2}$$

**d.** To determine the derivative, use the product rule:

$$y = (x^2 + 3)^2(4x^5 + 5x + 1)$$

$$\frac{dy}{dx} = 2(x^2 + 3)(2x)(4x^5 + 5x + 1) + (x^2 + 3)^2(20x^4 + 5)$$

$$= 4x(x^2 + 3)(4x^5 + 5x + 1) + (x^2 + 3)^2(20x^4 + 5)$$

**e.** To determine the derivative, use the quotient rule:

$$y = \frac{(4x^2 + 1)^5}{(3x - 2)^3}$$

$$\frac{dy}{dx} = \frac{5(4x^2 + 1)^4(8x)(3x - 2)^3}{(3x - 2)^6} - \frac{3(3x - 2)^2(3)(4x^2 + 1)^5}{(3x - 2)^6}$$

$$= (4x^2 + 1)^4(3x - 2)^2 \times \frac{40x(3x - 2) - 9(4x^2 + 1)}{(3x - 2)^6}$$

$$= \frac{(4x^2 + 1)^4(120x^2 - 80x - 36x^2 - 9)}{(3x - 2)^4}$$

$$= \frac{(4x^2 + 1)^4(84x^2 - 80x - 9)}{(3x - 2)^4}$$

**f.** $y = [x^2 + (2x + 1)^3]^5$

Use the chain rule

$$\frac{dy}{dx} = 5[x^2 + (2x + 1)^3]^4[2x + 6(2x + 1)^2]$$

**11.** To determine the equation of the tangent line, we need to determine its slope at the point $(1, 2)$. To do this, determine the derivative of $y$ and evaluate for $x = 1$:

$$y = \frac{18}{(x + 2)^2}$$

$$= 18(x + 2)^{-2}$$

$$\frac{dy}{dx} = -36(x + 2)^{-3}$$

$$= \frac{-36}{(x + 2)^3}$$

$$m = \frac{-36}{(x + 2)^3}$$

$$= \frac{-36}{27} = \frac{-4}{3}$$

Since we have a given point and we know the slope, use point-slope form to write the equation of the tangent line:

$$y - 2 = \frac{-4}{3}(x - 1)$$

$$3y - 6 = -4x + 4$$

$$4x + 3y - 10 = 0$$

**12.** The intersection point of the two curves occurs when

$$x^2 + 9x + 9 = 3x$$

$$x^2 + 6x + 9 = 0$$

$$(x + 3)^2 = 0$$

$$x = -3.$$

At a point $x$, the slope of the line tangent to the curve $y = x^2 + 9x + 9$ is given by

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 9x + 9)$$

$$= 2x + 9.$$

At $x = -3$, this slope is $2(-3) + 9 = 3$.

**13. a.** $p'(t) = \frac{d}{dt}(2t^2 + 6t + 1100)$

$$= 4t + 6$$

**b.** 1990 is 10 years after 1980, so the rate of change of population in 1990 corresponds to the value

$$p'(10) = 4(10) + 6$$

$= 46$ people per year.

**c.** The rate of change of the population will be 110 people per year when

$$4t + 6 = 110$$

$$t = 26.$$

This corresponds to 26 years after 1980, which is the year 2006.

**14. a.** $f'(x) = \frac{d}{dx}(x^5 - 5x^3 + x + 12)$

$$= 5x^4 - 15x^2 + 1$$

$$f''(x) = \frac{d}{dx}(5x^4 - 15x^2 + 1)$$

$$= 20x^3 - 30x$$

**b.** $f(x)$ can be rewritten as $f(x) = -2x^{-2}$

$$f'(x) = \frac{d}{dx}(-2x^{-2}) = 4x^{-3} = \frac{4}{x^3}$$

$$f''(x) = \frac{d}{dx}(4x^{-3}) = -12x^{-4} = -\frac{12}{x^4}$$

**c.** $f(x)$ can be rewritten as $f(x) = 4x^{-\frac{1}{2}}$

$$f'(x) = \frac{d}{dx}(4x^{-\frac{1}{2}}) = -2x^{-\frac{3}{2}} = -\frac{2}{\sqrt{x^3}}$$

$$f''(x) = \frac{d}{dx}(-2x^{-\frac{3}{2}}) = 3x^{-\frac{5}{2}} = \frac{3}{\sqrt{x^5}}$$

**d.** $f(x)$ can be rewritten as $f(x) = x^4 - x^{-4}$

$$f'(x) = \frac{d}{dx}(x^4 - x^{-4}) = 4x^3 + 4x^{-5} = 4x^3 + \frac{4}{x^5}$$

$$f''(x) = \frac{d}{dx}(4x^3 + 4x^{-5}) = 12x^2 - 20x^{-6} = 12x^2 - \frac{20}{x^6}$$

**15.** Extreme values of a function on an interval will only occur at the endpoints of the interval or at a critical point of the function.

**a.** $f'(x) = \frac{d}{dx}(1 + (x + 3)^2)$

$= 2(x + 3)$

The only place where $f'(x) = 0$ is at $x = -3$, but that point is outside of the interval in question. The extreme values therefore occur at the endpoints of the interval:

$f(-2) = 1 + (-2 + 3)^2 = 2$

$f(6) = 1 + (6 + 3)^2 = 82$

The maximum value is 82, and the minimum value is 6

**b.** $f(x)$ can be rewritten as $f(x) = x + x^{-\frac{1}{2}}$

$$f'(x) = \frac{d}{dx}(x + x^{-\frac{1}{2}}) = 1 + -\frac{1}{2}x^{-\frac{3}{2}} = 1 - \frac{1}{2\sqrt{x^3}}$$

On this interval, $x \geq 1$, so the fraction on the right is always less than or equal to $\frac{1}{2}$. This means that $f'(x) > 0$ on this interval and so the extreme values occur at the endpoints.

$$f(1) = 1 + \frac{1}{\sqrt{1}} = 2$$

$$f(9) = 9 + \frac{1}{\sqrt{9}} = 9\frac{1}{3}$$

The maximum value is $9\frac{1}{3}$, and the minimum value is 2.

**c.** $f'(x) = \frac{d}{dx}\left(\frac{e^x}{1 + e^x}\right)$

$$= \frac{(1 + e^x)(e^x) - (e^x)(e^x)}{(1 + e^x)^2} = \frac{e^x}{(1 + e^x)^2}$$

Since $e^x$ is never equal to zero, $f'(x)$ is never zero, and so the extreme values occur at the endpoints of the interval.

$$f(0) = \frac{e^0}{1 + e^0} = \frac{1}{2}$$

$$f(4) = \frac{e^4}{1 + e^4}$$

The maximum value is $\frac{e^4}{1 + e^4}$, and the minimum value is $\frac{1}{2}$.

**d.** $f'(x) = \frac{d}{dx}(2\sin(4x) + 3)$

$= 8\cos(4x)$

Cosine is 0 when its argument is a multiple of $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

$$4x = \frac{\pi}{2} + 2k\pi \text{ or } 4x = \frac{3\pi}{2} + 2k\pi$$

$$x = \frac{\pi}{8} + \frac{\pi}{2}k \qquad x = \frac{3\pi}{8} + \frac{\pi}{2}k$$

Since $x \in [0, \pi]$, $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$.

Also test the function at the endpoints of the interval.

$f(0) = 2\sin 0 + 3 = 3$

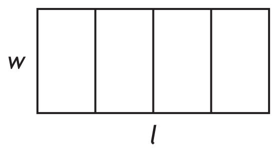
$$f\left(\frac{\pi}{8}\right) = 2\sin\frac{\pi}{2} + 3 = 5$$

$$f\left(\frac{3\pi}{8}\right) = 2\sin\frac{3\pi}{2} + 3 = 1$$
$$f\left(\frac{5\pi}{8}\right) = 2\sin\frac{5\pi}{2} + 3 = 5$$
$$f\left(\frac{7\pi}{8}\right) = 2\sin\frac{7\pi}{2} + 3 = 1$$
$$f(\pi) = 2\sin(4\pi) + 3 = 3$$
The maximum value is 5, and the minimum value is 1.

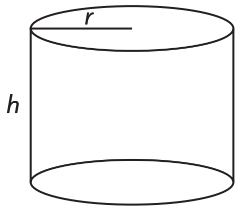
**16. a.** The velocity of the particle is given by
$$\begin{aligned} v(t) &= s'(t) \\ &= \frac{d}{dt}(3t^3 - 40.5t^2 + 162t) \\ &= 9t^2 - 81t + 162. \end{aligned}$$
The acceleration is
$$\begin{aligned} a(t) &= v'(t) \\ &= \frac{d}{dt}(9t^2 - 81t + 162) \\ &= 18t - 81 \end{aligned}$$
**b.** The object is stationary when $v(t) = 0$:
$9t^2 - 81t + 162 = 0$
$9(t - 6)(t - 3) = 0$
$t = 6$ or $t = 3$
The object is advancing when $v(t) > 0$ and retreating when $v(t) < 0$. Since $v(t)$ is the product of two linear factors, its sign can be determined using the signs of the factors:

| *t*-values | *t* − 3 | *t* − 6 | *v*(*t*) | Object |
|---|---|---|---|---|
| $0 < t < 3$ | $< 0$ | $< 0$ | $> 0$ | Advancing |
| $3 < t < 6$ | $> 0$ | $< 0$ | $< 0$ | Retreating |
| $6 < t < 8$ | $> 0$ | $> 0$ | $> 0$ | Advancing |

**c.** The velocity of the object is unchanging when the acceleration is 0; that is, when
$$\begin{aligned} a(t) = 18t - 81 &= 0 \\ t &= 4.5 \end{aligned}$$
**d.** The object is decelerating when $a(t) < 0$, which occurs when
$18t - 81 < 0$
$0 \le t < 4.5$
**e.** The object is accelerating when $a(t) > 0$, which occurs when
$18t - 81 > 0$
$4.5 < t \le 8$

**17.**



Let the length and width of the field be $l$ and $w$, as shown. The total amount of fencing used is then $2l + 5w$. Since there is 750 m of fencing available, this gives
$2l + 5w = 750$
$$l = 375 - \frac{5}{2}w$$
The total area of the pens is
$$\begin{aligned} A &= lw \\ &= 375w - \frac{5}{2}w^2 \end{aligned}$$
The maximum value of this area can be found by expressing $A$ as a function of $w$ and examining its derivative to determine critical points.
$A(w) = 375w - \frac{5}{2}w^2$, which is defined for $0 \le w$ and $0 \le l$. Since $l = 375 - \frac{5}{2}w$, $0 \le l$ gives the restriction $w \le 150$. The maximum area is therefore the maximum value of the function $A(w)$ on the interval $0 \le w \le 150$.
$$\begin{aligned} A'(w) &= \frac{d}{dw}\left(375w - \frac{5}{2}w^2\right) \\ &= 375 - 5w \end{aligned}$$
Setting $A'(w) = 0$ shows that $w = 75$ is the only critical point of the function. The only values of interest are therefore:
$$A(0) = 375(0) - \frac{5}{2}(0)^2 = 0$$
$$A(75) = 375(75) - \frac{5}{2}(75)^2 = 14\,062.5$$
$$A(150) = 375(150) - \frac{5}{2}(150)^2 = 0$$
The maximum area is 14 062.5 $m^2$

**18.**



Let the height and radius of the can be $h$ and $r$, as shown. The total volume of the can is then $\pi r^2 h$. The volume of the can is also give at 500 mL, so
$$\begin{aligned} \pi r^2 h &= 500 \\ h &= \frac{500}{\pi r^2} \end{aligned}$$

The total surface area of the can is

$A = 2\pi rh + 2\pi r^2$

$= \frac{1000}{r} + 2\pi r^2$

The minimum value of this surface area can be found by expressing $A$ as a function of $r$ and examining its derivative to determine critical points.

$A(r) = \frac{1000}{r} + 2\pi r^2$, which is defined for $0 < r$ and $0 < h$. Since $h = \frac{500}{\pi r^2}$, $0 < h$ gives no additional restriction on $r$. The maximum area is therefore the maximum value of the function $A(r)$ on the interval $0 < r$.

$$A'(r) = \frac{d}{dr}\left(\frac{1000}{r} + 2\pi r^2\right)$$
$$= -\frac{1000}{r^2} + 4\pi r$$

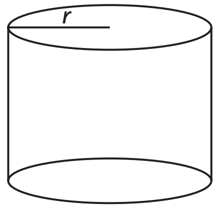
The critical points of $A(r)$ can be found by setting $A'(r) = 0$:

$$-\frac{1000}{r^2} + 4\pi r = 0$$
$$4\pi r^3 = 1000$$
$$r = \sqrt[3]{\frac{1000}{4\pi}} \doteq 4.3 \text{ cm}$$

So $r = 4.3$ cm is the only critical point of the function. This gives the value

$$h = \frac{500}{\pi(4.3)^2} \doteq 8.6 \text{ cm.}$$

**19.**




Let the radius be $r$ and the height $h$.
Minimize the cost:

$C = 2\pi r^2(0.005) + 2\pi rh(0.0025)$

$V = \pi r^2 h = 4000$

$$h = \frac{4000}{\pi r^2}$$

$$C(r) = 2\pi r^2(0.005) + 2\pi r\left(\frac{4000}{\pi r^2}\right)(0.0025)$$
$$= 0.01\pi r^2 + \frac{20}{r}, 1 \le r \le 36$$

$$C'(r) = 0.02\pi r - \frac{20}{r^2}.$$

For a maximum or minimum value, let $C'(r) = 0$.

$$0.02\pi r^2 - \frac{20}{r^2} = 0$$
$$r^3 = \frac{20}{0.02\pi}$$
$$r \doteq 6.8$$

Using the max min algorithm:
$C(1) = 20.03$, $C(6.8) = 4.39$, $C(36) = 41.27$.
The dimensions for the cheapest container are a radius of 6.8 cm and a height of 27.5 cm.

**20. a.** Let the length, width, and depth be $l$, $w$, and $d$, respectively. Then, the given information is that $l = x$, $w = x$, and $l + w + d = 140$. Substituting gives

$$2x + d = 140$$
$$d = 140 - 2x$$

**b.** The volume of the box is $V = lwh$. Substituting in the values from part **a**. gives

$V = (x)(x)(140 - 2x)$
$= 140x^2 - 2x^3$

In order for the dimensions of the box to make sense, the inequalities $l \ge 0$, $w \ge 0$, and $h \ge 0$ must be satisfied. The first two give $x \ge 0$, the third requires $x \le 70$. The maximum volume is therefore the maximum value of $V(x) = 140x^2 - 2x^3$ on the interval $0 \le x \le 70$, which can be found by determining the critical points of the derivative $V'(x)$.

$$V'(x) = \frac{d}{dx}(140x^2 - 2x^3)$$
$$= 280x - 6x^2$$
$$= 2x(140 - 3x)$$

Setting $V'(x) = 0$ shows that $x = 0$ and $x = \frac{140}{3} \doteq 46.7$ are the critical points of the function.

The maximum value therefore occurs at one of these points or at one of the endpoints of the interval:

$V(0) = 140(0)^2 - 2(0)^3 = 0$

$V(46.7) = 140(46.7)^2 - 2(46.7)^3 = 101\,629.5$

$V(0) = 140(70)^2 - 2(70)^3 = 0$

So the maximum volume is 101 629.5 $\text{cm}^3$, from a box with length and width 46.7 cm and depth $140 - 2(46.7) = 46.6$ cm.

**21.** The revenue function is

$R(x) = x(50 - x^2)$
$= 50x - x^3$. Its maximum for $x \ge 0$ can be found by examining its derivative to determine critical points.

$$R'(x) = \frac{d}{dx}(50x - x^3)$$
$$= 50 - 3x^2$$

The critical points can be found by setting $R'(x) = 0$:

$$50 - 3x^2 = 0$$
$$x = \pm\sqrt{\frac{50}{3}} \doteq \pm 4.1$$

Only the positive root is of interest since the number of MP3 players sold must be positive. The number must also be an integer, so both $x = 4$ and $x = 5$ must be tested to see which is larger.

$R(4) = 50(4) - 4^3 = 136$

$R(4) = 50(5) - 5^3 = 125$

So the maximum possible revenue is \$136, coming from a sale of 4 MP3 players.

**22.** Let $x$ be the fare, and $p(x)$ be the number of passengers per year. The given information shows that $p$ is a linear function of $x$ such that an increase of 10 in $x$ results in a decrease of 1000 in $p$. This means that the slope of the line described by $p(x)$ is $\frac{-1000}{10} = -100$. Using the initial point given,

$$p(x) = -100(x - 50) + 10\,000$$
$$= -100x + 15\,000$$

The revenue function can now be written:

$$R(x) = xp(x)$$
$$= x(-100x + 15\,000)$$
$$= 15\,000x - 100x^2$$

Its maximum for $x \geq 0$ can be found by examining its derivative to determine critical points.

$$R'(x) = \frac{d}{dx}(15\,000x - 100x^2)$$
$$= 15\,000 - 200x$$

Setting $R'(x) = 0$ shows that $x = 75$ is the only critical point of the function. The problem states that only \$10 increases in fare are possible, however, so the two nearest must be tried to determine the maximum possible revenue:

$R(70) = 15\,000(70) - 100(70)^2 = 560\,000$

$R(80) = 15\,000(80) - 100(80)^2 = 560\,000$

So the maximum possible revenue is \$560 000, which can be achieved by a fare of either \$70 or \$80.

**23.** Let the number of \$30 price reductions be $n$. The resulting number of tourists will be $80 + n$ where $0 \leq n \leq 70$. The price per tourist will be $5000 - 30n$ dollars. The revenue to the travel agency will be $(5000 - 30n)(80 + n)$ dollars. The cost to the agency will be $250\,000 + 300(80 + n)$ dollars.

Profit = Revenue − Cost

$$P(n) = (5000 - 30n)(80 + n) - 250\,000 - 300(80 + n), 0 \leq n \leq 70$$
$$P'(n) = -30(80 + n) + (5000 - 30n)(1) - 300$$
$$= 2300 - 60n$$
$$P'(n) = 0 \text{ when } n = 38\frac{1}{3}$$

Since $n$ must be an integer, we now evaluate $P(n)$ for $n = 0, 38, 39$, and 70. (Since $P(n)$ is a quadratic function whose graph opens downward with vertex at $38\frac{1}{3}$, we know $P(38) > P(39)$.)

$$P(0) = 126\,000$$
$$P(38) = (3860)(118) - 250\,000 - 300(118)$$
$$= 170\,080$$
$$P(39) = (3830)(119) - 250\,000 - 300(119)$$
$$= 170\,070$$
$$P(70) = (2900)(150) - 250\,000 - 300(150)$$
$$= 140\,000$$

The price per person should be lowered by \$1140 (38 decrements of \$30) to realize a maximum profit of \$170 080.

**24. a.** $$\frac{dy}{dx} = \frac{d}{dx}(-5x^2 + 20x + 2)$$
$$= -10x + 20$$

Setting $\frac{dy}{dx} = 0$ shows that $x = 2$ is the only critical number of the function.

| $x$ | $x < 2$ | $x = 2$ | $x > 2$ |
|---|---|---|---|
| $y'$ | + | 0 | − |
| Graph | Inc. | Local Max | Dec. |

**b.** $$\frac{dy}{dx} = \frac{d}{dx}(6x^2 + 16x - 40)$$
$$= 12x + 16$$

Setting $\frac{dy}{dx} = 0$ shows that $x = -\frac{4}{3}$ is the only critical number of the function.

| $x$ | $x < -\frac{4}{3}$ | $x = -\frac{4}{3}$ | $x > -\frac{4}{3}$ |
|---|---|---|---|
| $y'$ | − | 0 | + |
| Graph | Dec. | Local Min | Inc. |

**c.** $$\frac{dy}{dx} = \frac{d}{dx}(2x^3 - 24x)$$
$$= 6x^2 - 24$$

The critical numbers are found by setting $\frac{dy}{dx} = 0$:

$$6x^2 - 24 = 0$$
$$6x^2 = 24$$
$$x = \pm 2$$

| $x$ | $x < -2$ | $x = -2$ | $-2 < x < 2$ | $x = 2$ | $x > 2$ |
|---|---|---|---|---|---|
| $y'$ | + | 0 | − | 0 | + |
| Graph | Inc. | Local Max | Dec. | Local Min | Inc. |

**d.** $$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{x - 2}\right)$$
$$= \frac{(x - 2)(1) - x(1)}{(x - 2)^2}$$
$$= \frac{-2}{(x - 2)^2}$$

This derivative is never equal to zero, so the function has no critical numbers. Since the numerator is always negative and the denominator is never negative, the derivative is always negative. This means that the function is decreasing everywhere it is defined, that is, $x \neq 2$.

**25. a.** This function is discontinuous when

$x^2 - 9 = 0$

$x = \pm 3$. The numerator is non-zero at these points, so these are the equations of the vertical asymptotes.

To check for a horizontal asymptote:

$$\lim_{x\to\infty} \frac{8}{x^2 - 9} = \lim_{x\to\infty} \frac{8}{x^2\left(1 - \frac{9}{x^2}\right)}$$
$$= \frac{\lim_{x\to\infty}(8)}{\lim_{x\to\infty} x^2\left(1 - \frac{9}{x^2}\right)}$$
$$= \frac{\lim_{x\to\infty}(8)}{\lim_{x\to\infty}(x)^2 \times \lim_{x\to\infty}\left(1 - \frac{9}{x^2}\right)}$$
$$= \lim_{x\to\infty} \frac{1}{x^2} \times \frac{8}{1 - 0}$$
$$= 0$$

Similarly, $\lim_{x\to-\infty} \frac{8}{x^2 - 9} = 0$, so $y = 0$ is a horizontal asymptote of the function.

There is no oblique asymptote because the degree of the numerator does not exceed the degree of the denominator by 1.

Local extrema can be found by examining the derivative to determine critical points:

$$y' = \frac{(x^2 - 9)(0) - (8)(2x)}{(x^2 - 9)^2}$$
$$= \frac{-16x}{(x^2 - 9)^2}$$

Setting $y' = 0$ shows that $x = 0$ is the only critical point of the function.

| $x$ | $x < 0$ | $x = 0$ | $x > 0$ |
|---|---|---|---|
| $y'$ | + | 0 | + |
| Graph | Inc. | Local Max | Dec. |

So $(0, -\frac{8}{9})$ is a local maximum.

**b.** This function is discontinuous when

$x^2 - 1 = 0$

$x = \pm 1$. The numerator is non-zero at these points, so these are the equations of the vertical asymptotes.

To check for a horizontal asymptote:

$$\lim_{x\to\infty} \frac{4x^3}{x^2 - 1} = \lim_{x\to\infty} \frac{x^3(4)}{x^2\left(1 - \frac{1}{x^2}\right)}$$
$$= \lim_{x\to\infty} \frac{x(4)}{1 - \frac{1}{x^2}}$$
$$= \frac{\lim_{x\to\infty}(x(4))}{\lim_{x\to\infty}\left(1 - \frac{1}{x^2}\right)}$$
$$= \frac{\lim_{x\to\infty}(x) \times \lim_{x\to\infty}(4)}{\lim_{x\to\infty}\left(1 - \frac{1}{x^2}\right)}$$
$$= \lim_{x\to\infty}(x) \times \frac{4}{1 - 0}$$
$$= \infty$$

Similarly, $\lim_{x\to-\infty} \frac{4x^3}{x^2 - 1} = \lim_{x\to-\infty}(x) = -\infty$, so this function has no horizontal asymptote.

To check for an oblique asymptote:

$$\begin{array}{r} 4x \phantom{+ 0x^2 + 0x + 0} \\ x^2 - 1\overline{)4x^3 + 0x^2 + 0x + 0} \\ \underline{4x^3 + 0x^2 - 4x \phantom{+ 0}} \\ 0 \quad + 0 \quad + 4x + 0 \end{array}$$

So $y$ can be written in the form

$y = 4x + \frac{4x}{x^2 - 1}$. Since

$$\lim_{x\to\infty} \frac{4x}{x^2 - 1} = \lim_{x\to\infty} \frac{x(4)}{x^2\left(1 - \frac{1}{x^2}\right)}$$
$$= \lim_{x\to\infty} \frac{4}{x\left(1 - \frac{1}{x^2}\right)}$$
$$= \frac{\lim_{x\to\infty}(4)}{\lim_{x\to\infty}\left(x\left(1 - \frac{1}{x^2}\right)\right)}$$
$$= \frac{\lim_{x\to\infty}(4)}{\lim_{x\to\infty}(x) \times \lim_{x\to\infty}\left(1 - \frac{1}{x^2}\right)}$$
$$= \lim\left(\frac{1}{x}\right) \times \frac{4}{1 - 0}$$
$$= 0,$$

and similarly $\lim_{x\to-\infty} \frac{4x}{x^2 - 1} = 0$, the line $y = 4x$ is an asymptote to the function $y$.

Local extrema can be found by examining the derivative to determine critical points:

$$y' = \frac{(x^2 - 1)(12x^2) - (4x^3)(2x)}{(x^2 - 1)^2}$$
$$= \frac{12x^4 - 12x^2 - 8x^4}{(x^2 - 1)^2}$$
$$= \frac{4x^4 - 12x^2}{(x^2 - 1)^2}$$

Setting $y' = 0$:
$4x^4 - 12x^2 = 0$
$x^2(x^2 - 3) = 0$
so $x = 0$, $x = \pm\sqrt{3}$ are the critical points of the function
$(-\sqrt{3}, -6\sqrt{3})$ is a local maximum, $(\sqrt{3}, 6\sqrt{3})$ is a local minimum, and $(0, 0)$ is neither.

| $x$ | $x < -\sqrt{3}$ | $x = -\sqrt{3}$ | $-\sqrt{3} < x < 0$ | $x = 0$ |
|---|---|---|---|---|
| $y'$ | + | 0 | − | 0 |
| Graph | Inc. | Local Max | Dec. | Horiz. |

| $x$ | $0 < x < \sqrt{3}$ | $x = \sqrt{3}$ | $x > \sqrt{3}$ |
|---|---|---|---|
| $y'$ | − | 0 | − |
| Graph | Dec. | Local Min | Inc. |

**26. a.** This function is continuous everywhere, so it has no vertical asymptotes. To check for a horizontal asymptote:

$$\lim_{x\to\infty} (4x^3 + 6x^2 - 24x - 2)$$
$$= \lim_{x\to\infty} x^3\left(4 + \frac{6}{x} - \frac{24}{x^2} - \frac{2}{x^3}\right)$$
$$= \lim_{x\to\infty} (x^3) \times \lim_{x\to\infty}\left(4 + \frac{6}{x} - \frac{24}{x^2} - \frac{2}{x^3}\right)$$
$$= \lim_{x\to\infty} (x^3) \times (4 + 0 - 0 - 0)$$
$$= \infty$$

Similarly,
$\lim_{x\to-\infty} (4x^3 + 6x^2 - 24x - 2) = \lim_{x\to-\infty} (x^3) = -\infty$,
so this function has no horizontal asymptote.

The $y$-intercept can be found by letting $x = 0$, which gives $y = 4(0)^3 + 6(0)^2 - 24(0) - 2$
$= -2$

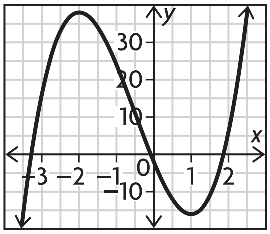
The derivative is of the function is

$$y' = \frac{d}{dx}(4x^3 + 6x^2 - 24x - 2)$$
$$= 12x^2 + 12x - 24$$
$= 12(x + 2)(x - 1)$, and the second derivative is
$$y'' = \frac{d}{dx}(12x^2 + 12x - 24)$$
$$= 24x + 12$$

Letting $f'(x) = 0$ shows that $x = -2$ and $x = 1$ are critical points of the function. Letting $y'' = 0$ shows that $x = -\frac{1}{2}$ is an inflection point of the function.

| $x$ | $x < -2$ | $x = -2$ | $-2 < x$ | $x = -\frac{1}{2}$ |
|---|---|---|---|---|
| $y'$ | + | 0 | − | − |
| Graph | Inc. | Local Max | Dec. | Dec. |
| $y''$ | − | − | − | 0 |
| Concavity | Down | Down | Down | Infl. |

| $x$ | $-\frac{1}{2} < x < 1$ | $x = 1$ | $x > 1$ |
|---|---|---|---|
| $y'$ | − | 0 | + |
| Graph | Dec. | Local Min | Inc. |
| $y''$ | + | + | + |
| Concavity | Up | Up | Up |



$y = 4x^3 + 6x^2 - 24x - 2$

**b.** This function is discontinuous when
$x^2 - 4 = 0$
$(x + 2)(x - 2) = 0$
$x = 2$ or $x = -2$. The numerator is non-zero at these points, so the function has vertical asymptotes at both of them. The behaviour of the function near these asymptotes is:

| $x$-values | $3x$ | $x + 2$ | $x - 2$ | $y$ | $\lim_{x\to\infty} y$ |
|---|---|---|---|---|---|
| $x \to -2^-$ | $< 0$ | $< 0$ | $< 0$ | $< 0$ | $-\infty$ |
| $x \to -2^+$ | $< 0$ | $> 0$ | $< 0$ | $> 0$ | $+\infty$ |
| $x \to 2^-$ | $> 0$ | $> 0$ | $< 0$ | $< 0$ | $-\infty$ |
| $x \to 2^+$ | $> 0$ | $> 0$ | $> 0$ | $> 0$ | $+\infty$ |

To check for a horizontal asymptote:

$$\lim_{x\to\infty} \frac{3x}{x^2 - 4} = \lim_{x\to\infty} \frac{x(3)}{x^2\left(1 - \frac{4}{x^2}\right)}$$
$$= \lim_{x\to\infty} \frac{3}{x\left(1 - \frac{4}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty}(3)}{\lim_{x\to\infty}\left(x\left(1 - \frac{4}{x^2}\right)\right)}$$

$$= \frac{\lim_{x\to\infty}(3)}{\lim_{x\to\infty}(x) \times \lim_{x\to\infty}\left(1 - \frac{4}{x^2}\right)}$$

$$= \lim_{x\to\infty}\frac{1}{x} \times \frac{3}{1 - 0}$$

$$= 0$$

Similarly, $\lim_{x\to\infty}\frac{3x}{x^2 - 4} = 0$, so $y = 0$ is a horizontal asymptote of the function.

This function has $y = 0$ when $x = 0$, so the origin is both the $x$- and $y$-intercept.

The derivative is

$$y' = \frac{(x^2 - 4)(3) - (3x)(2x)}{(x^2 - 4)^2}$$

$$= \frac{-3x^2 - 12}{(x^2 - 4)^2}$$, and the second derivative is

$$y'' = \frac{(x^2 - 4)^2(-6x)}{(x^2 - 4)^4} - \frac{(-3x^2 - 12)(2(x^2 - 4)(2x))}{(x^2 - 4)^4}$$

$$= \frac{-6x^3 + 24x + 12x^3 + 48x}{(x^2 - 4)^3}$$

$$= \frac{6x^3 + 72x}{(x^2 - 4)^3}$$

The critical points of the function can be found by letting $y' = 0$, so

$-3x^2 - 12 = 0$

$x^2 + 4 = 0$. This has no real solutions, so the function $y$ has no critical points.

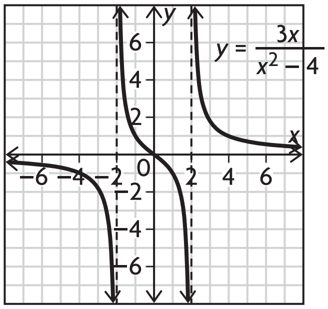
The inflection points can be found by letting $y'' = 0$, so

$6x^3 + 72x = 0$

$6x(x^2 + 12) = 0$

The only real solution to this equation is $x = 0$, so that is the only possible inflection point.

| **x** | $x < -2$ | $-2 < x < 0$ | $x = 0$ | $0 < x < 2$ | $x > 2$ |
|---|---|---|---|---|---|
| **$y'$** | − | − | − | − | − |
| **Graph** | Dec. | Dec. | Dec. | Dec. | Dec. |
| **$y''$** | − | + | 0 | − | + |
| **Concavity** | Down | Up | Infl. | Down | Up |




**27. a.** $f'(x) = \frac{d}{dx}((-4)e^{5x+1})$

$$= (-4)e^{5x+1} \times \frac{d}{dx}(5x + 1)$$

$$= (-20)e^{5x+1}$$

**b.** $f'(x) = \frac{d}{dx}(xe^{3x})$

$$= xe^{3x} \times \frac{d}{dx}(3x) + (1)e^{3x}$$

$$= e^{3x}(3x + 1)$$

**c.** $y' = \frac{d}{dx}(6^{3x-8})$

$$= (\ln 6)6^{3x-8} \times \frac{d}{dx}(3x - 8)$$

$$= (3\ln 6)6^{3x-8}$$

**d.** $y' = \frac{d}{dx}(e^{\sin x})$

$$= e^{\sin x} \times \frac{d}{dx}(\sin x)$$

$$= (\cos x)e^{\sin x}$$

**28.** The slope of the tangent line at $x = 1$ can be found by evaluating the derivative $\frac{dy}{dx}$ for $x = 1$:

$$\frac{dy}{dx} = \frac{d}{dx}(e^{2x-1})$$

$$= e^{2x-1} \times \frac{d}{dx}(2x - 1)$$

$$= 2e^{2x-1}$$

Substituting $x = 1$ shows that the slope is $2e$. The value of the original function at $x = 1$ is $e$, so the equation of the tangent line at $x = 1$ is

$y = 2e(x - 1) + e$.

**29. a.** The maximum of the function modelling the number of bacteria infected can be found by examining its derivative.

$$N'(t) = \frac{d}{dt}((15t)e^{-\frac{t}{5}})$$

$$= 15te^{-\frac{t}{5}} \times \frac{d}{dt}\left(-\frac{t}{5}\right) + (15)e^{-\frac{t}{5}}$$

$$= e^{-\frac{t}{5}}(15 - 3t)$$

Setting $N'(t) = 0$ shows that $t = 5$ is the only critical point of the function (since the exponential function is never zero). The maximum number of infected bacteria therefore occurs after 5 days.

**b.** $N(5) = (15(5))e^{-\frac{5}{5}}$
$= 27$ bacteria

**30. a.** $\frac{dy}{dx} = \frac{d}{dx}(2 \sin x - 3 \cos 5x)$
$= 2 \cos x - 3(-\sin 5x) \times \frac{d}{dx}(5x)$
$= 2 \cos x + 15 \sin 5x$

**b.** $\frac{dy}{dx} = \frac{d}{dx}(\sin 2x + 1)^4$
$= 4(\sin 2x + 1)^3 \times \frac{d}{dx}(\sin 2x + 1)$
$= 4(\sin 2x + 1)^3 \times (\cos 2x) \times \frac{d}{dx}(2x)$
$= 8 \cos 2x(\sin 2x + 1)^3$

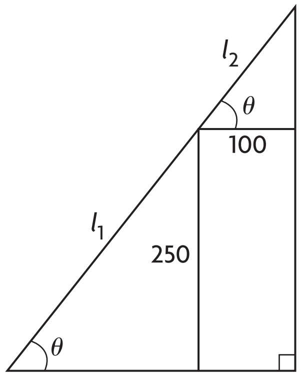
**c.** $y$ can be rewritten as $y = (x^2 + \sin 3x)^{\frac{1}{2}}$. Then,
$\frac{dy}{dx} = \frac{d}{dx}(x^2 + \sin 3x)^{\frac{1}{2}}$
$= \frac{1}{2}(x^2 + \sin 3x)^{-\frac{1}{2}} \times \frac{d}{dx}(x^2 + \sin 3x)$
$= \frac{1}{2}(x^2 + \sin 3x)^{-\frac{1}{2}}$
$\times \left(2x + \cos 3x \times \frac{d}{dx}(3x)\right)$
$= \frac{2x + 3 \cos 3x}{2\sqrt{x^2 + \sin 3x}}$

**d.** $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{\cos x + 2}\right)$
$= \frac{(\cos x + 2)(\cos x) - (\sin x)(-\sin x)}{(\cos x + 2)^2}$
$= \frac{\cos^2 x + \sin^2 x + 2 \cos x}{(\cos x + 2)^2}$
$= \frac{1 + 2 \cos x}{(\cos x + 2)^2}$

**e.** $\frac{dy}{dx} = \frac{d}{dx}(\tan x^2 - \tan^2 x)$
$= \frac{d}{dx}\sec^2 x^2 \times \frac{d}{dx}(x^2)$
$- 2 \tan x \times \frac{d}{dx}(\tan x)$
$= 2x \sec^2 x^2 - 2 \tan x \sec^2 x$

**f.** $\frac{dy}{dx} = \frac{d}{dx}(\sin(\cos x^2))$
$= \cos(\cos x^2) \times \frac{d}{dx}(\cos x^2)$
$= \cos(\cos x^2) \times (-\sin x^2) \times \frac{d}{dx}(x^2)$
$= -2x \sin x^2 \cos(\cos x^2)$

**31.**



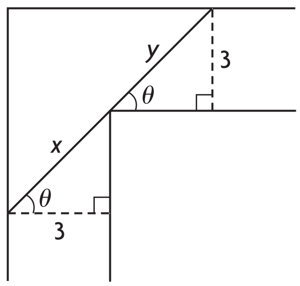

As shown in the diagram, let $\theta$ be the angle between the ladder and the ground, and let the total length of the ladder be $l = l_1 + l_2$, where $l_1$ is the length from the ground to the top corner of the shed and $l_2$ is the length from the corner of the shed to the wall.

$\sin \theta = \frac{250}{l_1}$ $\quad\quad$ $\cos \theta = \frac{100}{l_2}$
$l_1 = 250 \csc \theta$ $\quad\quad$ $l_2 = 100 \sec \theta$
$l = 250 \csc \theta + 100 \sec \theta$
$\frac{dl}{d\theta} = -250 \csc \theta \cot \theta + 100 \sec \theta \tan \theta$
$= -\frac{250 \cos \theta}{\sin^2 \theta} + \frac{100 \sin \theta}{\cos^2 \theta}$

To determine the minimum, solve $\frac{dl}{d\theta} = 0$.

$\frac{250 \cos \theta}{\sin^2 \theta} = \frac{100 \sin \theta}{\cos^2 \theta}$
$250 \cos^3 \theta = 100 \sin^3 \theta$
$2.5 = \tan^3 \theta$
$\tan \theta = \sqrt[3]{2.5}$
$\theta \doteq 0.94$
At $\theta = 0.94$, $l = 250 \csc 0.94 + 100 \sec 0.94$
$\doteq 479$ cm
The shortest ladder is about 4.8 m long.

**32.** The longest rod that can fit around the corner is determined by the minimum value of $x + y$. So, determine the minimum value of $l = x + y$.




From the diagram, $\sin\theta = \frac{3}{y}$ and $\cos\theta = \frac{3}{x}$. So,

$l = \frac{3}{\cos\theta} + \frac{3}{\sin\theta}$, for $0 \le \theta \le \frac{\pi}{2}$.

$$\begin{aligned}\frac{dl}{d\theta} &= \frac{3\sin\theta}{\cos^2\theta} - \frac{3\cos\theta}{\sin^2\theta}\\ &= \frac{3\sin^3\theta - 3\cos^3\theta}{\cos^2\theta\sin^2\theta}\end{aligned}$$

Solving $\frac{dl}{d\theta} = 0$ yields:

$$\begin{aligned}3\sin^3\theta - 3\cos^3\theta &= 0\\ \tan^3\theta &= 1\\ \tan\theta &= 1\\ \theta &= \frac{\pi}{4}\end{aligned}$$

So $l = \frac{3}{\cos\frac{\pi}{4}} + \frac{3}{\sin\frac{\pi}{4}}$

$= 3\sqrt{2} + 3\sqrt{2}$

$= 6\sqrt{2}$

When $\theta = 0$ or $\theta = \frac{\pi}{2}$, the longest possible rod would have a length of 3 m. Therefore the longest rod that can be carried horizontally around the corner is one of length $6\sqrt{2}$, or about 8.5 m.

# Chapter 5: Derivatives of Exponential and Trigonometric Functions

## Review of Prerequisite Skills, pp. 224–225

**1. a.** $3^{-2} = \frac{1}{3^2}$

$= \frac{1}{9}$

**b.** $32^{\frac{2}{5}} = \left(\sqrt[5]{32}\right)^2$

$= 2^2$

$= 4$

**c.** $27^{-\frac{2}{3}} = \frac{1}{\left(\sqrt[3]{27}\right)^2}$

$= \frac{1}{3^2}$

$= \frac{1}{9}$

**d.** $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2$

$= \frac{9}{4}$

**2. a.** $\log_5 625 = 4$

**b.** $\log_4 \frac{1}{16} = -2$

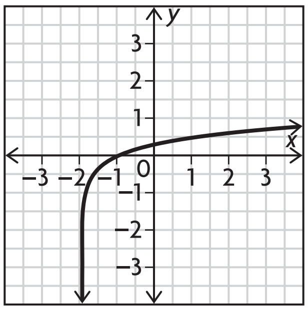
**c.** $\log_x 3 = 3$

**d.** $\log_{10} 450 = w$

**e.** $\log_3 z = 8$

**f.** $\log_a T = b$

**3. a.**




The $x$-intercept occurs where $y = 0$.

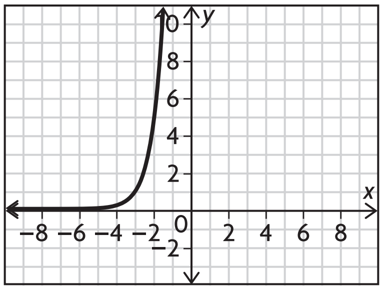
$0 = \log_{10}(x + 2)$

$10^0 = x + 2$

$x = -1$

The $x$-intercept is $(-1, 0)$.

**b.**




An exponential function is always positive, so there is no $x$-intercept.

**4. a.** $\sin\theta = \frac{y}{r}$

**b.** $\cos\theta = \frac{x}{r}$

**c.** $\tan\theta = \frac{y}{x}$

**5.** To convert to radian measure from degree measure, multiply the degree measure by $\frac{\pi}{180°}$.

**a.** $360° \times \frac{\pi}{180°} = 2\pi$

**b.** $45° \times \frac{\pi}{180°} = \frac{\pi}{4}$

**c.** $-90° \times \frac{\pi}{180°} = -\frac{\pi}{2}$

**d.** $30° \times \frac{\pi}{180°} = \frac{\pi}{6}$

**e.** $270° \times \frac{\pi}{180°} = \frac{3\pi}{2}$

**f.** $-120° \times \frac{\pi}{180°} = -\frac{2\pi}{3}$

**g.** $225° \times \frac{\pi}{180°} = \frac{5\pi}{4}$

**h.** $330° \times \frac{\pi}{180°} = \frac{11\pi}{6}$

**6.** For the unit circle, sine is associated with the $y$-coordinate of the point where the terminal arm of the angle meets the circle, and cosine is associated with the $x$-coordinate.

**a.** $\sin\theta = b$

**b.** $\tan\theta = \frac{b}{a}$

**c.** $\cos\theta = a$

**d.** $\sin\left(\frac{\pi}{2} - \theta\right) = a$

**e.** $\cos\left(\frac{\pi}{2} - \theta\right) = b$

**f.** $\sin(-\theta) = -b$

**7. a.** The angle is in the second quadrant, so cosine and tangent will be negative.

$\cos\theta = -\frac{12}{13}$

$\tan\theta = -\frac{5}{12}$

**b.** The angle is in the third quadrant, so sine will be negative and tangent will be positive.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin^2\theta + \frac{4}{9} = 1$$
$$\sin^2\theta = \frac{5}{9}$$
$$\sin\theta = -\frac{\sqrt{5}}{3}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$= \frac{\sqrt{5}}{2}$$

**c.** The angle is in the fourth quadrant, so cosine will be positive and sine will be negative.
Because $\tan\theta = -2$, the point $(1, -2)$ is on the terminal arm of the angle. The reference triangle for this angle has a hypotenuse of $\sqrt{2^2 + 1^2}$ or $\sqrt{5}$.

$\sin\theta = -\frac{2}{\sqrt{5}}$

$\cos\theta = \frac{1}{\sqrt{5}}$

**d.** The sine is only equal to 1 for one angle between 0 and $\pi$, so $\theta = \frac{\pi}{2}$.

$\cos\frac{\pi}{2} = 0$

$\tan\frac{\pi}{2}$ is undefined

**8. a.** The period is $\frac{2\pi}{2}$ or $\pi$. The amplitude is 1.

**b.** The period is $\frac{2\pi}{\frac{1}{2}}$ or $4\pi$. The amplitude is 2.

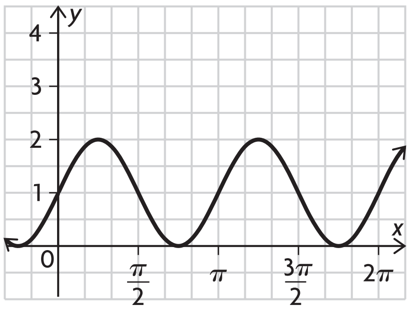
**c.** The period is $\frac{2\pi}{\pi}$ or 2. The amplitude is 3.

**d.** The period is $\frac{2\pi}{12}$ or $\frac{\pi}{6}$. The amplitude is $\frac{2}{7}$.

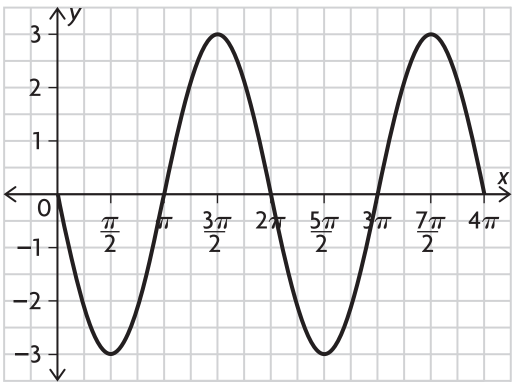
**e.** The period is $2\pi$. The amplitude is 5.

**f.** The period is $2\pi$. Because of the absolute value being taken, the amplitude is $\frac{3}{2}$.

**9. a.** The period is $\frac{2\pi}{2}$ or $\pi$. Graph the function from $x = 0$ to $x = 2\pi$.




**b.** The period is $2\pi$, so graph the function from $x = 0$ to $x = 4\pi$.




**10. a.** $\tan x + \cot x = \sec x \csc x$

$$\text{LS} = \tan x + \cot x$$
$$= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$
$$= \frac{\sin^2 x + \cos^2 x}{\cos x + \sin x}$$

$$= \frac{1}{\cos x + \sin x}$$
$$\text{RS} = \sec x + \csc x$$
$$= \frac{1}{\cos x} \cdot \frac{1}{\sin x}$$
$$= \frac{1}{\cos x \sin x}$$
Therefore, $\tan x + \cot x = \sec x \csc x$.

**b.** $\frac{\sin x}{1 - \sin^2 x} = \tan x + \sec x$
$$\text{LS} = \frac{\sin x}{1 - \sin^2 x}$$
$$= \frac{\sin x}{\cos^2 x}$$
$$\text{RS} = \tan x \sec x$$
$$= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$
$$= \frac{\sin x}{\cos^2 x}$$
Therefore, $\frac{\sin x}{1 - \sin^2 x} = \tan x \sec x$.

**11. a.** $3 \sin x = \sin x + 1$
$$2 \sin x = 1$$
$$\sin x = \frac{1}{2}$$
$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$
**b.** $\cos x - 1 = -\cos x$
$$2 \cos x = 1$$
$$\cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

## 5.1 Derivatives of Exponential Functions, $y = e^x$, pp. 232–234

**1.** You can only use the power rule when the term containing variables is in the base of the exponential expression. In the case of $y = e^x$, the exponent contains a variable.

**2. a.** $y = e^{3x}$
$$\frac{dy}{dx} = 3e^{3x}$$
**b.** $s = e^{3t-5}$
$$\frac{ds}{dt} = 3e^{3t-5}$$
**c.** $y = 2e^{10t}$
$$\frac{dy}{dt} = 20e^{10t}$$
**d.** $y = e^{-3x}$
$$\frac{dy}{dx} = -3e^{-3x}$$
**e.** $y = e^{5-6x+x^2}$
$$\frac{dy}{dx} = (-6 + 2x)e^{5-6x+x^2}$$
**f.** $y = e^{\sqrt{x}}$
$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}e^{\sqrt{x}}$$
**3. a.** $y = 2e^{x^3}$
$$\frac{dy}{dx} = 2(3x^2)e^{x^3}$$
$$= 6x^2e^{x^3}$$
**b.** $\frac{dy}{dx} = \frac{d(xe^{3x})}{dx}$
$$= (x)(3e^{3x}) + (e^{3x})(1)$$
$$= 3xe^{3x} + e^{3x}$$
$$= e^{3x}(3x + 1)$$
**c.** $f(x) = \frac{e^{-x^3}}{x}$
$$f'(x) = \frac{-3x^2e^{-x^3}(x) - e^{-x^3}}{x^2}$$
**d.** $f(x) = \sqrt{x}e^x$
$$f'(x) = \sqrt{x}e^x + e^x\left(\frac{1}{2\sqrt{x}}\right)$$
**e.** $h(t) = e^{t^2} + 3e^{-t}$
$$h'(t) = 2te^{t^2} - 3e^{-t}$$
**f.** $g(t) = \frac{e^{2t}}{1 + e^{2t}}$
$$g'(t) = \frac{2e^{2t}(1 + e^{2t}) - 2e^{2t}(e^{2t})}{(1 + e^{2t})^2}$$
$$= \frac{2e^{2t}}{(1 + e^{2t})^2}$$
**4. a.** $f'(x) = \frac{1}{3}(3e^{3x} - 3e^{-3x})$
$$= e^{3x} - e^{-3x}$$
$$f'(1) = e^3 - e^{-3}$$
**b.** $f(x) = e^{-\frac{1}{x+1}}$
$$f'(x) = e^{-\frac{1}{x+1}}\left(\frac{1}{(x + 1)^2}\right)$$
$$f'(0) = e^{-1}(1)$$
$$= \frac{1}{e}$$

$$= \frac{1}{\cos x + \sin x}$$
$$\text{RS} = \sec x + \csc x$$
$$= \frac{1}{\cos x} \cdot \frac{1}{\sin x}$$
$$= \frac{1}{\cos x \sin x}$$
Therefore, $\tan x + \cot x = \sec x \csc x$.

**b.** $\frac{\sin x}{1 - \sin^2 x} = \tan x + \sec x$
$$\text{LS} = \frac{\sin x}{1 - \sin^2 x}$$
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$$= e^{3x}(3x + 1)$$
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$$f'(x) = \sqrt{x}e^x + e^x\left(\frac{1}{2\sqrt{x}}\right)$$
**e.** $h(t) = e^{t^2} + 3e^{-t}$
$$h'(t) = 2te^{t^2} - 3e^{-t}$$
**f.** $g(t) = \frac{e^{2t}}{1 + e^{2t}}$
$$g'(t) = \frac{2e^{2t}(1 + e^{2t}) - 2e^{2t}(e^{2t})}{(1 + e^{2t})^2}$$
$$= \frac{2e^{2t}}{(1 + e^{2t})^2}$$
**4. a.** $f'(x) = \frac{1}{3}(3e^{3x} - 3e^{-3x})$
$$= e^{3x} - e^{-3x}$$
$$f'(1) = e^3 - e^{-3}$$
**b.** $f(x) = e^{-\frac{1}{x+1}}$
$$f'(x) = e^{-\frac{1}{x+1}}\left(\frac{1}{(x + 1)^2}\right)$$
$$f'(0) = e^{-1}(1)$$
$$= \frac{1}{e}$$

**c.** $h'(z) = 2z(1 + e^{-z}) + z^2(-e^{-z})$

$$\begin{aligned} h'(-1) &= 2(-1)(1 + e) + (-1)^2(-e^1) \\ &= -2 - 2e - e \\ &= -2 - 3e \end{aligned}$$

**5. a.** $y = \dfrac{2e^x}{1 + e^x}$

$$\frac{dy}{dx} = \frac{(1 + e^x)2e^x - 2e^x(e^x)}{(1 + e^x)^2}$$

$$\frac{dy}{dx} = \frac{2(2) - 2(1)(1)}{2^2} = \frac{1}{2}$$

When $x = 0$,

the slope of the tangent is $\frac{1}{2}$.

The equation of the tangent is $y = \frac{1}{2}x + 1$, since the $y$-intercept was given as $(0, 1)$.

**b.**

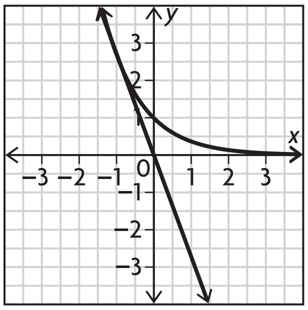

**c.** The answers agree very well; the calculator does not show a slope of exactly 0.5, due to internal rounding.

**6.** $y = e^{-x}$

$$\frac{dy}{dx} = -e^{-x}$$

When $x = -1$, $\dfrac{dy}{dx} = -e$. And when $x = -1$, $y = e$.

The equation of the tangent is $y - e = -e(x + 1)$ or $ex + y = 0$.




**7.** The slope of the tangent line at any point is given by

$$\begin{aligned} \frac{dy}{dx} &= (1)(e^{-x}) + x(-e^{-x}) \\ &= e^{-x}(1 - x). \end{aligned}$$

At the point $(1, e^{-1})$, the slope is $e^{-1}(0) = 0$. The equation of the tangent line at the point $A$ is

$y - e^{-1} = 0(x - 1)$ or $y = \frac{1}{e}$.

**8.** The slope of the tangent line at any point on the curve is

$$\begin{aligned} \frac{dy}{dx} &= 2xe^{-x} + x^2(e^{-x}) \\ &= (2x - x^2)(e^{-x}) \\ &= \frac{2x - x^2}{e^x}. \end{aligned}$$

Horizontal lines have slope equal to 0.

We solve $\dfrac{dy}{dx} = 0$

$$\frac{x(2 - x)}{e^x} = 0.$$

Since $e^x > 0$ for all $x$, the solutions are $x = 0$ and $x = 2$. The points on the curve at which the tangents are horizontal are $(0, 0)$ and $\left(2, \frac{4}{e^2}\right)$.

**9.** If $y = \dfrac{5}{2}(e^{\frac{x}{5}} + e^{-\frac{x}{5}})$, then

$$y' = \frac{5}{2}\left(\frac{1}{5}e^{\frac{x}{5}} - \frac{1}{5}e^{-\frac{x}{5}}\right), \text{ and}$$

$$\begin{aligned} y'' &= \frac{5}{2}\left(\frac{1}{25}e^{\frac{x}{5}} + \frac{1}{25}e^{-\frac{x}{5}}\right) \\ &= \frac{1}{25}\left[\frac{5}{2}(e^{\frac{x}{5}} + e^{-\frac{x}{5}})\right] \\ &= \frac{1}{25}y. \end{aligned}$$

**10. a.** $y = e^{-3x}$

$$\frac{dy}{dx} = -3e^{-3x}$$

$$\frac{d^2y}{dx^2} = 9e^{-3x}$$

$$\frac{d^3y}{dx^3} = -27e^{-3x}$$

**b.** $\dfrac{d^ny}{dx^n} = (-1)^n(3^n)e^{-3x}$

**11. a.** $\dfrac{dy}{dx} = \dfrac{d(-3e^x)}{dx} = -3e^x$

$$\frac{d^2y}{dx^2} = -3e^x$$

**b.** 
$$\begin{aligned} \frac{dy}{dx} &= \frac{d(xe^{2x})}{dx} \\ &= (x)(2e^{2x}) + (e^{2x})(1) \\ &= 2xe^{2x} + e^{2x} \\ &= e^{2x}(2x + 1) \end{aligned}$$

$$\frac{d^2y}{dx^2} = e^{2x}(2) + (2x + 1)(2e^{2x})$$
$$= 4xe^{2x} + 4e^{2x}$$

**c.** $\frac{dy}{dx} = \frac{d(e^x(4 - x))}{dx}$
$$= (e^x)(-1) + (4 - x)(e^x)$$
$$= -e^x + 4e^x - xe^x$$
$$= 3e^x - xe^x$$
$$= e^x(3 - x)$$
$$\frac{d^2y}{dx^2} = e^x(-1) + (3 - x)(e^x)$$
$$= 2e^x - xe^x$$
$$= e^x(2 - x)$$

**12. a.** When $t = 0$, $N = 1000[30 + e^0] = 31\,000$.

**b.** $\frac{dN}{dt} = 1000\left[0 - \frac{1}{30}e^{\frac{-t}{30}}\right] = -\frac{100}{3}e^{-\frac{t}{30}}$

**c.** When $t = 20h$, $\frac{dN}{dt} = -\frac{100}{3}e^{-\frac{2}{3}} \doteq -17$ bacteria/h.

**d.** Since $e^{-\frac{t}{30}} > 0$ for all $t$, there is no solution to $\frac{dN}{dt} = 0$.
Hence, the maximum number of bacteria in the culture occurs at an endpoint of the interval of domain.
When $t = 50$, $N = 1000[30 + e^{-\frac{5}{3}}] \doteq 30\,189$.
The largest number of bacteria in the culture is 31 000 at time $t = 0$.

**e.** The number of bacteria is constantly decreasing as time passes.

**13. a.** $v = \frac{ds}{dt} = 160\left(\frac{1}{4} - \frac{1}{4}e^{-\frac{t}{4}}\right)$
$$= 40(1 - e^{-\frac{t}{4}})$$

**b.** $a = \frac{dv}{dt} = 40\left(\frac{1}{4}e^{-\frac{t}{4}}\right) = 10e^{-\frac{t}{4}}$

From **a.**, $v = 40(1 - e^{-\frac{t}{4}})$, which gives $e^{-\frac{t}{4}} = 1 - \frac{v}{40}$.

Thus, $a = 10\left(1 - \frac{v}{40}\right) = 10 - \frac{1}{4}v$.

**c.** $v_T = \lim_{t\to\infty} v$
$$v_T = \lim_{t\to\infty} 40(1 - e^{-\frac{t}{4}})$$
$$= 40 \lim_{t\to\infty}\left(1 - \frac{1}{e^{\frac{t}{4}}}\right)$$
$$= 40(1), \text{ since } \lim_{t\to\infty}\frac{1}{e^{\frac{t}{4}}} = 0$$
The terminal velocity of the skydiver is 40 m/s.

**d.** 95% of the terminal velocity is
$\frac{95}{100}(40) = 38$ m/s.
To determine when this velocity occurs, we solve
$40(1 - e^{-\frac{t}{4}}) = 38$
$$1 - e^{-\frac{t}{4}} = \frac{38}{40}$$
$$e^{-\frac{t}{4}} = \frac{1}{20}$$
$$e^{\frac{t}{4}} = 20$$
$$\text{and } \frac{t}{4} = \ln 20,$$
which gives $t = 4 \ln 20 \doteq 12$ s.
The skydiver's velocity is 38 m/s, 12 s after jumping.
The distance she has fallen at this time is
$$S = 160(\ln 20 - 1 + e^{-20})$$
$$= 160\left(\ln 20 - 1 + \frac{1}{20}\right)$$
$$\doteq 327.3 \text{ m.}$$

**14. a. i.** Let $f(x) = \left(1 + \frac{1}{x}\right)^x$. Then,

| $x$ | $f(x)$ |
|---|---|
| 1 | 2 |
| 10 | 2.5937 |
| 100 | 2.7048 |
| 1000 | 2.7169 |
| 10 000 | 2.7181 |

So, from the table one can see that $\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e$.

**ii.** Let $f(x) = (1 + x)^{\frac{1}{x}}$.

| $x$ | $f(x)$ |
|---|---|
| −0.1 | 2.8680 |
| −0.01 | 2.7320 |
| −0.001 | 2.7196 |
| −0.0001 | 2.7184 |
| ? | ? |
| 0.0001 | 2.7181 |
| 0.001 | 2.7169 |
| 0.01 | 2.7048 |
| 0.1 | 2.5937 |

So, from the table one can see that $\lim_{x\to 0}(1 + x)^{\frac{1}{x}} = e$.
That is, the limit approaches the value of $e = 2.718\,281\,828\ldots$

**b.** The limits have the same value because as $x \to \infty$, $\frac{1}{x} \to 0$.

**15. a.** The given limit can be rewritten as

$$\lim_{h\to 0}\frac{e^h - 1}{h} = \lim_{h\to 0}\frac{e^{0+h} - e^0}{h}$$

This expression is the limit definition of the derivative at $x = 0$ for $f(x) = e^x$.

$$\left[f'(0) = \lim_{h\to 0}\frac{e^{0+h} - e^0}{h}\right]$$

Since $f'(x) = \frac{de^x}{dx} = e^x$, the value of the given limit is $e^0 = 1$.

**b.** Again, $\lim_{h\to 0}\frac{e^{2+h} - e^2}{h}$ is the derivative of $e^x$ at $x = 2$.

Thus, $\lim_{h\to 0}\frac{e^{2+h} - e^2}{h} = e^2$.

**16.** For $y = Ae^{ex}$, $\frac{dy}{dt} = Ame^{ex}$ and $\frac{d^2y}{dt^2} = Am^2e^{ex}$.

Substituting in the differential equation gives

$$Am^2e^{ex} + Ame^{ex} - 6Ae^{ex} = 0$$
$$Ae^{ex}(m^2 + m - 6) = 0.$$

Since $Ae^{ex} \neq 0$, $m^2 + m - 6 = 0$

$$(m + 3)(m - 2) = 0$$
$$m = -3 \text{ or } m = 2.$$

**17. a.**
$$\frac{d}{dx}\sinh x = \frac{d}{dx}\left[\frac{1}{2}(e^x - e^{-x})\right] = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

**b.**
$$\frac{d}{dx}\cosh x = \frac{1}{2}(e^x - e^{-x}) = \sinh x$$

**c.** Since $\tanh x = \frac{\sinh x}{\cosh x}$,

$\frac{d}{dx}\tanh x$

$$= \frac{\left(\frac{d}{dx}\sinh x\right)(\cosh x) - (\sinh x)\left(\frac{d}{dx}\cosh x\right)}{(\cosh x)^2}$$

$$= \frac{\frac{1}{2}(e^x + e^{-x})\left(\frac{1}{2}\right)(\cosh x)^2(e^x + e^{-x})}{(\cosh x)^2} - \frac{\frac{1}{2}(e^x - e^{-x})\left(\frac{1}{2}\right)(e^x - e^{-x})}{(\cosh x)^2}$$

$$= \frac{\frac{1}{4}\left[(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})\right]}{(\cosh x)^2}$$

$$= \frac{\frac{1}{4}(4)}{(\cosh x)^2}$$

$$= \frac{1}{(\cosh x)^2}$$

**18. a.** Four terms:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} = 2.666\,66\overline{6}$$

Five terms:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 2.708\,33\overline{3}$$

Six terms:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = 2.716\,66\overline{6}$$

Seven terms:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} = 2.718\,05\overline{5}$$

**b.** The expression for $e$ in part **a.** is a special case of $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ in that it is the case when $x = 1$. Then $e^x = e^1 = e$ is in fact $e^1 = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$. The value of $x$ is 1.

## 5.2 Derivatives of the General Exponential Function, $y = b^x$, p. 240

**1. a.**
$$\frac{dy}{dx} = \frac{d(2^{3x})}{dx} = 3(2^{3x})\ln 2$$

**b.**
$$\frac{dy}{dx} = \frac{d(3.1^x + x^3)}{dx} = \ln 3.1(3.1)^x + 3x^2$$

**c.**
$$\frac{ds}{dt} = \frac{d(10^{3t-5})}{dt} = 3(10^{3t-5})\ln 10$$

**d.**
$$\frac{dw}{dn} = \frac{d(10^{5-6n+n^2})}{dn} = (-6 + 2n)(10^{5-6n+n^2})\ln 10$$

**e.**
$$\frac{dy}{dx} = \frac{d(3^{x^2+2})}{dx} = 2x(3^{x^2+2})\ln 3$$

**f.**
$$\frac{dy}{dx} = \frac{d(400(2)^{x+3})}{dx} = 400(2)^{x+3}\ln 2$$

**15. a.** The given limit can be rewritten as

$$\lim_{h\to 0}\frac{e^h-1}{h}=\lim_{h\to 0}\frac{e^{0+h}-e^0}{h}$$

This expression is the limit definition of the derivative at $x = 0$ for $f(x) = e^x$.

$$\left[f'(0)=\lim_{h\to 0}\frac{e^{0+h}-e^0}{h}\right]$$

Since $f'(x) = \frac{de^x}{dx} = e^x$, the value of the given limit is $e^0 = 1$.

**b.** Again, $\lim_{h\to 0}\frac{e^{2+h}-e^2}{h}$ is the derivative of $e^x$ at $x = 2$.

Thus, $\lim_{h\to 0}\frac{e^{2+h}-e^2}{h} = e^2$.

**16.** For $y = Ae^{ex}$, $\frac{dy}{dt} = Ame^{ex}$ and $\frac{d^2y}{dt^2} = Am^2e^{ex}$.

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$$(m+3)(m-2) = 0$$
$$m = -3 \text{ or } m = 2.$$

**17. a.** $\frac{d}{dx}\sinh x = \frac{d}{dx}\left[\frac{1}{2}(e^x - e^{-x})\right]$

$$= \frac{1}{2}(e^x + e^{-x})$$
$$= \cosh x$$

**b.** $\frac{d}{dx}\cosh x = \frac{1}{2}(e^x - e^{-x})$

$$= \sinh x$$

**c.** Since $\tanh x = \frac{\sinh x}{\cosh x}$,

$\frac{d}{dx}\tanh x$

$$= \frac{\left(\frac{d}{dx}\sinh x\right)(\cosh x) - (\sinh x)\left(\frac{d}{dx}\cosh x\right)}{(\cosh x)^2}$$

$$= \frac{\frac{1}{2}(e^x + e^{-x})\left(\frac{1}{2}\right)(\cosh x)^2\,(e^x + e^{-x})}{(\cosh x)^2}$$

$$- \frac{\frac{1}{2}(e^x - e^{-x})\left(\frac{1}{2}\right)(e^x - e^{-x})}{(\cosh x)^2}$$

$$= \frac{\frac{1}{4}\left[(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})\right]}{(\cosh x)^2}$$

$$= \frac{\frac{1}{4}(4)}{(\cosh x)^2}$$

$$= \frac{1}{(\cosh x)^2}$$

**18. a.** Four terms:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} = 2.666\,66\overline{6}$$

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$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = 2.716\,66\overline{6}$$

Seven terms:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} = 2.718\,05\overline{5}$$

**b.** The expression for $e$ in part **a.** is a special case of $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ in that it is the case when $x = 1$. Then $e^x = e^1 = e$ is in fact $e^1 = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$ The value of $x$ is 1.

## 5.2 Derivatives of the General Exponential Function, $y = b^x$, p. 240

**1. a.** $\frac{dy}{dx} = \frac{d(2^{3x})}{dx}$

$$= 3(2^{3x})\ln 2$$

**b.** $\frac{dy}{dx} = \frac{d(3.1^x + x^3)}{dx}$

$$= \ln 3.1(3.1)^x + 3x^2$$

**c.** $\frac{ds}{dt} = \frac{d(10^{3t-5})}{dt}$

$$= 3(10^{3t-5})\ln 10$$

**d.** $\frac{dw}{dn} = \frac{d(10^{5-6n+n^2})}{dn}$

$$= (-6 + 2n)(10^{5-6n+n^2})\ln 10$$

**e.** $\frac{dy}{dx} = \frac{d(3^{x^2+2})}{dx}$

$$= 2x(3^{x^2+2})\ln 3$$

**f.** $\frac{dy}{dx} = \frac{d(400(2)^{x+3})}{dx}$

$$= 400(2)^{x+3}\ln 2$$

**2. a.** $\frac{dy}{dx} = \frac{d(x^5 \times (5)^x)}{dx}$
$= (x^5)((5)^x(\ln 5)) + ((5)^x)(5x^4)$
$= 5^x[(x^5 \times \ln 5) + 5x^4]$

**b.** $\frac{dy}{dx} = \frac{d(x(3)^{x^2})}{dx}$
$= (x)(2x(3)^{x^2} \ln 3) + (3)^{x^2}(1)$
$= (3)^{x^2}[(2x^2 \ln 3) + 1]$

**c.** $v = (2^t)(t^{-1})$
$\frac{dv}{dt} = \frac{d((2^t)(t^{-1}))}{dx}$
$= (2^t)(-1t^{-2}) + (t^{-1})(2^t \ln 2)$
$= -\frac{2^t}{t^2} + \frac{2^t \ln 2}{t}$

**d.** $f(x) = \frac{3^{\frac{x}{2}}}{x^2}$
$f'(x) = \frac{\frac{1}{2}\ln 3(3^{\frac{x}{2}})(x^2) - 2x(3^{\frac{x}{2}})}{x^4}$
$= \frac{x \ln 3(3^{\frac{x}{2}}) - 4(3^{\frac{x}{2}})}{x^4}$
$= \frac{3^{\frac{x}{2}}[x \ln 3 - 4]}{x^3}$

**3.** $f(t) = 10^{3t-5} \cdot e^{2t^2}$
$f'(t) = (10^{3t-5})(4te^{2t^2}) + (e^{2t^2})(3(10)^{3t-5} \ln 10)$
$= 10^{3t-5}e^{2t^2}(4t + 3 \ln 10)$
Now, set $f'(t) = 0$.
So, $f'(t) = 0 = 10^{3t-5}e^{2t^2}(4t + 3\ln 10)$
So $10^{3t-5}e^{2t^2} = 0$ and $4t + 3 \ln 10 = 0$.
The first equation never equals zero because solving would force one to take the natural log of both sides, but ln 0 is undefined. So the first equation does not produce any values for which $f'(t) = 0$. The second equation does give one value.
$4t + 3 \ln 10 = 0$
$4t = -3 \ln 10$
$t = -\frac{3 \ln 10}{4}$

**4.** When $x = 3$, the function $y = f(x)$ evaluated at 3 is $f(3) = 3(2^3) = 3(8) = 24$. Also,
$\frac{dy}{dx} = \frac{d(3(2)^x)}{dx}$
$= 3(2^x)\ln 2$
So, at $x = 3$,
$\frac{dy}{dx} = 3(2^3)(\ln 2) = 24(\ln 2) \doteq 16.64$
Therefore, $y - 24 = 16.64(x - 3)$
$y - 24 = 16.64x - 49.92$
$-16.64x + y + 25.92 = 0$

**5.** $\frac{dy}{dx} = \frac{d(10^x)}{dx}$
$= 10^x \ln 10$
So, at $x = 1$,
$\frac{dy}{dx} = 10^1 \ln 10 = 10(\ln 10) \doteq 23.03$
Therefore, $y - 10 = 23.03(x - 1)$
$y - 10 = 23.03x - 23.03$
$-23.03x + y + 13.03 = 0$

**6. a.** The half-life of the substance is the time required for half of the substance to decay. That is, it is when 50% of the substance is left, so $P(t) = 50$.
$50 = 100(1.2)^{-t}$
$\frac{1}{2} = (1.2)^{-t}$
$\frac{1}{2} = \frac{1}{(1.2)^t}$
$(1.2)^t = 2$
$t(\ln 1.2) = \ln 2$
$t = \frac{\ln 1.2}{\ln 2}$
$t \doteq 3.80$ years
Therefore, the half-life of the substance is about 3.80 years.

**b.** The problem asks for the rate of change when $t \doteq 3.80$ years.
$P'(t) = -100(1.2)^{-t}(\ln 1.2)$
$P'(3.80) = -100(1.2)^{-(3.80)}(\ln 1.2)$
$\doteq -9.12$
So, the substance is decaying at a rate of about $-9.12$ percent/year at the time 3.80 years where the half-life is reached.

**7.** $P = 0.5(10^9)e^{0.200\,15t}$

**a.** $\frac{dP}{dt} = 0.5(10^9)(0.200\,15)e^{0.200\,15t}$
In 1968, $t = 1$ and $\frac{dP}{dt} = 0.5(10^9)(0.200\,15)e^{0.200\,15} \doteq$ $0.122\,25 \times 10^9$ dollars/annum
In 1978, $t = 11$ and
$\frac{dP}{dt} = 0.5(10^9)(0.200\,15)e^{11 \times 0.200\,15}$
$\doteq 0.904\,67 \times 10^9$ dollars/annum.
In 1978, the rate of increase of debt payments was \$904 670 000/annum compared to \$122 250 000/annum in 1968. As a ratio, $\frac{\text{Rate in 1978}}{\text{Rate in 1968}} = \frac{7.4}{1}$. The rate of increase for 1978 is 7.4 times larger than that for 1968.

**b.** In 1988, $t = 21$ and

$$\frac{dP}{dt} = 0.5(10^9)(0.20015)e^{21 \times 0.200\,15}$$
$$\doteq 6.694\,69 \times 10^9 \text{ dollars/annum}$$

In 1998, $t = 31$ and

$$\frac{dP}{dt} = 0.5(10^9)(0.20015)e^{31 \times 0.200\,15}$$
$$\doteq 49.541\,69 \times 10^9 \text{ dollars/annum}$$

As a ratio, $\frac{\text{Rate in 1998}}{\text{Rate in 1988}} = \frac{7.4}{1}$. The rate of increase for 1998 is 7.4 times larger than that for 1988.

**c.** Answers may vary. For example, data from the past are not necessarily good indicators of what will happen in the future. Interest rates change, borrowing may decrease, principal may be paid off early.

**8.** When $x = 0$, the function $y = f(x)$ evaluated at 0 is $f(0) = 2^{-0^2} = 2^0 = 1$. Also,

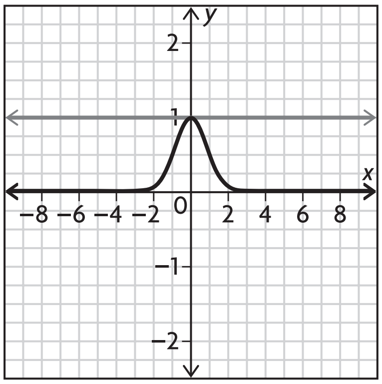
$$\frac{dy}{dx} = \frac{d(2^{-x^2})}{dx}$$
$$= -2x(2^{-x^2})\ln 2$$

So, at $x = 0$,

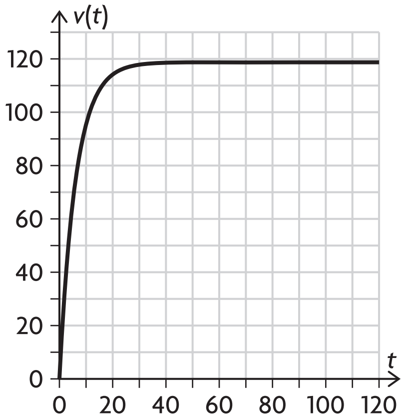
$$\frac{dy}{dx} = -2(0)(2^{-0^2})\ln 2 = 0$$

Therefore, $y - 1 = 0(x - 0)$

So, $y - 1 = 0$ or $y = 1$.



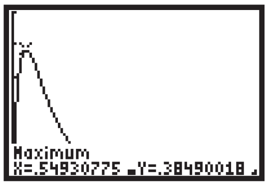

**9.**

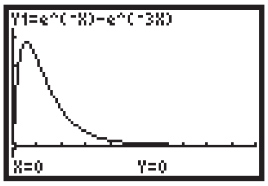



From the graph, one can notice that the values of $v(t)$ quickly rise in the range of about $0 \le t \le 15$. The slope for these values is positive and steep. Then as the graph nears $t = 20$ the steepness of the slope decreases and seems to get very close to 0. One can reason that the car quickly accelerates for the first 20 units of time. Then, it seems to maintain a constant acceleration for the rest of the time. To verify this, one could differentiate and look at values where $v'(t)$ is increasing.

## 5.3 Optimization Problems Involving Exponential Functions, pp. 245–247

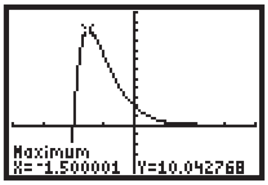
**1. a.**

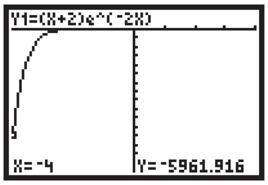



The maximum value is about 0.3849. The minimum value is 0.

**b.**




The maximum value is about 10.043. The minimum value is about $-5961.916$.

**2. a.** $f(x) = e^{-x} - e^{-3x}$ on $0 \le x \le 10$

$f'(x) = -e^{-x} + 3e^{-3x}$

Let $f'(x) = 0$, therefore $e^{-x} + 3e^{-3x} = 0$.

Let $e^{-x} = w$, when $-w + 3w^3 = 0$.

$w(-1 + 3w^2) = 0$.

Therefore, $w = 0$ or $w^2 = \frac{1}{3}$

$$w = \pm\frac{1}{\sqrt{3}}.$$

But $w \ge 0$, $w = +\frac{1}{\sqrt{3}}$.

**b.** In 1988, $t = 21$ and

$$\frac{dP}{dt} = 0.5(10^9)(0.20015)e^{21 \times 0.200\,15}$$
$$\doteq 6.694\,69 \times 10^9 \text{ dollars/annum}$$

In 1998, $t = 31$ and

$$\frac{dP}{dt} = 0.5(10^9)(0.20015)e^{31 \times 0.200\,15}$$
$$\doteq 49.541\,69 \times 10^9 \text{ dollars/annum}$$

As a ratio, $\frac{\text{Rate in 1998}}{\text{Rate in 1988}} = \frac{7.4}{1}$. The rate of increase for 1998 is 7.4 times larger than that for 1988.

**c.** Answers may vary. For example, data from the past are not necessarily good indicators of what will happen in the future. Interest rates change, borrowing may decrease, principal may be paid off early.

**8.** When $x = 0$, the function $y = f(x)$ evaluated at 0 is $f(0) = 2^{-0^2} = 2^0 = 1$. Also,

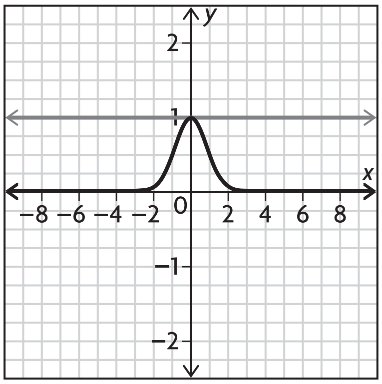
$$\frac{dy}{dx} = \frac{d(2^{-x^2})}{dx}$$
$$= -2x(2^{-x^2})\ln 2$$

So, at $x = 0$,

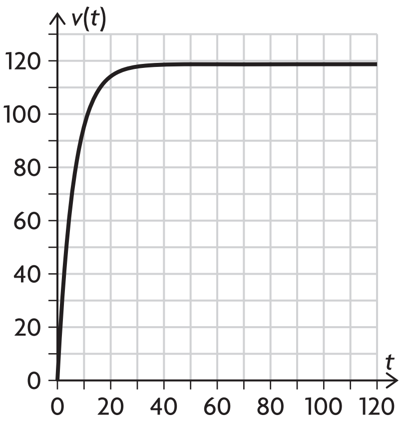
$$\frac{dy}{dx} = -2(0)(2^{-0^2})\ln 2 = 0$$

Therefore, $y - 1 = 0(x - 0)$

So, $y - 1 = 0$ or $y = 1$.



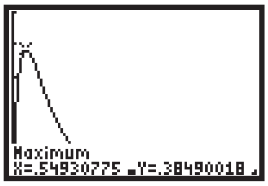

**9.**

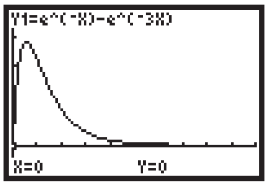



From the graph, one can notice that the values of $v(t)$ quickly rise in the range of about $0 \le t \le 15$. The slope for these values is positive and steep. Then as the graph nears $t = 20$ the steepness of the slope decreases and seems to get very close to 0. One can reason that the car quickly accelerates for the first 20 units of time. Then, it seems to maintain a constant acceleration for the rest of the time. To verify this, one could differentiate and look at values where $v'(t)$ is increasing.

## 5.3 Optimization Problems Involving Exponential Functions, pp. 245–247

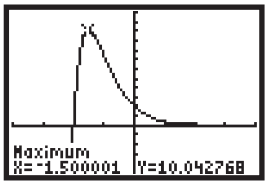
**1. a.**

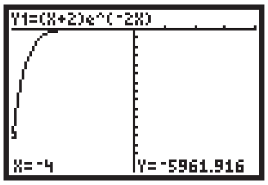



The maximum value is about 0.3849. The minimum value is 0.

**b.**




The maximum value is about 10.043. The minimum value is about −5961.916.

**2. a.** $f(x) = e^{-x} - e^{-3x}$ on $0 \le x \le 10$

$f'(x) = -e^{-x} + 3e^{-3x}$

Let $f'(x) = 0$, therefore $e^{-x} + 3e^{-3x} = 0$.

Let $e^{-x} = w$, when $-w + 3w^3 = 0$.

$w(-1 + 3w^2) = 0$.

Therefore, $w = 0$ or $w^2 = \frac{1}{3}$

$$w = \pm\frac{1}{\sqrt{3}}.$$

But $w \ge 0$, $w = +\frac{1}{\sqrt{3}}$.

When $w = \frac{1}{\sqrt{3}}$, $e^{-x} = \frac{1}{\sqrt{3}}$,

$$-x \ln e = \ln 1 - \ln \sqrt{3}$$
$$x = \frac{\ln \sqrt{3} - \ln 1}{1}$$
$$= \ln \sqrt{3}$$
$$\doteq 0.55.$$

$$f(0) = e^0 - e^0 = 0$$
$$f(0.55) \doteq 0.3849$$
$$f(10) = e^{-10} - e^{-30} \doteq 0.00005$$

Absolute maximum is about 0.3849 and absolute minimum is 0.

$m(x) = (x + 2)e^{-2x}$ on $-4 \le x \le 4$

$m'(x) = e^{-2x} + (-2)(x + 2)e^{-2x}$

Let $m'(x) = 0$.

$e^{-2x} \neq 0$, therefore, $1 + (-2)(x + 2) = 0$

$$x = \frac{-3}{2} = -1.5.$$

$$m(-4) = -2e^8 \doteq -5961$$
$$m(-1.5) = 0.5e^3 \doteq 10$$
$$m(4) = 6e^{-8} \doteq 0.0002$$

The maximum value is about 10 and the minimum value is about $-5961$.

**b.** The graphing approach seems to be easier to use for the functions. It is quicker and it gives the graphs of the functions in a good viewing rectangle. The only problem may come in the second function, $m(x)$, because for $x < 1.5$ the function quickly approaches values in the negative thousands.

**3. a.** $P(t) = \frac{20}{1 + 3e^{-0.02t}}$

$$P(0) = \frac{20}{1 + 3e^{-0.02(0)}} = \frac{20}{1 + 3e^0} = \frac{20}{4} = 5$$

So, the population at the start of the study when $t = 0$ is 500 squirrels.

**b.** The question asks for $\lim_{t\to\infty} P(t)$.

As $t$ approaches $\infty$, $e^{-0.02t} = \frac{1}{e^{0.02t}}$ approaches 0.

So, $\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{20}{1 + 3e^{-0.02t}}$

$$= \frac{20}{1 + 3(0)} = 20.$$

Therefore, the largest population of squirrels that the forest can sustain is 2000 squirrels.

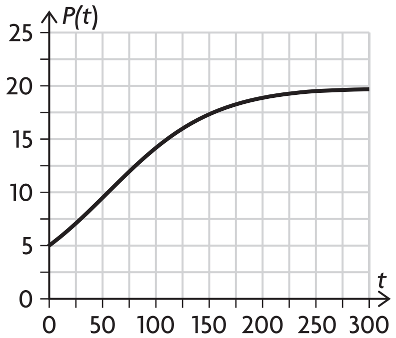
**c.** A point of inflection can only occur when $P''(t) = 0$ and concavity changes around the point.

$$P(t) = \frac{20}{1 + 3e^{-0.02t}}$$
$$P(t) = 20(1 + 3e^{-0.02t})^{-1}$$
$$P'(t) = 20(-(1 + 3e^{-0.02t})^{-2}(-0.06e^{-0.02t}))$$
$$= (1.2e^{-0.02t})(1 + 3e^{-0.02t})^{-2}$$
$$P''(t) = [(1.2e^{-0.02t})(-2(1 + 3e^{-0.02t})^{-3}(-0.06e^{-0.02t})] + (1 + 3e^{-0.02t})^{-2}(-0.024e^{-0.02t})$$
$$= \frac{0.144e^{-0.04t}}{(1 + 3e^{-0.02t})^3} - \frac{0.024e^{-0.02t}}{(1 + 3e^{-0.02t})^2}$$

$P''(0)$ when $\frac{0.144e^{-0.04t}}{(1 + 3e^{-0.02t})^3} - \frac{0.024e^{-0.02t}}{(1 + 3e^{-0.02t})^2} = 0$

Solving for $t$ after setting the second derivative equal to 0 is very tedious. Use a graphing calculator to determine the value of $t$ for which the second derivative is 0, 54.9. Evaluate $P(54.9)$. The point of inflection is (54.9, 10).

**d.**




**e.** $P$ grows exponentially until the point of inflection, then the growth rate decreases and the curve becomes concave down.

**4. a.** $P(x) = 10^6[1 + (x - 1)e^{-0.001x}]$, $0 \le x \le 2000$

Using the Algorithm for Extreme Values, we have

$$P(0) = 10^6[1 - 1] = 0$$
$$P(2000) = 10^6[1 + 1999e^{-2}] \doteq 271.5 \times 10^6.$$

Now,

$$P'(x) = 10^6[(1)e^{-0.001x} + (x - 1)(-0.001)e^{-0.001x}]$$
$$= 10^6e^{-0.001x}(1 - 0.001x + 0.001)$$

Since $e^{-0.001x} > -$ for all $x$,
$P'(x) = 0$ when $1.001 - 0.001x = 0$

$$x = \frac{1.001}{0.001} = 1001.$$

$P(1001) = 10^6[1 + 1000e^{-1.001}] \doteq 368.5 \times 10^6$
The maximum monthly profit will be $368.5 \times 106$ dollars when 1001 items are produced and sold.
**b.** The domain for $P(x)$ becomes $0 \leq x \leq 500$.
$P(500) = 10^6[1 + 499e^{-0.5}] = 303.7 \times 10^6$
Since there are no critical values in the domain, the maximum occurs at an endpoint. The maximum monthly profit when 500 items are produced and sold is $303.7 \times 10^6$ dollars.
**5.** $R(x) = 40x^2e^{-0.4x} + 30, 0 \leq x \leq 8$
We use the Algorithm for Extreme Values:

$$R'(x) = 80xe^{-0.4x} + 40x^2(-0.4)e^{-0.4x}$$
$$= 40xe^{-0.4x}(2 - 0.4x)$$

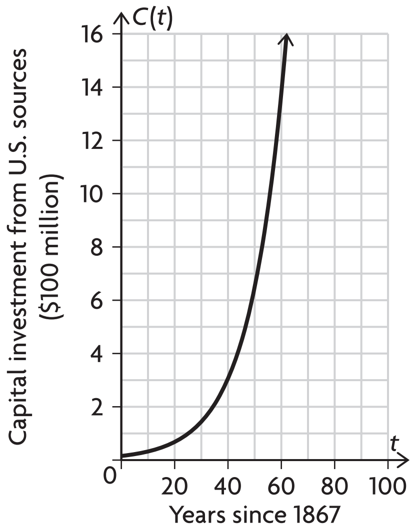
Since $e^{-0.4x} > 0$ for all $x$, $R'(x) = 0$ when
$x = 0$ or $2 - 0.4x = 0$
$x = 5.$
$R(0) = 30$
$R(5) \doteq 165.3$
$R(8) \doteq 134.4$
The maximum monthly revenue of 165.3 thousand dollars is achieved when 500 units are produced and sold.
**6.** $P(t) = 100(e^{-t} - e^{-4t}), 0 \leq t \leq 3$

$$P'(t) = 100(-e^{-t} + 4e^{-4t})$$
$$= 100e^{-t}(-1 + 4e^{-3t})$$

Since $e^{-t} > 0$ for all $t$, $P'(t) = 0$ when

$$4e^{-3t} = 1$$
$$e^{-3t} = \frac{1}{4}$$
$$-3t = \ln(0.25)$$
$$t = \frac{-\ln(0.25)}{3}$$
$$= 0.462.$$

$P(0) = 0$
$P(0.462) \doteq 47.2$
$P(3) \doteq 4.98$
The highest percentage of people spreading the rumour is 47.2% and occurs at the 0.462 h point.

**7.** $C = 0.015 \times 10^9 e^{0.075\,33t}, 0 \leq t \leq 100$
**a.**



**b.** $\frac{dC}{dt} = 0.015 \times 10^9 \times 0.07533e^{0.075\,33t}$
In 1947, $t = 80$ and the growth rate was

$$\frac{dC}{dt} = 0.468\,05 \times 10^9 \text{ dollars/year.}$$

In 1967, $t = 100$ and the growth rate was

$$\frac{dC}{dt} = 2.1115 \times 10^9 \text{ dollars/year.}$$

The ratio of growth rates of 1967 to that of 1947 is

$$\frac{2.1115 \times 10^9}{0.468\,05 \times 10^9} = \frac{4.511}{1}.$$

The growth rate of capital investment grew from 468 million dollars per year in 1947 to 2.112 billion dollars per year in 1967.
**c.** In 1967, the growth rate of investment as a percentage of the amount invested is

$$\frac{2.1115 \times 10^9}{28.0305 \times 10^9} \times 100 = 7.5\%.$$

**d.** In 1977, $t = 110$
$C = 59.537 \times 10^9$ dollars

$$\frac{dC}{dt} = 4.4849 \times 10^9 \text{ dollars/year.}$$

**e.** Statistics Canada data shows the actual amount of U.S. investment in 1977 was $62.5 \times 10^9$ dollars. The error in the model is 3.5%.
**f.** In 2007, $t = 140$.
The expected investment and growth rates are
$C = 570.490 \times 10^9$ dollars and $\frac{dC}{dt} = 42.975 \times 10^9$ dollars/year.

**8. a.** The growth function is $N = 2^{\frac{t}{5}}$.

The number killed is given by $K = e^{\frac{t}{3}}$.
After 60 minutes, $N = 2^{12}$.
Let $T$ be the number of minutes after 60 minutes.
The population of the colony at any time, $T$ after the first 60 minutes is

$$P = N - k$$
$$= 2^{\frac{60+T}{5}} - e^{\frac{T}{3}}, T \geq 0$$
$$\frac{dP}{dt} = 2^{\frac{60+T}{5}}\left(\frac{1}{5}\right)\ln 2 - \frac{1}{3}e^{\frac{T}{3}}$$
$$= 2^{\frac{12+T}{5}}\left(\frac{\ln 2}{5}\right) - \frac{1}{3}e^{\frac{T}{3}}$$
$$= 2^{12} \cdot 2^{\frac{T}{5}}\left(\frac{\ln 2}{5}\right) - \frac{1}{3}e^{\frac{T}{3}}$$

$\frac{dP}{dt} = 0$ when $2^{12}\frac{\ln 2}{5}2^{\frac{T}{5}} = \frac{1}{3}e^{\frac{T}{3}}$ or

$3\frac{\ln 2}{5} \cdot 2^{12}\, 2^{\frac{T}{5}} = e^{\frac{T}{3}}$.

We take the natural logarithm of both sides:

$$\ln\left(3.2^{12}\frac{\ln 2}{5}\right) + \frac{T}{5}\ln 2 = \frac{T}{3}$$
$$7.4404 = T\left(\frac{1}{3} - \frac{\ln 2}{5}\right)$$
$$T = \frac{7.4404}{0.1947} = 38.2 \text{ min.}$$

At $T = 0$, $P = 2^{12} = 4096$.
At $T = 38.2$, $P = 478\,158$.

For $T > 38.2$, $\frac{dP}{dt}$ is always negative.
The maximum number of bacteria in the colony occurs 38.2 min after the drug was introduced. At this time the population numbers 478 158.

**b.** $P = 0$ when $2^{\frac{60+T}{5}} = e^{\frac{T}{3}}$

$$\frac{60 + T}{5}\ln 2 = \frac{T}{3}$$
$$12 \ln 2 = T\left(\frac{1}{3} - \frac{\ln 2}{5}\right)$$
$$T = 42.72$$

The colony will be obliterated 42.72 minutes after the drug was introduced.

**9.** Let $t$ be the number of minutes assigned to study for the first exam and $30 - t$ minutes assigned to study for the second exam. The measure of study effectiveness for the two exams is given by

$$E(t) = E_1(t) + E_2(30 - t), 0 \leq t \leq 30$$
$$= 0.5\left(10 + te^{-\frac{t}{10}}\right) + 0.6\left(9 + (30 - t)e^{-\frac{30-t}{20}}\right)$$

$$E'(t) = 0.5\left(e^{-\frac{t}{10}} - \frac{1}{10}te^{-\frac{t}{10}}\right)$$
$$+ 0.6\left(-e^{-\frac{30-t}{5}} + \frac{1}{20}(30 - t)e^{-\frac{30-t}{20}}\right)$$
$$= 0.05e^{-\frac{t}{10}}(10 - t) + 0.03e^{-\frac{30-t}{5}}$$
$$(-20 + 30 - t)$$
$$= \left(0.05e^{-\frac{t}{10}} + 0.03e^{-\frac{30-t}{5}}\right)(10 - t)$$

$E'(t) = 0$ when $10 - t = 0$
$t = 10$ (The first factor is always a positive number.)
$E(0) = 5 + 5.4 + 18e^{-\frac{3}{2}} = 14.42$
$E(10) = 16.65$
$E(30) = 11.15$
For maximum study effectiveness, 10 h of study should be assigned to the firs exam and 20 h of study for the second exam.

**10.** Use the algorithm for finding extreme values. First, find the derivative $f'(x)$. Then, find any critical points by setting $f'(x) = 0$ and solving for $x$. Also, find the values of $x$ for which $f'(x)$ is undefined. Together these are the critical values. Now, evaluate $f(x)$ for the critical values and the endpoints 2 and $-2$. The highest value will be the absolute maximum on the interval and the lowest value will be the absolute minimum on the interval.

**11. a.** $f'(x) = (x^2)(e^x) + (e^x)(2x)$
$= e^x(x^2 + 2x)$

The function is increasing when $f'(x) > 0$ and decreasing when $f'(x) < 0$. First, find the critical values of $f'(x)$. Solve $e^x = 0$ and $(x^2 + 2x) = 0$
$e^x$ is never equal to zero.
$x^2 + 2x = 0$
$x(x + 2) = 0$.
So, the critical values are 0 and $-2$.

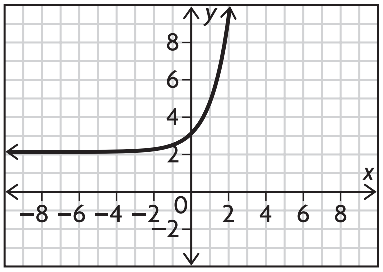
| Interval | $e^x(x^2 + 2x)$ |
|---|---|
| $x < -2$ | + |
| $-2 < x < 0$ | − |
| $0 < x$ | + |

So, $f(x)$ is increasing on the intervals $(-\infty, -2)$ and $(0, \infty)$.
Also, $f(x)$ is decreasing on the interval $(-2, 0)$.

**b.** At $x = 0$, $f'(x)$ switches from decreasing on the left of zero to increasing on the right of zero. So, $x = 0$ is a minimum. Since it is the only critical point that is a minimum, it is the $x$-coordinate of the absolute minimum value of $f(x)$. The absolute minimum value is $f(0) = 0$.

**12. a.** $y' = e^x$

Setting $e^x = 0$ yields no solutions for $x$. $e^x$ is a function that is always increasing. So, there is no maximum or minimum value for $y = e^x + 2$.




**b.** $y' = (x)(e^x) + (e^x)(1)$

$= e^x(x + 1)$

Solve $e^x = 0$ and $(x + 1) = 0$

$e^x$ is never equal to zero.

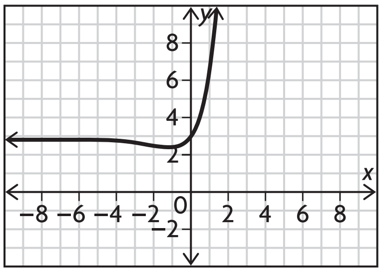
$x + 1 = 0$

$x = -1.$

So there is one critical point: $x = -1$.

| Interval | $e^x(x + 1)$ |
|---|---|
| $x < -1$ | − |
| $x > -1$ | + |

So $y$ is decreasing on the left of $x = -1$ and increasing on the right of $x = -1$. So $x = -1$ is the $x$-coordinate of the minimum of $y$. The minimum value is $-e^{-1} + 3 \doteq 2.63$. There is no maximum value.




**c.** $y' = (2x)(2e^{2x}) + (e^{2x})(2)$

$= 2e^{2x}(2x + 1)$

Solve $2e^{2x} = 0$ and $(2x + 1) = 0$

$2e^{2x}$ is never equal to zero.

$2x + 1 = 0$

$$x = -\frac{1}{2}$$

So there is one critical point: $x = -\frac{1}{2}$.

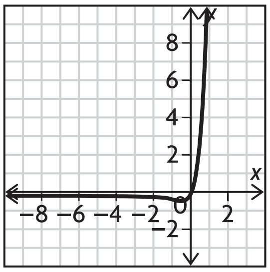
| Interval | $2e^{2x}(2x + 1)$ |
|---|---|
| $x < -\frac{1}{2}$ | − |
| $x > -\frac{1}{2}$ | + |

So $y$ is decreasing on the left of $x = -\frac{1}{2}$ and increasing on the right of $x = -\frac{1}{2}$. So $x = -\frac{1}{2}$ is the $x$-coordinate of the minimum of $y$. The minimum value is

$$2\left(-\frac{1}{2}\right)(e^{2(-\frac{1}{2})})$$

$= -e^{-1}$

$\doteq -0.37.$ There is no maximum value.

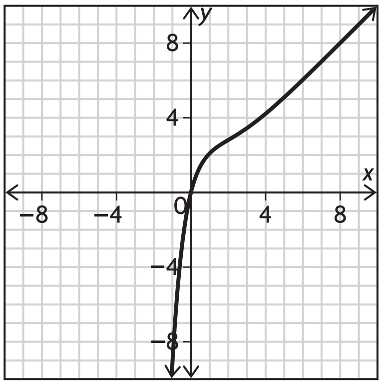



**d.** $y' = (3x)(-e^{-x}) + (e^{-x})(3) + 1$

$= 3e^{-x}(1 - x) + 1$

Solve $3e^{-x}(1 - x) + 1 = 0$.

This gives no real solutions. By looking at the graph of $y = f(x)$, one can see that the function is always increasing. So, there is no maximum or minimum value for $y = 3xe^{-x} + x$.




**13.** $P'(x) = (x)(-xe^{-0.5x^2}) + (e^{-0.5x^2})(1)$

$= e^{-0.5x^2}(-x^2 + 1)$

Solve $e^{-0.5x^2} = 0$ and $(1 - x^2) = 0$.

$e^{-0.5x^2}$ gives no critical points.

$1 - x^2 = 0$

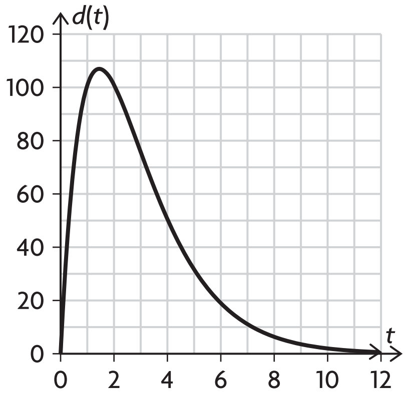
$(1 - x)(1 + x) = 0$

So $x = 1$ and $x = -1$ are the critical points.
So $P(x)$ is decreasing on the left of $x = -1$ and on

| Interval | $e^{-0.5x^2}(-x^2 + 1)$ |
|---|---|
| $x < -1$ | $-$ |
| $-1 < x < 1$ | $+$ |
| $1 < x$ | $-$ |

the right of $x = 1$ and it is increasing between $x = -1$ and $x = 1$. So $x = -1$ is the $x$-coordinate of the minimum of $P(x)$. Also, $x = 1$ is the $x$-coordinate of the maximum of $P(x)$. The minimum value is
$P(-1) = (-1)(e^{-0.5(-1)^2}) = -e^{-0.5} \doteq -0.61$.
The maximum value is
$P(1) = (1)(e^{-0.5(1)^2}) = e^{-0.5} \doteq 0.61$.

**14. a.**




**b.** The speed is increasing when $d'(t) > 0$ and the speed is decreasing when $d'(t) < 0$.

$$d'(t) = (200t)(-2^{-t})(\ln 2) + (2^{-t})(200)$$
$$= 200(2)^{-t}(-t\ln 2 + 1)$$

Solve $200(2)^{-t} = 0$ and $-t\ln 2 + 1 = 0$.
$200(2)^{-t}$ gives no critical points.

$$-t\ln 2 + 1 = 0$$
$$t = \frac{1}{\ln 2} \doteq 1.44$$

So $t = \frac{1}{\ln 2}$ is the critical point.

| Interval | $200(2)^{-t}(-t\ln 2 + 1)$ |
|---|---|
| $t < \frac{1}{\ln 2}$ | $+$ |
| $t > \frac{1}{\ln 2}$ | $-$ |

So the speed of the closing door is increasing when $0 < t < \frac{1}{\ln 2}$ and decreasing when $t > \frac{1}{\ln 2}$.

**c.** There is a maximum at $t = \frac{1}{\ln 2}$ since $d'(t) < 0$ for $t < \frac{1}{\ln 2}$ and $d'(t) > 0$ for $t > \frac{1}{\ln 2}$.

The maximum speed is

$$d\left(\frac{1}{\ln 2}\right) = 200\left(\frac{1}{\ln 2}\right)(2)^{-\frac{1}{\ln 2}} \doteq 106.15 \text{ degrees/s}$$

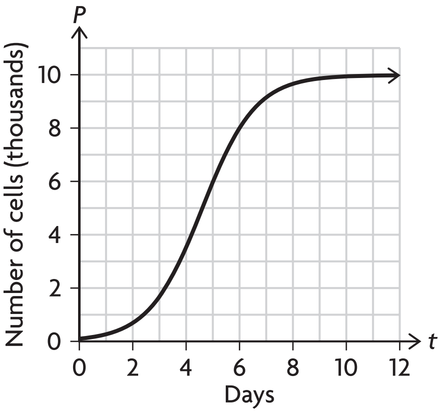
**d.** The door seems to be closed for $t > 10$ s.

**15.** The solution starts in a similar way to that of 9. The effectiveness function is
$E(t) = 0.5\left(10 + te^{-\frac{t}{10}}\right) + 0.6\left(9 + (25 - t)e^{-\frac{25-t}{20}}\right)$.
The derivative simplifies to
$E'(t) = 0.05e^{-\frac{t}{10}}(10 - t) + 0.03e^{-\frac{25-t}{20}}(5 - t)$.
This expression is very difficult to solve analytically. By calculation on a graphing calculator, we can determine the maximum effectiveness occurs when $t = 8.16$ hours.

**16.** $P = \dfrac{aL}{a + (L - a)e^{-kLt}}$

**a.** We are given $a = 100$, $L = 10\,000$, $k = 0.0001$.

$$P = \frac{10^6}{100 + 9900e^{-t}} = \frac{10^4}{1 + 99e^{-t}}$$
$$= 10^4(1 + 99e^{-t})^{-1}$$




**b.** We need to determine when the derivative of the growth rate $\left(\frac{dP}{dt}\right)$ is zero, i.e., when $\frac{d^2P}{dt^2} = 0$.

$$\frac{dP}{dt} = \frac{-10^4(-99e^{-t})}{(1 + 99e^{-t})^2} = \frac{990000e^{-t}}{(1 + 99e^{-t})^2}$$

$$\frac{d^2P}{dt^2} = \frac{-990000e^{-t}(1 + 99e^{-t})^2 - 990000e^{-t}}{(1 + 99e^{-t})^4}$$
$$\times \frac{(2)(1 + 99e^{-t})(-99e^{-t})}{(1 + 99e^{-t})^4}$$
$$= \frac{-990000e^{-t}(1 + 99e^{-t}) + 198(990000)e^{-2t}}{(1 + 99e^{-t})^3}$$

$\frac{d^2P}{dt^2} = 0$ when

$$990000e^{-t}(-1 - 99e^{-t} + 198e^{-t}) = 0$$
$$99e^{-t} = 1$$
$$e^t = 99$$
$$t = \ln 99$$
$$\doteq 4.6$$

After 4.6 days, the rate of change of the growth rate is zero. At this time the population numbers 5012.

**c.** When $t = 3$, $\frac{dP}{dt} = \frac{990\,000e^{-3}}{(1 + 99e^{-3})^2} \doteq 1402$ cells/day.

When $t = 8$, $\frac{dP}{dt} = \frac{990\,000e^{-8}}{(1 + 99e^{-8})^2} \doteq 311$ cells/day.

The rate of growth is slowing down as the colony is getting closer to its limiting value.

## Mid-Chapter Review, pp. 248–249

**1. a.**
$$\frac{dy}{dx} = \frac{d(5e^{-3x})}{dx}$$
$$= (5e^{-3x})(-3x)'$$
$$= (5e^{-3x})(-3)$$
$$= -15e^{-3x}$$

**b.**
$$\frac{dy}{dx} = \frac{d\left(7e^{\frac{1}{7}x}\right)}{dx}$$
$$= \left(7e^{\frac{1}{7}x}\right)\left(\frac{1}{7}x\right)'$$
$$= \left(7e^{\frac{1}{7}x}\right)\left(\frac{1}{7}\right)$$
$$= e^{\frac{1}{7}x}$$

**c.**
$$\frac{dy}{dx} = (x^3)(e^{-2x})' + (x^3)'(e^{-2x})$$
$$= (x^3)((e^{-2x})(-2x)') + (3x^2)(e^{-2x})$$
$$= (x^3)((e^{-2x}))(-2) + 3x^2e^{-2x}$$
$$= -2x^3e^{-2x} + 3x^2e^{-2x}$$
$$= e^{-2x}(-2x^3 + 3x^2)$$

**d.**
$$\frac{dy}{dx} = (x-1)^2(e^x)' + ((x-1)^2)'(e^x)$$
$$= (x-1)^2(e^x) + (2(x-1))(e^x)$$
$$= (x^2 - 2x + 1)(e^x) + (2x - 2)(e^x)$$
$$= (e^x)(x^2 - 2x + 1 + 2x - 2)$$
$$= (e^x)(x^2 - 1)$$

**e.**
$$\frac{dy}{dx} = 2(x - e^{-x})(x - e^{-x})'$$
$$= 2(x - e^{-x})(1 - (e^{-x})(-x)')$$
$$= 2(x - e^{-x})(1 - (e^{-x})(-1))$$
$$= 2(x - e^{-x})(1 + e^{-x})$$
$$= 2(x + xe^{-x} - e^{-x} - e^{-x+-x})$$
$$= 2(x + xe^{-x} - e^{-x} - e^{-2x})$$

**f.**
$$\frac{dy}{dx} = \frac{(e^x + e^{-x})(e^x - e^{-x})'}{(e^x + e^{-x})^2} - \frac{(e^x - e^{-x})(e^x + e^{-x})'}{(e^x + e^{-x})^2}$$
$$= \frac{(e^x + e^{-x})(e^x - (e^{-x})(-x)')}{(e^x + e^{-x})^2} - \frac{(e^x - e^{-x})(e^x + (e^{-x})(-x)')}{(e^x + e^{-x})^2}$$
$$= \frac{(e^x + e^{-x})(e^x - (e^{-x})(-1))}{(e^x + e^{-x})^2} - \frac{(e^x - e^{-x})(e^x + (e^{-x})(-1))}{(e^x + e^{-x})^2}$$
$$= \frac{(e^x + e^{-x})(e^x + e^{-x})}{(e^x + e^{-x})^2} - \frac{(e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$
$$= \frac{e^{2x} + e^0 + e^0 + e^{-2x}}{(e^x + e^{-x})^2} - \frac{(e^{2x} - e^0 - e^0 + e^{-2x})}{(e^x + e^{-x})^2}$$
$$= \frac{e^{2x} + e^0 + e^0 + e^{-2x} - e^{2x}}{(e^x + e^{-x})^2} + \frac{e^0 + e^0 - e^{-2x}}{(e^x + e^{-x})^2}$$
$$= \frac{4}{(e^x + e^{-x})^2}$$

**2. a.**
$$\frac{dP}{dt} = 100e^{-5t}(-5t)'$$
$$= 100e^{-5t}(-5)$$
$$= -500e^{-5t}$$

**b.** The time is needed for when the sample of the substance is at half of the original amount. So, find $t$ when $P = \frac{1}{2}$.

$$P = 100e^{-5t}$$
$$\frac{1}{2} = 100e^{-5t}$$
$$\frac{1}{200} = e^{-5t}$$
$$\ln\frac{1}{200} = -5t$$
$$\frac{\ln\frac{1}{200}}{-5} = t$$

$$\frac{d^2P}{dt^2} = 0 \text{ when}$$

$$990000e^{-t}(-1 - 99e^{-t} + 198e^{-t}) = 0$$
$$99e^{-t} = 1$$
$$e^{t} = 99$$
$$t = \ln 99$$
$$\doteq 4.6$$

After 4.6 days, the rate of change of the growth rate is zero. At this time the population numbers 5012.

**c.** When $t = 3$, $\frac{dP}{dt} = \frac{990\,000e^{-3}}{(1 + 99e^{-3})^2} \doteq 1402$ cells/day.

When $t = 8$, $\frac{dP}{dt} = \frac{990\,000e^{-8}}{(1 + 99e^{-8})^2} \doteq 311$ cells/day.

The rate of growth is slowing down as the colony is getting closer to its limiting value.

## Mid-Chapter Review, pp. 248–249

**1. a.** $\frac{dy}{dx} = \frac{d(5e^{-3x})}{dx}$
$= (5e^{-3x})(-3x)'$
$= (5e^{-3x})(-3)$
$= -15e^{-3x}$

**b.** $\frac{dy}{dx} = \frac{d\left(7e^{\frac{1}{7}x}\right)}{dx}$
$= \left(7e^{\frac{1}{7}x}\right)\left(\frac{1}{7}x\right)'$
$= \left(7e^{\frac{1}{7}x}\right)\left(\frac{1}{7}\right)$
$= e^{\frac{1}{7}x}$

**c.** $\frac{dy}{dx} = (x^3)(e^{-2x})' + (x^3)'(e^{-2x})$
$= (x^3)((e^{-2x})(-2x)') + (3x^2)(e^{-2x})$
$= (x^3)((e^{-2x}))(-2) + 3x^2e^{-2x}$
$= -2x^3e^{-2x} + 3x^2e^{-2x}$
$= e^{-2x}(-2x^3 + 3x^2)$

**d.** $\frac{dy}{dx} = (x - 1)^2(e^x)' + ((x - 1)^2)'(e^x)$
$= (x - 1)^2(e^x) + (2(x - 1))(e^x)$
$= (x^2 - 2x + 1)(e^x) + (2x - 2)(e^x)$
$= (e^x)(x^2 - 2x + 1 + 2x - 2)$
$= (e^x)(x^2 - 1)$

**e.** $\frac{dy}{dx} = 2(x - e^{-x})(x - e^{-x})'$
$= 2(x - e^{-x})(1 - (e^{-x})(-x)')$
$= 2(x - e^{-x})(1 - (e^{-x})(-1))$
$= 2(x - e^{-x})(1 + e^{-x})$
$= 2(x + xe^{-x} - e^{-x} - e^{-x+-x})$
$= 2(x + xe^{-x} - e^{-x} - e^{-2x})$

**f.** $\frac{dy}{dx} = \frac{(e^x + e^{-x})(e^x - e^{-x})'}{(e^x + e^{-x})^2} - \frac{(e^x - e^{-x})(e^x + e^{-x})'}{(e^x + e^{-x})^2}$
$= \frac{(e^x + e^{-x})(e^x - (e^{-x})(-x)')}{(e^x + e^{-x})^2} - \frac{(e^x - e^{-x})(e^x + (e^{-x})(-x)')}{(e^x + e^{-x})^2}$
$= \frac{(e^x + e^{-x})(e^x - (e^{-x})(-1))}{(e^x + e^{-x})^2} - \frac{(e^x - e^{-x})(e^x + (e^{-x})(-1))}{(e^x + e^{-x})^2}$
$= \frac{(e^x + e^{-x})(e^x + e^{-x})}{(e^x + e^{-x})^2} - \frac{(e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$
$= \frac{e^{2x} + e^0 + e^0 + e^{-2x}}{(e^x + e^{-x})^2} - \frac{(e^{2x} - e^0 - e^0 + e^{-2x})}{(e^x + e^{-x})^2}$
$= \frac{e^{2x} + e^0 + e^0 + e^{-2x} - e^{2x}}{(e^x + e^{-x})^2} + \frac{e^0 + e^0 - e^{-2x}}{(e^x + e^{-x})^2}$
$= \frac{4}{(e^x + e^{-x})^2}$

**2. a.** $\frac{dP}{dt} = 100e^{-5t}(-5t)'$
$= 100e^{-5t}(-5)$
$= -500e^{-5t}$

**b.** The time is needed for when the sample of the substance is at half of the original amount. So, find $t$ when $P = \frac{1}{2}$.

$$P = 100e^{-5t}$$
$$\frac{1}{2} = 100e^{-5t}$$
$$\frac{1}{200} = e^{-5t}$$
$$\ln \frac{1}{200} = -5t$$
$$\frac{\ln \frac{1}{200}}{-5} = t$$

Now, the question asks for $\frac{dP}{dt} = P'$ when

$$t = \frac{\ln \frac{1}{200}}{-5} \doteq 1.06$$

$$P'\left(\frac{\ln \frac{1}{200}}{-5}\right) = -2.5 \text{ (using a calculator)}$$

**3.** $\frac{dy}{dx} = (-x)(e^x)' + (e^x)(-x)'$

$= (-x)(e^x) + (e^x)(-1)$

$= -xe^x - e^x$

At the point $x = 0$,

$\frac{dy}{dx} = -0e^0 - e^0 = -1.$

At the point $x = 0$,

$y = 2 - 0e^0 = 2$

So, an equation of the tangent to the curve at the point $x = 0$ is

$y - 2 = -1(x - 0)$

$y - 2 = -x$

$y = -x + 2$

$x + y - 2 = 0$

**4. a.** $y' = -3(e^x)'$

$= -3e^x$

$y'' = -3e^x$

**b.** $y' = (x)(e^{2x})' + (e^{2x})(x)'$

$= (x)((e^{2x}) + (2x)') + (e^{2x})(1)$

$= (x)((e^{2x})(2)) + e^{2x}$

$= 2xe^{2x} + e^{2x}$

$y'' = (2x)(e^{2x})' + (e^{2x})(2x)' + e^{2x}(2x)'$

$= (2x)((e^{2x})(2x)') + (e^{2x})(2) + (e^{2x})(2)$

$= (2x)((e^{2x})(2)) + 2e^{2x} + 2e^{2x}$

$= 4xe^{2x} + 4e^{2x}$

**c.** $y' = (e^x)(4 - x)' + (4 - x)(e^x)'$

$= (e^x)(-1) + (4 - x)(e^x)$

$= -e^x + 4e^x - xe^x$

$= 3e^x - xe^x$

$y'' = (3e^x)' - [(x)(e^x)' + (e^x)(x)']$

$= 3e^x - [xe^x + (e^x)(1)]$

$= 3e^x - xe^x - e^x$

$= 2e^x - xe^x$

**5. a.** $\frac{dy}{dx} = (8^{2x+5})(\ln 8)(2x + 5)'$

$= (8^{2x+5})(\ln 8)(2)$

$= 2(\ln 8)(8^{2x+5})$

**b.** $\frac{dy}{dx} = 3.2((10)^{0.2x})(\ln 10)(0.2x)'$

$= 3.2((10)^{0.2x})(\ln 10)(0.2)$

$= 0.64(\ln 10)((10)^{.2x})$

**c.** $f'(x) = (x^2)(2^x)' + (2^x)(x^2)'$

$= (x^2)(2^x)(\ln 2) + (2^x)(2x)$

$= (\ln 2)(x^2 2^x) + 2x2^x$

$= 2^x((\ln 2)(x^2) + 2x)$

**d.** $H'(x) = 300((5)^{3x-1})(\ln 5)(3x - 1)'$

$= 300((5)^{3x-1})(\ln 5)(3)$

$= 900(\ln 5)(5)^{3x-1}$

$= 900(\ln 5)(5)^{3x-1}$

**e.** $q'(x) = (1.9)^x \cdot (\ln 1.9) + 1.9(x)^{1.9-1}$

$= (1.9)^x \cdot (\ln 1.9) + 1.9(x)^{0.9}$

$= (\ln 1.9)(1.9)^x + 1.9x^{0.9}$

**f.** $f'(x) = (x - 2)^2(4^x)' + (4^x)((x - 2)^2)'$

$= (x - 2)^2(4^x)(\ln 4) + (4^x)(2(x - 2))$

$= (\ln 4)(4^x)(x - 2)^2 + (4^x)(2x - 4)$

$= 4^x((\ln 4)(x - 2)^2 + 2x - 4)$

**6. a.** The initial number of rabbits in the forest is given by the time $t = 0$.

$R(0) = 500\left(10 + e^{-\frac{0}{10}}\right)$

$= 500(10 + 1)$

$= 500(11)$

$= 5500$

**b.** The rate of change is the derivative, $\frac{dR}{dt}$.

$R(t) = 5000 + 500\left(e^{-\frac{t}{10}}\right)$

$\frac{dR}{dt} = 0 + 500\left(e^{-\frac{t}{10}}\right)\left(-\frac{t}{10}\right)'$

$= 500\left(e^{-\frac{t}{10}}\right)\left(-\frac{1}{10}\right)$

$= -50\left(e^{-\frac{t}{10}}\right)$

**c.** 1 year = 12 months

The question asks for $\frac{dR}{dt} = R'$ when $t = 12$.

$R'(12) = -50\left(e^{-\frac{12}{10}}\right)$

$\doteq -15.06$

**d.** To find the maximum number of rabbits, optimize the function.

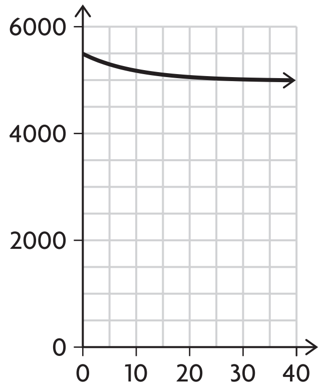
$R'(t) = -50\left(e^{-\frac{t}{10}}\right)$

$0 = -50\left(e^{-\frac{t}{10}}\right)$

$0 = e^{-\frac{t}{10}}$

When solving, the natural log (ln) of both sides must be taken, but (ln 0) does not exist. So there are no solutions to the equation. The function is therefore always decreasing. So, the largest number of rabbits will exist at the earliest time in the interval at time $t = 0$. To check, compare $R(0)$ and $R(36)$. $R(0) = 5500$ and $R(36) \doteq 5013$. So, the largest number of rabbits in the forest during the first 3 years is 5500.

**e.**




The graph is constantly decreasing. The $y$-intercept is at the point (0, 5500). Rabbit populations normally grow exponentially, but this population is shrinking exponentially. Perhaps a large number of rabbit predators such as snakes recently began to appear in the forest. A large number of predators would quickly shrink the rabbit population.

**7.** The highest concentration of the drug can be found by optimizing the given function.

$$C(t) = 10e^{-2t} - 10e^{-3t}$$

$$\begin{aligned} C'(t) &= (10e^{-2t})(-2t)' - (10e^{-3t})(-3t)' \\ &= (10e^{-2t})(-2) - (10e^{-3t})(-3) \\ &= -20e^{-2t} + 30e^{-3t} \end{aligned}$$

Set the derivative of the function equal to zero and find the critical points.

$$\begin{aligned} 0 &= -20e^{-2t} + 30e^{-3t} \\ 20e^{-2t} &= 30e^{-3t} \\ \frac{2}{3}e^{-2t} &= e^{-3t} \\ \frac{2}{3} &= \frac{e^{-3t}}{e^{-2t}} \\ \frac{2}{3} &= (e^{-3t})(e^{2t}) \\ \frac{2}{3} &= e^{-3t+2t} \\ \frac{2}{3} &= e^{-t} \\ \ln\frac{2}{3} &= -t \\ -\left(\ln\frac{2}{3}\right) &= t \end{aligned}$$

Therefore, $t = -\left(\ln\frac{2}{3}\right) \doteq 0.41$ is the critical value.
Now, use the algorithm for finding extreme values.

$$\begin{aligned} C(0) &= 10(e^0 - e^0) = 0 \\ C\left(-\left(\ln\frac{2}{3}\right)\right) &\doteq 1.48 \text{ (using a calculator)} \\ C(5) &= 0.0005 \end{aligned}$$

So, the function has a maximum when $t = -\left(\ln\frac{2}{3}\right) \doteq 0.41$. Therefore, during the first five hours, the highest concentration occurs at about 0.41 hours.

**8.** $y = ce^{kx}$
$y' = cke^{kx}$

The original function is increasing when its derivative is positive and decreasing when its derivative is negative.

$e^{kx} > 0$ for all $k, x \in \mathbf{R}$.

So, the original function represents growth when $ck > 0$, meaning that $c$ and $k$ must have the same sign. The original function represents decay when $c$ and $k$ have opposite signs.

**9. a.** $A(t) = 5000e^{0.02t}$
$= 5000e^{0.02(0)}$
$= 5000$

The initial population is 5000.

**b.** at $t = 7$
$A(7) = 5000e^{0.02(7)} = 5751$

After a week, the population is 5751.

**c.** at $t = 30$
$A(30) = 5000e^{0.02(30)} = 9111$

After 30 days, the population is 9111.

**10. a.** $P(5) = 760e^{-0.125(5)}$
$\doteq 406.80$ mm Hg

**b.** $P(7) = 760e^{-0.125(7)}$
$\doteq 316.82$ mm Hg

**c.** $P(9) = 760e^{-0.125(9)}$
$\doteq 246.74$ mm Hg

**11.**

$$\begin{aligned} A &= 100e^{-0.3x} \\ A' &= 100e^{-0.3x}(-0.3) \\ &= -30e^{-0.3x} \end{aligned}$$

When 50% of the substance is gone, $y = 50$

$$\begin{aligned} 50 &= 100e^{-0.3x} \\ 0.5 &= e^{-0.3x} \\ \ln(0.5) &= \ln e^{-0.3x} \\ \ln(0.5) &= -0.3x \ln e \end{aligned}$$

$$\frac{\ln 0.5}{\ln e} = -0.3x$$
$$-\frac{\ln 0.5}{0.3 \ln e} = x$$
$$x = 2.31$$
$$A' = -30e^{-0.3x}$$
$$A'(2.31) = -30e^{-0.3(2.31)}$$
$$A' \doteq -15$$

When 50% of the substance is gone, the rate of decay is 15% per year.

**12.** $f(x) = xe^x$

$$f'(x) = xe^x + (1)e^x$$
$$= e^x(x + 1)$$

So $e^x > 0$

$x + 1 > 0$

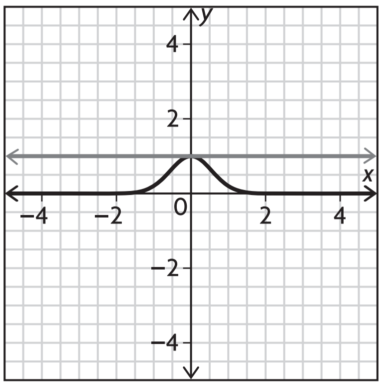
$x > -1$

This means that the function is increasing when $x > -1$.

**13.** $y = 5^{-x^2}$

When $x = 1$,

$$y = \frac{1}{5}$$
$$y' = 5^{-x^2}(-2x) \ln 5$$
$$y' = -\frac{2}{5} \ln 5$$
$$5y - \frac{1}{5} = -\frac{2}{5} \ln 5(x - 1)$$
$$5y - 1 = -2 \ln 5(x - 1)$$
$$5y - 1 = (-2 \ln 5)x + 2 \ln 5$$
$$(2 \ln 5)x + 5y = 2 \ln 5 + 1$$




**14. a.** $A = P(1 + i)^t$

$$A(t) = 1000(1 + 0.06)^t$$
$$= 1000(1.06)^t$$

**b.** $A'(t) = 1000(1.06)^t(1) \ln (1.06)$

$$= 1000(1.06)^t \ln 1.06$$

**c.** $A'(2) = 1000(1.06)^2 \ln 1.06$

$$= \$65.47$$
$$A'(5) = 1000(1.06)^5 \ln 1.06$$
$$= \$77.98$$
$$A'(10) = 1000(1.06)^{10} \ln 1.06$$
$$= \$104.35$$

**d.** No, the rate is not constant.

**e.** $\frac{A'(2)}{A(2)} = \ln 1.06$

$$\frac{A'(5)}{A(5)} = \ln 1.06$$
$$\frac{A'(10)}{A(10)} = \ln 1.06$$

**f.** All the ratios are equivalent (they equal ln 1.06, which is about 0.058 27), which means that $\frac{A'(t)}{A(t)}$ is constant.

**15.** $y = ce^x$

$$y' = c(e^x) + (0)e^x$$
$$= ce^x$$
$$y = y' = ce^x$$

## 5.4 The Derivatives of $y = \sin x$ and $y = \cos x$, pp. 256–257

**1. a.** $\frac{dy}{dx} = (\cos 2x) \cdot \frac{d(2x)}{dx}$

$$= 2 \cos 2x$$

**b.** $\frac{dy}{dx} = -2 (\sin 3x) \cdot \frac{d(3x)}{dx}$

$$= -6 \sin 3x$$

**c.** $\frac{dy}{dx} = (\cos (x^3 - 2x + 4)) \cdot \frac{d(x^3 - 2x + 4)}{dx}$

$$= (3x^2 - 2)(\cos (x^3 - 2x + 4))$$

**d.** $\frac{dy}{dx} = -2 \sin (-4x) \cdot \frac{d(-4x)}{dx}$

$$= 8 \sin (-4x)$$

**e.** $\frac{dy}{dx} = \cos (3x) \cdot \frac{d(3x)}{dx} + \sin (4x) \cdot \frac{d(4x)}{dx}$

$$= 3 \cos (3x) + 4 \sin (4x)$$

**f.** $\frac{dy}{dx} = 2^x(\ln 2) + 2 \cos x + 2 \sin x$

**g.** $\frac{dy}{dx} = \cos (e^x) \cdot \frac{d(e^x)}{dx}$

$$= e^x\cos (e^x)$$

**h.** $\frac{dy}{dx} = 3 \cos (3x + 2\pi) \cdot \frac{d(3x + 2\pi)}{dx}$

$$= 9 \cos (3x + 2\pi)$$

$$\frac{\ln 0.5}{\ln e} = -0.3x$$
$$-\frac{\ln 0.5}{0.3 \ln e} = x$$
$$x = 2.31$$
$$A' = -30e^{-0.3x}$$
$$A'(2.31) = -30e^{-0.3(2.31)}$$
$$A' \doteq -15$$

When 50% of the substance is gone, the rate of decay is 15% per year.

**12.** $f(x) = xe^x$
$$f'(x) = xe^x + (1)e^x$$
$$= e^x(x + 1)$$
So $e^x > 0$
$x + 1 > 0$
$x > -1$
This means that the function is increasing when $x > -1$.

**13.** $y = 5^{-x^2}$
When $x = 1$,
$$y = \frac{1}{5}$$
$$y' = 5^{-x^2}(-2x) \ln 5$$
$$y' = -\frac{2}{5} \ln 5$$
$$5y - \frac{1}{5} = -\frac{2}{5} \ln 5(x - 1)$$
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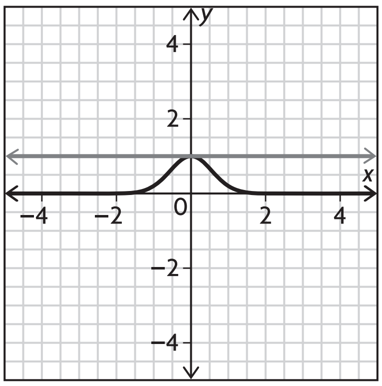
$$5y - 1 = (-2 \ln 5)x + 2 \ln 5$$
$$(2 \ln 5)x + 5y = 2 \ln 5 + 1$$




**14. a.** $A = P(1 + i)^t$
$$A(t) = 1000(1 + 0.06)^t$$
$$= 1000(1.06)^t$$
**b.** $A'(t) = 1000(1.06)^t(1) \ln (1.06)$
$$= 1000(1.06)^t \ln 1.06$$
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$$= \$65.47$$
$$A'(5) = 1000(1.06)^5 \ln 1.06$$
$$= \$77.98$$
$$A'(10) = 1000(1.06)^{10} \ln 1.06$$
$$= \$104.35$$
**d.** No, the rate is not constant.

**e.** $\frac{A'(2)}{A(2)} = \ln 1.06$
$$\frac{A'(5)}{A(5)} = \ln 1.06$$
$$\frac{A'(10)}{A(10)} = \ln 1.06$$
**f.** All the ratios are equivalent (they equal ln 1.06, which is about 0.058 27), which means that $\frac{A'(t)}{A(t)}$ is constant.

**15.** $y = ce^x$
$$y' = c(e^x) + (0)e^x$$
$$= ce^x$$
$$y = y' = ce^x$$

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**1. a.** $\frac{dy}{dx} = (\cos 2x) \cdot \frac{d(2x)}{dx}$
$$= 2 \cos 2x$$
**b.** $\frac{dy}{dx} = -2 (\sin 3x) \cdot \frac{d(3x)}{dx}$
$$= -6 \sin 3x$$
**c.** $\frac{dy}{dx} = (\cos (x^3 - 2x + 4)) \cdot \frac{d(x^3 - 2x + 4)}{dx}$
$$= (3x^2 - 2)(\cos (x^3 - 2x + 4))$$
**d.** $\frac{dy}{dx} = -2 \sin (-4x) \cdot \frac{d(-4x)}{dx}$
$$= 8 \sin (-4x)$$
**e.** $\frac{dy}{dx} = \cos (3x) \cdot \frac{d(3x)}{dx} + \sin (4x) \cdot \frac{d(4x)}{dx}$
$$= 3 \cos (3x) + 4 \sin (4x)$$
**f.** $\frac{dy}{dx} = 2^x(\ln 2) + 2 \cos x + 2 \sin x$

**g.** $\frac{dy}{dx} = \cos (e^x) \cdot \frac{d(e^x)}{dx}$
$$= e^x \cos (e^x)$$
**h.** $\frac{dy}{dx} = 3 \cos (3x + 2\pi) \cdot \frac{d(3x + 2\pi)}{dx}$
$$= 9 \cos (3x + 2\pi)$$

**i.** $\frac{dy}{dx} = 2x - \sin x + 0$
$= 2x - \sin x$

**j.** $\frac{dy}{dx} = \cos\left(\frac{1}{x}\right) \cdot \frac{d\left(\frac{1}{x}\right)}{dx}$
$= -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$

**2. a.** $\frac{dy}{dx} = (2\sin x)(-\sin x) + (\cos x)(2\cos x)$
$= -2\sin^2 x + 2\cos^2 x$
$= 2(\cos^2 x - \sin^2 x)$
$= 2\cos(2x)$

**b.** $y = (x^{-1})(\cos 2x)$
$\frac{dy}{dx} = (x^{-1})(-2\sin 2x) + (\cos 2x)(-x^{-2})$
$= -\frac{2\sin 2x}{x} - \frac{\cos 2x}{x^2}$

**c.** $\frac{dy}{dx} = -\sin(\sin 2x) \cdot \frac{d(\sin 2x)}{dx}$
$= -\sin(\sin 2x) \cdot 2\cos 2x$

**d.** $y = (\sin x)(1 + \cos x)^{-1}$
$\frac{dy}{dx} = (\sin x)(-(1 + \cos x)^{-2} \cdot (-\sin x)$
$+ (1 + \cos x)^{-1}(\cos x)$
$= \frac{-\sin^2 x}{-(1 + \cos x)^2} + \frac{\cos x}{1 + \cos x}$
$= \frac{\sin^2 x}{(1 + \cos x)^2} + \frac{\cos x(1 + \cos x)}{(1 + \cos x)^2}$
$= \frac{\sin^2 x + \cos^2 x + \cos x}{(1 + \cos x)^2}$
$= \frac{1 + \cos x}{(1 + \cos x)^2}$
$= \frac{1}{1 + \cos x}$

**e.** $\frac{dy}{dx} = (e^x)(-\sin x + \cos x) + (\cos x + \sin x)(e^x)$
$= e^x(-\sin x + \cos x + \cos x + \sin x)$
$= e^x(2\cos x)$

**f.** $\frac{dy}{dx} = (2x^3)(\cos x) + (\sin x)(6x^2)$
$-[(3x)(-\sin x) + (\cos x)(3)]$
$= 2x^3\cos x + 6x^2\sin x + 3x\sin x - 3\cos x$

**3. a.** When $x = \frac{\pi}{3}$, $f(x) = f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.
$f'(x) = \cos x$
$f'\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3}$
$= \frac{1}{2}$

So an equation for the tangent at the point $x = \frac{\pi}{3}$ is

$y - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(x - \frac{\pi}{3}\right)$
$2y - \sqrt{3} = x - \frac{\pi}{3}$
$-x + 2y + \left(\frac{\pi}{3} - \sqrt{3}\right) = 0$

**b.** When $x = 0$, $f(x) = f(0) = 0 + \sin(0) = 0$.
$f'(x) = 1 + \cos x$
$f'(0) = 1 + \cos(0)$
$= 1 + 1$
$= 2$

So an equation for the tangent at the point $x = 0$ is
$y - 0 = 2(x - 0)$
$y = 2x$
$-2x + y = 0$

**c.** When $x = \frac{\pi}{4}$, $f(x) = f\left(\frac{\pi}{4}\right) = \cos\left(4 \cdot \frac{\pi}{4}\right)$
$= \cos(\pi)$
$= -1$
$f'(x) = -\sin(4x) \cdot \frac{d(4x)}{dx}$
$= -4\sin(4x)$
$f'\left(\frac{\pi}{4}\right) = -4\sin\left(4 \cdot \frac{\pi}{4}\right)$
$= -4\sin(\pi)$
$= 0$

So an equation for the tangent at the point $x = \frac{\pi}{4}$ is

$y - (-1) = 0\left(x - \frac{\pi}{4}\right)$
$y + 1 = 0$
$y = -1$

**d.** $f(x) = \sin 2x + \cos x$, $x = \frac{\pi}{2}$

The point of contact is $\left(\frac{\pi}{2}, 0\right)$. The slope of the tangent line at any point is $f'(x) = 2\cos 2x - \sin x$. At $\left(\frac{\pi}{2}, 0\right)$, the slope of the tangent line is

$2\cos\pi - \sin\frac{\pi}{2} = -3$.

An equation of the tangent line is $y = -3\left(x - \frac{\pi}{2}\right)$.

**e.** $f(x) = \cos\left(2x + \frac{\pi}{3}\right), x = \frac{\pi}{4}$

The point of tangency is $\left(\frac{\pi}{4}, -\frac{\sqrt{3}}{2}\right)$. The slope of the tangent line at any point is $f'(x) = -2\sin\left(2x + \frac{\pi}{3}\right)$.

At $\left(\frac{\pi}{4}, -\frac{\sqrt{3}}{2}\right)$, the slope of the tangent line is

$-2\sin\left(\frac{5\pi}{6}\right) = -1$.

An equation of the tangent line is

$y + \frac{\sqrt{3}}{2} = -\left(x - \frac{\pi}{4}\right)$.

**f.** When $x = \frac{\pi}{2}$, $f(x) = f\left(\frac{\pi}{2}\right) = 2\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)$

$$= 2(1)(0)$$
$$= 0$$

$$\begin{aligned} f'(x) &= (2\sin x)(-\sin x) + (\cos x)(2\cos x) \\ &= -2\sin^2 x + 2\cos^2 x \\ &= 2(\cos^2 x - \sin^2 x) \\ &= 2\cos(2x) \\ f'\left(\frac{\pi}{2}\right) &= 2\cos\left(2 \cdot \frac{\pi}{2}\right) \\ &= 2\cos\pi \\ &= -2 \end{aligned}$$

So an equation for the tangent when $x = \frac{\pi}{2}$ is

$$y - 0 = -2\left(x - \frac{\pi}{2}\right)$$
$$y = -2x + \pi$$

$2x + y - \pi = 0$

**4. a.** One could easily find $f'(x)$ and $g'(x)$ to see that they both equal $2(\sin x)(\cos x)$. However, it is easier to notice a fundamental trigonometric identity. It is known that $\sin^2 x + \cos^2 x = 1$. So, $\sin^2 x = 1 - \cos^2 x$.

Therefore, one can notice that $f(x)$ is in fact equal to $g(x)$. So, because $f(x) = g(x)$, $f'(x) = g'(x)$.

**b.** $f'(x)$ and $g'(x)$ are negatives of each other. That is, $f'(x) = 2(\sin x)(\cos x)$ while $g'(x) = -2(\sin x)(\cos x)$.

**5. a.** $v(t) = (\sin(\sqrt{t}))^2$

$$\begin{aligned} v'(t) &= 2\sin(\sqrt{t}) \cdot \frac{d(\sin(\sqrt{t}))}{dt} \\ &= 2\sin(\sqrt{t}) \cdot \cos(\sqrt{t}) \cdot \frac{d(\sqrt{t})}{dt} \\ &= 2\sin(\sqrt{t}) \cdot \cos(\sqrt{t}) \cdot \frac{1}{2}t^{-\frac{1}{2}} \\ &= 2\sin(\sqrt{t}) \cdot \cos(\sqrt{t}) \cdot \frac{1}{2\sqrt{t}} \\ &= \frac{\sin(\sqrt{t})\cos(\sqrt{t})}{\sqrt{t}} \end{aligned}$$

**b.** $v(t) = (1 + \cos t + \sin^2 t)^{\frac{1}{2}}$

$$\begin{aligned} v'(t) &= \frac{1}{2}(1 + \cos t + \sin^2 t)^{-\frac{1}{2}} \\ &\quad \times \frac{d(1 + \cos t + (\sin t)^2)}{dt} \\ &= \frac{-\sin t + 2(\sin t) \cdot \frac{d(\sin t)}{dt}}{2\sqrt{1 + \cos t + \sin^2 t}} \\ &= \frac{-\sin t + 2(\sin t)(\cos t)}{2\sqrt{1 + \cos t + \sin^2 t}} \end{aligned}$$

**c.** $h(x) = \sin x \sin 2x \sin 3x$

So, treat $\sin x \sin 2x$ as one function, say $f(x)$ and treat $\sin 3x$ as another function, say $g(x)$.

Then, the product rule may be used with the chain rule:

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= (\sin x \sin 2x)(3\cos 3x) \\ &\quad + (\sin 3x)[(\sin x)(2\cos 2x) \\ &\quad + (\sin 2x)(\cos x)] \\ &= 3\sin x \sin 2x \cos 3x \\ &\quad + 2\sin x \sin 3x \cos 2x \\ &\quad + \sin 2x \sin 3x \cos x \end{aligned}$$

**d.** 
$$\begin{aligned} m'(x) &= 3(x^2 + \cos^2 x)^2 \cdot \frac{d(x^2 + (\cos x)^2)}{dx} \\ &= 3(x^2 + \cos^2 x)^2 \cdot (2x + 2(\cos x)(-\sin x)) \\ &= 3(x^2 + \cos^2 x)^2 \cdot (2x - 2\sin x \cos x) \end{aligned}$$

**6.** By the algorithm for finding extreme values, the maximum and minimum values occur at points on the graph where $f'(x) = 0$, or at an endpoint of the interval.

**a.** $\frac{dy}{dx} = -\sin x + \cos x$

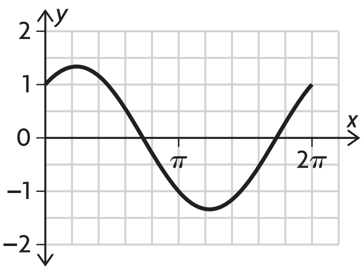
Set $\frac{dy}{dx} = 0$ and solve for $x$ to find any critical points.

$$\begin{aligned} \cos x - \sin x &= 0 \\ \cos x &= \sin x \\ 1 &= \frac{\sin x}{\cos x} \\ 1 &= \tan x \\ x &= \frac{\pi}{4}, \frac{5\pi}{4} \end{aligned}$$

Evaluate $f(x)$ at the critical numbers, including the endpoints of the interval.

| $x$ | $0$ | $\frac{\pi}{4}$ | $\frac{5\pi}{4}$ | $2\pi$ |
|---|---|---|---|---|
| $f(x) = \cos x + \sin x$ | $1$ | $\sqrt{2}$ | $-\sqrt{2}$ | $1$ |

So, the absolute maximum value on the interval is $\sqrt{2}$ when $x = \frac{\pi}{4}$ and the absolute minimum value on the interval is $-\sqrt{2}$ when $x = \frac{5\pi}{4}$.




**b.** $\frac{dy}{dx} = 1 - 2\sin x$

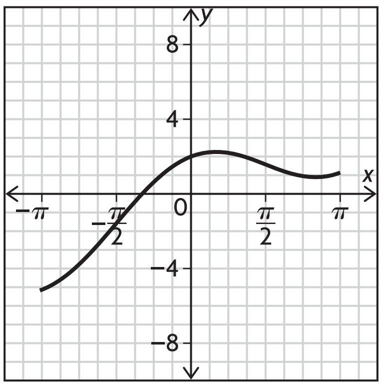
Set $\frac{dy}{dx} = 0$ and solve for $x$ to find any critical points.

$$1 - 2\sin x = 0$$
$$1 = 2\sin x$$
$$\frac{1}{2} = \sin x$$
$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

Evaluate $f(x)$ at the critical numbers, including the endpoints of the interval.

| $x$ | $-\pi$ | $-\frac{\pi}{6}$ | $\frac{\pi}{6}$ | $\pi$ |
|---|---|---|---|---|
| $f(x) = x + 2\cos x$ | $-\pi - 2$ $\doteq -5.14$ | $-\frac{\pi}{6} + \sqrt{3}$ $\doteq 1.21$ | $\frac{\pi}{6} + \sqrt{3}$ $\doteq 2.26$ | $\pi - 2$ $\doteq 1.14$ |

So, the absolute maximum value on the interval is 2.26 when $x = \frac{\pi}{6}$ and the absolute minimum value on the interval is $-5.14$ when $x = -\pi$.




**c.** $\frac{dy}{dx} = \cos x + \sin x$

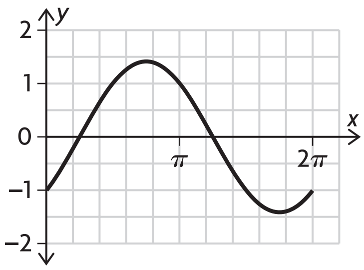
Set $\frac{dy}{dx} = 0$ and solve for $x$ to find any critical points.

$$\cos x + \sin x = 0$$
$$\sin x = -\cos x$$
$$\frac{\sin x}{\cos x} = -1$$
$$\tan x = -1$$
$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

Evaluate $f(x)$ at the critical numbers, including the endpoints of the interval.

| $x$ | $0$ | $\frac{3\pi}{4}$ | $\frac{7\pi}{4}$ | $2$ |
|---|---|---|---|---|
| $f(x) = \sin x - \cos x$ | $-1$ | $\sqrt{2}$ | $-\sqrt{2}$ | $-1$ |

So, the absolute maximum value on the interval is $\sqrt{2}$ when $x = \frac{3\pi}{4}$ and the absolute minimum value on the interval is $-\sqrt{2}$ when $x = \frac{7\pi}{4}$.




**d.** $\frac{dy}{dx} = 3\cos x - 4\sin x$

Set $\frac{dy}{dx} = 0$ and solve for $x$ to find any critical points.

$$3\cos x - 4\sin x = 0$$
$$3\cos x = 4\sin x$$
$$\frac{3}{4} = \frac{\sin x}{\cos x}$$
$$\frac{3}{4} = \tan x$$
$$\tan^{-1}\left(\frac{3}{4}\right) = \tan^{-1}(\tan x)$$

Using a calculator, $x \doteq 0.6435$.

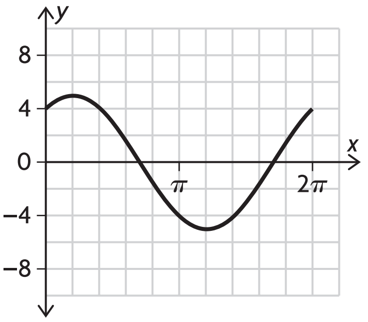
This is a critical value, but there is also one more in the interval $0 \le x \le 2\pi$. The period of $\tan x$ is $\pi$, so adding $\pi$ to the one solution will give another solution in the interval.

$x = 0.6435 + \pi \doteq 3.7851$

Evaluate $f(x)$ at the critical numbers, including the endpoints of the interval.

| $x$ | 0 | 0.64 | 3.79 | $2\pi$ |
|---|---|---|---|---|
| $f(x) = 3\sin x + 4\cos x$ | 4 | 5 | $-5$ | 4 |

So, the absolute maximum value on the interval is 5 when $x \doteq 0.64$ and the absolute minimum value on the interval is $-5$ when $x \doteq 3.79$.




**7. a.** The particle will change direction when the velocity, $s'(t)$, changes from positive to negative.
$s'(t) = 16\cos 2t$
Set $s'(t) = 0$ and solve for $t$ to find any critical points.

$$0 = 16\cos 2t$$
$$0 = \cos 2t$$
$$\frac{\pi}{2}, \frac{3\pi}{2} = 2t$$
$$\frac{\pi}{4}, \frac{3\pi}{4} = t$$

Also, there is no given interval so it will be beneficial to locate all solutions.

Therefore, $t = \frac{\pi}{4} + \pi k, \frac{3\pi}{4} + \pi k$ for some positive integer $k$ constitutes all solutions.

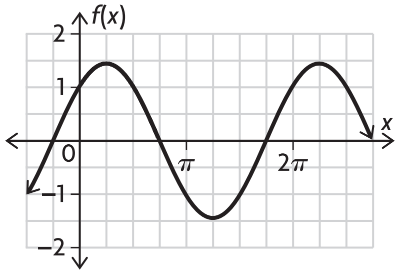
One can create a table and notice that on each side of any value of $t$, the function is increasing on one side and decreasing on the other. So, each $t$ value is either a maximum or a minimum.

| $t$ | $\frac{\pi}{4}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{4}$ | $\frac{7\pi}{4}$ |
|---|---|---|---|---|
| $s(t) = 8\sin 2t$ | 8 | $-8$ | 8 | $-8$ |

The table continues in this pattern for all critical values $t$. So, the particle changes direction at all critical values. That is, it changes direction for $t = \frac{\pi}{4} + \pi k, \frac{3\pi}{4} + \pi k$ for positive integers $k$.

**b.** From the table or a graph, one can see that the particle's maximum velocity is 8 at the time $t = \frac{\pi}{4} + \pi k$.

**8. a.**




**b.** The tangent to the curve $f(x)$ is horizontal at the point(s) where $f'(x)$ is zero.
$f'(x) = -\sin x + \cos x$
Set $f'(x) = 0$ and solve for $x$ to find any critical points.

$$\cos x - \sin x = 0$$
$$\cos x = \sin x$$
$$1 = \frac{\sin x}{\cos x}$$
$$1 = \tan x$$

$x = \frac{\pi}{4}$ (*Note:* The solution $x = \frac{5\pi}{4}$ is not in the interval $0 \le x \le \pi$ so it is not included.) When $x = \frac{\pi}{4}, f(x) = f\left(\frac{\pi}{4}\right) = \sqrt{2}$.

So, the coordinates of the point where the tangent to the curve of $f(x)$ is horizontal is $\left(\frac{\pi}{4}, \sqrt{2}\right)$.

**9.** $\csc x = \frac{1}{\sin x} = (\sin x)^{-1}$

$\sec x = \frac{1}{\cos x} = (\cos x)^{-1}$

Now, the power rule can be used to compute the derivates of $\csc x$ and $\sec x$.

$$((\sin x)^{-1})' = -(\sin x)^{-2} \cdot \frac{d(\sin x)}{dx}$$
$$= -(\sin x)^{-2} \cdot \cos x$$
$$= -\frac{\cos x}{(\sin x)^2}$$
$$((\sin x)^{-1})' = -(\sin x)^{-2} \cdot \frac{d(\sin x)}{dx}$$
$$= -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}$$
$$= -\csc x \cot x$$
$$((\cos x)^{-1})' = -(\cos x)^{-2} \cdot \frac{d(\cos x)}{dx}$$
$$= -(\cos x)^{-2} \cdot (-\sin x)$$
$$= \frac{\sin x}{(\cos x)^2}$$
$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$
$$= \sec x \tan x$$

**10.** $\frac{dy}{dx} = -2 \sin 2x$

At the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$,

$$\frac{dy}{dx} = -2 \sin \left(2 \cdot \frac{\pi}{6}\right)$$
$$= -2 \sin \left(\frac{\pi}{3}\right)$$
$$= -2\left(\frac{\sqrt{3}}{2}\right)$$
$$= -\sqrt{3}$$

Therefore, at the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$, the slope of the tangent to the curve $y = \cos 2x$ is $-\sqrt{3}$.

**11. a.** The particle will change direction when the velocity, $s'(t)$ changes from positive to negative.
$s'(t) = 16 \cos 4t$
Set $s'(t) = 0$ and solve for $t$ to find any critical points.
$0 = 16 \cos 4t$
$0 = \cos 4t$
$\frac{\pi}{2}, \frac{3\pi}{2} = 4t$
$\frac{\pi}{8}, \frac{3\pi}{8} = t$
Also, there is no given interval so it will be beneficial to locate all solutions.

Therefore, $t = \frac{\pi}{8} + \pi k, \frac{3\pi}{8} + \pi k$ for some positive integer $k$ constitutes all solutions.

One can create a table and notice that on each side of any value of $t$, the function is increasing on one side and decreasing on the other. So, each $t$ value is either a maximum or a minimum.

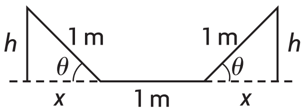
| $t$ | $\frac{\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{7\pi}{8}$ |
|---|---|---|---|---|
| $s(t) = 4 \sin 4t$ | 4 | $-4$ | 4 | $-4$ |

The table continues in this pattern for all critical values $t$. So, the particle changes direction at all critical values. That is, it changes direction for $t = \frac{\pi}{4} + \pi k, \frac{3\pi}{4} + \pi k$ for positive integers $k$.

**b.** From the table or a graph, one can see that the particle's maximum velocity is 4 at the time $t = \frac{\pi}{4} + \pi k$.

**c.** At $t = 0$, $s = 0$, so the minimum distance from the origin is 0. The maximum value of the sine function is 1, so the maximum distance from the origin is 4(1) or 4.

**12.**




Label the base of a triangle $x$ and the height $h$. So $\cos \theta = \frac{x}{1} = x$ and $\sin \theta = \frac{h}{1} = h$.
Therefore, $x = \cos \theta$ and $h = \sin \theta$.
The irrigation channel forms a trapezoid and the area of a trapezoid is $\frac{(b_1 + b_2)h}{2}$ where $b_1$ and $b_2$ are the bottom and top bases of the trapezoid and $h$ is the height.
$b_1 = 1$
$b_2 = x + 1 + x = \cos \theta + 1 + \cos \theta = 2 \cos \theta + 1$
$h = \sin \theta$
Therefore, the area equation is given by

$$A = \frac{(2 \cos \theta + 1 + 1) \sin \theta}{2}$$
$$= \frac{(2 \cos \theta + 2) \sin \theta}{2}$$
$$= \frac{2 \cos \theta \sin \theta + 2 \sin \theta}{2}$$
$$= \sin \theta \cos \theta + \sin \theta$$

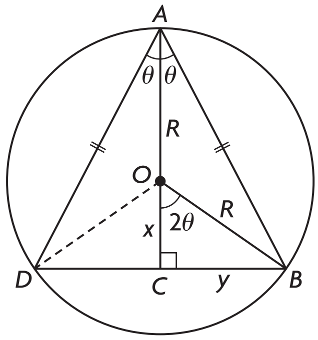
To maximize the cross-sectional area, differentiate:
$A' = (\sin \theta)(-\sin \theta) + (\cos \theta)(\cos \theta) + \cos \theta$
$= -\sin^2 \theta + \cos^2 \theta + \cos \theta$
Using the trig identity $\sin^2 \theta + \cos^2 \theta = 1$, use the fact that $\sin^2 \theta = 1 - \cos^2 \theta$.
$A' = -(1 - \cos^2 \theta) + \cos^2 \theta + \cos \theta$
$= -1 + \cos^2 \theta + \cos^2 \theta + \cos \theta$
$= 2 \cos^2 \theta + \cos \theta - 1$
Set $A' = 0$ to find the critical points.
$0 = 2 \cos^2 \theta + \cos \theta - 1$
$0 = (2 \cos \theta - 1)(\cos \theta + 1)$
Solve the two expressions for $\theta$.
$2 \cos \theta = 1$
$\cos \theta = \frac{1}{2}$
$\theta = \frac{\pi}{3}$
Also, $\cos \theta = -1$
$\theta = \pi$
(*Note:* The question only seeks an answer around $0 \le \theta \le \frac{\pi}{2}$. So, there is no need to find all solutions by adding $k\pi$ for all integer values of $k$.)
The area, $A$, when $\theta = \pi$ is 0 so that answer is disregarded for this problem.

When $\theta = \frac{\pi}{3}$,

$$A = \sin\frac{\pi}{3}\cos\frac{\pi}{3} + \sin\frac{\pi}{3}$$
$$= \left(\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\right) + \frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{3}}{4} + \frac{2\sqrt{3}}{4}$$
$$= \frac{3\sqrt{3}}{4}$$

The area is maximized by the angle $\theta = \frac{\pi}{3}$.

**13.** Let O be the centre of the circle with line segments drawn and labeled, as shown.




In $\triangle OCB$, $\angle COB = 2\theta$.

Thus, $\frac{y}{R} = \sin 2\theta$ and $\frac{x}{R} = \cos 2\theta$,

so $y = R\sin 2\theta$ and $x = R\cos 2\theta$.

The area $A$ of $\triangle ABD$ is

$$A = \frac{1}{2}|DB||AC|$$
$$= y(R + x)$$
$$= R\sin 2\theta(R + R\cos 2\theta)$$
$$= R^2(\sin 2\theta + \sin 2\theta\cos 2\theta), \text{ where } 0 < 2\theta < \pi$$

$$\frac{dA}{d\theta} = R^2(2\cos 2\theta + 2\cos 2\theta\cos 2\theta + \sin 2\theta(-2\sin 2\theta)).$$

We solve $\frac{dA}{d\theta} = 0$:

$$2\cos^2 2\theta - 2\sin^2 2\theta + 2\cos 2\theta = 0$$
$$2\cos^2 2\theta + \cos 2\theta - 1 = 0$$
$$(2\cos 2\theta - 1)(\cos 2\theta + 1) = 0$$
$$\cos 2\theta = \frac{1}{2} \text{ or } \cos 2\theta = -1$$
$$2\theta = \frac{\pi}{3} \text{ or } 2\theta = \pi \text{ (not in domain).}$$

As $2\theta \to 0$, $A \to 0$ and as $2\theta \to \pi$, $A \to 0$. The maximum area of the triangle is $\frac{3\sqrt{3}}{4}R^2$ when $2\theta = \frac{\pi}{3}$, i.e., $\theta = \frac{\pi}{6}$.

**14.** First find $y''$.

$$y = A\cos kt + B\sin kt$$
$$y' = -kA\sin kt + kB\cos kt$$
$$y'' = -k^2A\cos kt - k^2B\sin kt$$

So, $y'' + k^2y$

$$= -k^2A\cos kt - k^2B\sin kt + k^2(A\cos kt + B\sin kt)$$
$$= -k^2A\cos kt - k^2B\sin kt + k^2A\cos kt + k^2B\sin kt$$
$$= 0$$

Therefore, $y'' + k^2y = 0$.

## 5.5 The Derivative of $y = \tan x$, p. 260

**1. a.** $\frac{dy}{dx} = \sec^2 3x\left(\frac{d}{dx}3x\right)$
$= 3\sec^2 3x$

**b.** $\frac{dy}{dx} = 2\sec^2 x - \sec 2x\left(\frac{d}{dx}2x\right)$
$= 2\sec^2 x - 2\sec 2x$

**c.** $\frac{dy}{dx} = 2\tan(x^3)\left(\frac{d}{dx}\tan(x^3)\right)$

$\frac{dy}{dx} = 2\tan(x^3)\left(\frac{d}{dx}\tan(x^3)\right)$
$= 2\tan(x^3)\sec^2(x^3)\left(\frac{d}{dx}x^3\right)$
$= 6x^2\tan(x^3)\sec^2(x^3)$

**d.** $\frac{dy}{dx} = \frac{2x\tan\pi x - x^2\sec^2\pi x\left(\frac{d}{dx}\pi x\right)}{\tan^2\pi x}$
$= \frac{2x\tan\pi x - \pi x^2\sec^2\pi x}{\tan^2\pi x}$
$= \frac{x(2\tan\pi x - \pi x\sec^2\pi x)}{\tan^2\pi x}$

**e.** $\frac{dy}{dx} = \sec^2(x^2)\left(\frac{d}{dx}x^2\right) - 2\tan x\left(\frac{d}{dx}\right)(\tan x)$
$= 2x\sec^2(x^2) - 2\tan x\sec^2 x$

**f.** $\frac{dy}{dx} = \tan 5x(3\cos 5x)\left(\frac{d}{dx}5x\right) + 3\sin 5x\sec^2 5x\left(\frac{d}{dx}5x\right)$
$= 15\tan 5x\cos 5x + 15\sin 5x\sec^2 5x$
$= 15(\tan 5x\cos 5x + \sin 5x\sec^2 5x)$

**2. a.** The general equation for the line tangent to the function $f(x)$ at the point $(a, b)$ is
$y - b = f'(x)(x - a)$.

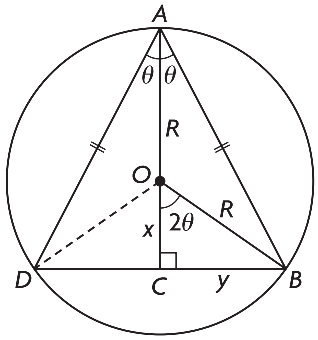
$f(x) = \tan x$
$f'(x) = \sec^2 x$

When $\theta = \frac{\pi}{3}$,

$$A = \sin\frac{\pi}{3}\cos\frac{\pi}{3} + \sin\frac{\pi}{3}$$
$$= \left(\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\right) + \frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{3}}{4} + \frac{2\sqrt{3}}{4}$$
$$= \frac{3\sqrt{3}}{4}$$

The area is maximized by the angle $\theta = \frac{\pi}{3}$.

**13.** Let O be the centre of the circle with line segments drawn and labeled, as shown.




In $\triangle OCB$, $\angle COB = 2\theta$.

Thus, $\frac{y}{R} = \sin 2\theta$ and $\frac{x}{R} = \cos 2\theta$,

so $y = R\sin 2\theta$ and $x = R\cos 2\theta$.

The area $A$ of $\triangle ABD$ is

$$A = \frac{1}{2}|DB||AC|$$
$$= y(R + x)$$
$$= R\sin 2\theta(R + R\cos 2\theta)$$
$$= R^2(\sin 2\theta + \sin 2\theta\cos 2\theta), \text{ where } 0 < 2\theta < \pi$$

$$\frac{dA}{d\theta} = R^2(2\cos 2\theta + 2\cos 2\theta\cos 2\theta + \sin 2\theta(-2\sin 2\theta)).$$

We solve $\frac{dA}{d\theta} = 0$:

$$2\cos^2 2\theta - 2\sin^2 2\theta + 2\cos 2\theta = 0$$
$$2\cos^2 2\theta + \cos 2\theta - 1 = 0$$
$$(2\cos 2\theta - 1)(\cos 2\theta + 1) = 0$$
$$\cos 2\theta = \frac{1}{2} \text{ or } \cos 2\theta = -1$$
$$2\theta = \frac{\pi}{3} \text{ or } 2\theta = \pi \text{ (not in domain).}$$

As $2\theta \to 0$, $A \to 0$ and as $2\theta \to \pi$, $A \to 0$. The maximum area of the triangle is $\frac{3\sqrt{3}}{4}R^2$ when $2\theta = \frac{\pi}{3}$, i.e., $\theta = \frac{\pi}{6}$.

**14.** First find $y''$.

$$y = A\cos kt + B\sin kt$$
$$y' = -kA\sin kt + kB\cos kt$$
$$y'' = -k^2A\cos kt - k^2B\sin kt$$

So, $y'' + k^2y$

$$= -k^2A\cos kt - k^2B\sin kt + k^2(A\cos kt + B\sin kt)$$
$$= -k^2A\cos kt - k^2B\sin kt + k^2A\cos kt + k^2B\sin kt$$
$$= 0$$

Therefore, $y'' + k^2y = 0$.

## 5.5 The Derivative of $y = \tan x$, p. 260

**1. a.** $\frac{dy}{dx} = \sec^2 3x\left(\frac{d}{dx}3x\right)$
$= 3\sec^2 3x$

**b.** $\frac{dy}{dx} = 2\sec^2 x - \sec 2x\left(\frac{d}{dx}2x\right)$
$= 2\sec^2 x - 2\sec 2x$

**c.** $\frac{dy}{dx} = 2\tan(x^3)\left(\frac{d}{dx}\tan(x^3)\right)$

$\frac{dy}{dx} = 2\tan(x^3)\left(\frac{d}{dx}\tan(x^3)\right)$
$= 2\tan(x^3)\sec^2(x^3)\left(\frac{d}{dx}x^3\right)$
$= 6x^2\tan(x^3)\sec^2(x^3)$

**d.** $\frac{dy}{dx} = \frac{2x\tan\pi x - x^2\sec^2\pi x\left(\frac{d}{dx}\pi x\right)}{\tan^2\pi x}$
$= \frac{2x\tan\pi x - \pi x^2\sec^2\pi x}{\tan^2\pi x}$
$= \frac{x(2\tan\pi x - \pi x\sec^2\pi x)}{\tan^2\pi x}$

**e.** $\frac{dy}{dx} = \sec^2(x^2)\left(\frac{d}{dx}x^2\right) - 2\tan x\left(\frac{d}{dx}\right)(\tan x)$
$= 2x\sec^2(x^2) - 2\tan x\sec^2 x$

**f.** $\frac{dy}{dx} = \tan 5x(3\cos 5x)\left(\frac{d}{dx}5x\right) + 3\sin 5x\sec^2 5x\left(\frac{d}{dx}5x\right)$
$= 15\tan 5x\cos 5x + 15\sin 5x\sec^2 5x$
$= 15(\tan 5x\cos 5x + \sin 5x\sec^2 5x)$

**2. a.** The general equation for the line tangent to the function $f(x)$ at the point $(a, b)$ is
$y - b = f'(x)(x - a)$.
$f(x) = \tan x$
$f'(x) = \sec^2 x$

$f\left(\frac{\pi}{4}\right) = 0$

$f'\left(\frac{\pi}{4}\right) = 2$

The equation for the line tangent to the function $f(x)$ at $x = \frac{\pi}{4}$ is $y = 2\left(x - \frac{\pi}{4}\right)$.

**b.** The general equation for the line tangent to the function $f(x)$ at the point $(a, b)$ is $y - b = f'(x)(x - a)$.

$f(x) = 6 \tan x - \tan 2x$

$f'(x) = 6 \sec^2 x - \sec^2 2x\left(\frac{d}{dx}2x\right)$

$f'(x) = 6 \sec^2 x - 2 \sec^2 2x$

$f(0) = 0$

$f'(0) = -2$

The equation for the line tangent to the function $f(x)$ at $x = 0$ is $y = -2x$.

**3. a.** $\frac{dy}{dx} = \sec^2 x(\sin x)\left(\frac{d}{dx}\sin x\right)$

$= \cos x \sec^2 (\sin x)$

**b.** $\frac{dy}{dx} = -2\,[\tan(x^2 - 1)]^{-3}\left(\frac{d}{dx}\tan(x^2 - 1)\right)$

$= -2\,[\tan(x^2 - 1)]^{-3} \sec^2 (x^2 - 1) \times \left(\frac{d}{dx}(x^2 - 1)\right)$

$= -4x\,[\tan(x^2 - 1)]^{-3} \sec^2 (x^2 - 1)$

**c.** $\frac{dy}{dx} = 2 \tan(\cos x)\left(\frac{d}{dx}\tan(\cos x)\right)$

$= 2 \tan(\cos x) \sec^2 (\cos x)\left(\frac{d}{dx}\cos x\right)$

$= -2 \tan(\cos x) \sec^2 (\cos x) \sin x$

**d.** $\frac{dy}{dx} = 2\,(\tan x + \cos x)\left(\frac{d}{dx}\tan x + \cos x\right)$

$= 2\,(\tan x + \cos x)(\sec^2 x - \sin x)$

**e.** $\frac{dy}{dx} = \tan x\,(3 \sin^2 x)\left(\frac{d}{dx}\sin x\right) + \sin^3 x \sec^2 x$

$= 3 \tan x \sin^2 x \cos x + \sin^3 x \sec^2 x$

$= \sin^2 x\,(3 \tan x \cos x + \sin x \sec^2 x)$

**f.** $\frac{dy}{dx} = e^{\tan\sqrt{x}}\left(\frac{d}{dx}\tan\sqrt{x}\right)$

$= e^{\tan\sqrt{x}}\,(\sec^2 \sqrt{x})\left(\frac{d}{dx}\sqrt{x}\right)$

$= \frac{1}{2\sqrt{x}}\, e^{\tan\sqrt{x}} \sec^2 \sqrt{x}$

**4. a.** $\frac{dy}{dx} = \tan x \cos x + \sin x \sec^2 x$

$= \frac{\sin x}{\cos x} \cdot \cos x + \sin x \cdot \frac{1}{\cos^2 x}$

$= \sin x + \frac{\sin x}{\cos^2 x}$

$\frac{d^2y}{dx^2} = \cos x + \frac{\cos^3 x}{\cos^4 x} - \frac{\sin x(2 \cos x)\left(\frac{d}{dx}\cos x\right)}{\cos^4 x}$

$= \cos x + \frac{\cos^3 x + 2 \sin^2 x \cos x}{\cos^4 x}$

$= \cos x + \frac{1}{\cos x} + \frac{2 \sin^2 x}{\cos^3 x}$

$= \cos x + \sec x + \frac{2 \sin^2 x}{\cos^3 x}$

**b.** $\frac{dy}{dx} = 2 \tan x\left(\frac{d}{dx}\tan x\right)$

$= 2 \tan x \sec^2 x$

$= \frac{2 \sin x}{\cos x} \cdot \frac{1}{\cos^2 x}$

$= \frac{2 \sin x}{\cos^3 x}$

$\frac{d^2y}{dx^2} = \frac{2 \cos^4 x - 6 \sin x \cos^2 x\left(\frac{d}{dx}\cos x\right)}{\cos^6 x}$

$= \frac{2 \cos^4 x + 6 \sin^2 x \cos^2 x}{\cos^6 x}$

$= \frac{2}{\cos^2 x} + \frac{6 \sin^2 x}{\cos^2 x} \cdot \frac{1}{\cos^2 x}$

$= 2 \sec^2 x + 6 \tan^2 x \sec^2 x$

$= 2 \sec^2 x(1 + 3 \tan^2 x)$

**5.** The slope of $f(x) = \sin x \tan x$ equals zero when the derivative equals zero.

$f(x) = \sin x \tan x$

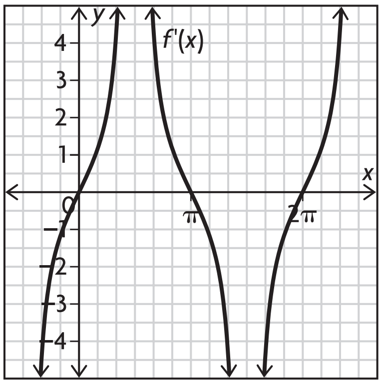
$f'(x) = \sin x(\sec^2 x) + \tan x(\cos x)$

$= \sin x(\sec^2 x) + \frac{\sin x}{\cos x}(\cos x)$

$= \sin x(\sec^2 x) + \sin x$

$= \sin x(\sec^2 x + 1)$

$\sec^2 x + 1$ is always positive, so the derivative is 0 only when $\sin x = 0$. So, $f'(x)$ equals 0 when $x = 0$, $x = \pi$, and $x = 2\pi$. The solutions can be verified by examining the graph of the derivative function shown below.




**6.** The local maximum point occurs when the derivative equals zero.

$$\frac{dy}{dx} = 2 - \sec^2 x$$

$$2 - \sec^2 x = 0$$
$$\sec^2 x = 2$$
$$\sec x = \pm\sqrt{2}$$
$$x = \pm\frac{\pi}{4}$$

$\frac{dy}{dx} = 0$ when $x = \pm\frac{\pi}{4}$, so the local maximum point occurs when $x = \pm\frac{\pi}{4}$. Solve for $y$ when $x = \frac{\pi}{4}$.

$$y = 2\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{4}\right)$$
$$y = \frac{\pi}{2} - 1$$
$$y = 0.57$$

Solve for $y$ when $x = -\frac{\pi}{4}$.

$$y = 2\left(-\frac{\pi}{4}\right) - \tan\left(-\frac{\pi}{4}\right)$$
$$y = -\frac{\pi}{2} + 1$$
$$y = -0.57$$

The local maximum occurs at the point $\left(\frac{\pi}{4}, 0.57\right)$.

**7.** $y = \sec x + \tan x$

$$= \frac{1}{\cos x} + \frac{\sin x}{\cos x}$$
$$= \frac{1 + \sin x}{\cos x}$$
$$\frac{dy}{dx} = \frac{\cos^2 x - (1 + \sin x)(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x - (-\sin x - \sin^2 x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1 + \sin x}{\cos^2 x}$$

The denominator is never negative. $1 + \sin x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, since $\sin x$ reaches its minimum of $-1$ at $x = \frac{\pi}{2}$. Since the derivative of the original function is always positive in the specified interval, the function is always increasing in that interval.

**8.** When $x = \frac{\pi}{4}$, $y = 2\tan\left(\frac{\pi}{4}\right)$

$$= 2$$
$$y' = 2\sec^2 x$$

When $x = \frac{\pi}{4}$, $y' = 2\left(\sec\frac{\pi}{4}\right)^2$

$$= 2(\sqrt{2})^2$$
$$= 4$$

So an equation for the tangent at the point $x = \frac{\pi}{4}$ is

$$y - 2 = 4\left(x - \frac{\pi}{4}\right)$$
$$y - 2 = 4x - \pi$$
$$-4x + y - (2 - \pi) = 0$$

**9.** Write $\tan x = \frac{\sin x}{\cos x}$ and use the quotient rule to derive the derivative of the tangent function.

**10.** $y = \cot x$

$$y = \frac{1}{\tan x}$$
$$\frac{dy}{dx} = \frac{\tan x(0) - (1)\sec^2 x}{\tan^2 x}$$
$$= \frac{-\sec^2 x}{\tan^2 x}$$
$$= \frac{\frac{-1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}}$$
$$= \frac{-1}{\sin^2 x}$$
$$= -\csc^2 x$$

**11.** Using the fact from question 10 that the derivative of $\cot x$ is $-\csc^2 x$,

$$f'(x) = -4\csc^2 x$$
$$= -4(\csc x)^2$$
$$f''(x) = -8(\csc x) \cdot \frac{d(\csc x)}{dx}$$
$$= -8(\csc x) \cdot (-\csc x \cot x)$$
$$= 8\csc^2 x \cot x$$

## Review Exercise, pp. 263–265

**1. a.** $y' = 0 - e^x$
$= -e^x$
**b.** $y' = 2 + 3e^x$
**c.** $y' = e^{2x+3} \cdot \frac{d(2x+3)}{dx}$
$= 2e^{2x+3}$
**d.** $y' = e^{-3x^2+5x} \cdot \frac{d(-3x^2+5x)}{dx}$
$= (-6x+5)e^{-3x^2+5x}$
**e.** $y' = (x)(e^x) + (e^x)(1)$
$= e^x(x+1)$
**f.** $s' = \frac{(e^t+1)(e^t) - (e^t-1)(e^t)}{(e^t+1)^2}$
$= \frac{e^{2t} + e^t - (e^{2t} - e^t)}{(e^t+1)^2}$
$= \frac{2e^t}{(e^t+1)^2}$

**2. a.** $\frac{dy}{dx} = 10^x \ln 10$
**b.** $\frac{dy}{dx} = 4^{3x^2} \cdot \ln 4 \cdot \frac{d(3x^2)}{dx}$
$= 6x(4^{3x^2})\ln 4$
**c.** $\frac{dy}{dx} = (5x)(5^x \ln 5) + (5^x)(5)$
$= 5 \cdot 5^x(x \ln 5 + 1)$
**d.** $\frac{dy}{dx} = (x^4)(2^x \ln 2) + (2^x)(4x^3)$
$= x^3 \cdot 2^x(x \ln 2 + 4)$
**e.** $y = (4x)(4^{-x})$
$\frac{dy}{dx} = (4x)(-4^{-x} \ln 4) + (4^{-x})(4)$
$= 4 \cdot 4^{-x}(-x \ln 4 + 1)$
$= \frac{4 - 4x \ln 4}{4^x}$
**f.** $y = (5^{\sqrt{x}})(x^{-1})$
$\frac{dy}{dx} = (5^{\sqrt{x}})(-x^{-2}) + (x^{-1})\left(5^{\sqrt{x}} \cdot \ln 5 \cdot \frac{d(\sqrt{x})}{dx}\right)$
$= (5^{\sqrt{x}})\left(-\frac{1}{x^2}\right) + (x^{-1})\left(5^{\sqrt{x}} \cdot \ln 5 \cdot \frac{1}{2\sqrt{x}}\right)$
$= 5^{\sqrt{x}}\left(-\frac{1}{x^2} + \frac{\ln 5}{2x\sqrt{x}}\right)$

**3. a.** $\frac{dy}{dx} = 3\cos(2x) \cdot \frac{d(2x)}{dx} + 4\sin(2x) \cdot \frac{d(2x)}{dx}$
$= 6\cos(2x) + 8\sin(2x)$
**b.** $\frac{dy}{dx} = \sec^2(3x) \cdot \frac{d(3x)}{dx}$
$= 3\sec^2(3x)$
**c.** $y = (2 - \cos x)^{-1}$
$\frac{dy}{dx} = -(2 - \cos x)^{-2} \cdot \frac{d(2 - \cos x)}{dx}$
$= -\frac{\sin x}{(2 - \cos x)^2}$
**d.** $\frac{dy}{dx} = (x)\left(\sec^2(2x) \cdot \frac{d(2x)}{dx}\right) + (\tan(2x))(1)$
$= 2x \sec^2(2x) + \tan 2x$
**e.** $\frac{dy}{dx} = (\sin 2x)\left(e^{3x} \cdot \frac{d(3x)}{dx}\right)$
$+ (e^{3x})\left(\cos 2x \cdot \frac{d(2x)}{dx}\right)$
$= 3e^{3x} \sin 2x + 2e^{3x} \cos 2x$
$= e^{3x}(3 \sin 2x + 2 \cos 2x)$
**f.** $y = (\cos(2x))^2$
$\frac{dy}{dx} = 2(\cos(2x)) \cdot \frac{d(\cos(2x))}{dx}$
$= 2(\cos(2x)) \cdot -\sin(2x) \cdot \frac{d(2x)}{dx}$
$= -4\cos(2x)\sin(2x)$

**4. a.** $f(x) = e^x \cdot x^{-1}$
$f'(x) = (e^x)(-x^{-2}) + (x^{-1})(e^x)$
$= e^x\left(-\frac{1}{x^2} + \frac{1}{x}\right)$
$= e^x\left(\frac{-x + x^2}{x^3}\right)$
Now, set $f'(x) = 0$ and solve for $x$.
$0 = e^x\left(\frac{-x + x^2}{x^3}\right)$
Solve $e^x = 0$ and $\frac{x^2 - x}{x^3} = 0$.
$e^x$ is never zero.
$\frac{x^2 - x}{x^3} = 0$
$x^2 - x = 0$
$x(x - 1) = 0$
So $x = 0$ or $x = 1$.
(Note, however, that $x$ cannot be zero because this would cause division by zero in the original function.)
So $x = 1$.
**b.** The function has a horizontal tangent at $(1, e)$.

**5. a.** $f'(x) = (x)\left(e^{-2x} \cdot \frac{d(-2x)}{dx}\right) + (e^{-2x})(1)$

$$= -2xe^{-2x} + e^{-2x}$$
$$= e^{-2x}(-2x + 1)$$
$$f'\left(\frac{1}{2}\right) = e^{-2\frac{1}{2}}\left(-2 \cdot \frac{1}{2} + 1\right)$$
$$= e^{-1}(-1 + 1)$$
$$= 0$$

**b.** This means that the slope of the tangent to $f(x)$ at the point with $x$-coordinate $\frac{1}{2}$ is 0.

**6. a.** $y' = (x)(e^x) + (e^x)(1) - e^x$

$$= xe^x$$
$$y'' = (x)(e^x) + (e^x)(1)$$
$$= xe^x + e^x$$
$$= e^x(x + 1)$$

**b.** $y' = (x)(10e^{10x}) + (e^{10x})(1)$

$$= 10xe^{10x} + e^{10x}$$
$$y'' = (10x)(10e^{10x}) + (e^{10x})(10) + 10e^{10x}$$
$$= 100xe^{10x} + 10e^{10x} + 10e^{10x}$$
$$= 100xe^{10x} + 20e^{10x}$$
$$= 20e^{10x}(5x + 1)$$

**7.** $y = \frac{e^{2x} - 1}{e^{2x} + 1}$

$$\frac{dy}{dx} = \frac{2e^{2x}(e^{2x} + 1) - (e^{2x} - 1)(2e^{2x})}{(e^{2x} + 1)^2}$$
$$= \frac{2e^{4x} + 2e^{2x} - 2e^{4x} + 2e^{2x}}{(e^{2x} + 1)^2}$$
$$= \frac{4e^{2x}}{(e^{2x} + 1)^2}$$

Now, $1 - y^2 = 1 - \frac{e^{4x} - 2e^{2x} + 1}{(e^{2x} + 1)^2}$

$$= \frac{e^{4x} + 2e^{2x} + 1 - e^{4x} + 2e^{2x} - 1}{(e^{2x} + 1)^2}$$
$$= \frac{4e^{2x}}{(3^{2x} + 1)^2} = \frac{dy}{dx}$$

**8.** The slope of the required tangent line is 3.
The slope at any point on the curve is given by $\frac{dy}{dx} = 1 + e^{-x}$.
To find the point(s) on the curve where the tangent has slope 3, we solve:

$$1 + e^{-x} = 3$$
$$e^{-x} = 2$$
$$-x = \ln 2$$
$$x = -\ln 2.$$

The point of contact of the tangent is $(-\ln 2, -\ln 2 - 2)$.
The equation of the tangent line is
$y + \ln 2 + 2 = 3(x + \ln 2)$ or
$3x - y + 2\ln 2 - 2 = 0.$

**9.** When $x = \frac{\pi}{2}$,

$$y = f(x) = f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2}(1) = \frac{\pi}{2}$$
$$y' = f'(x) = (x)(\cos x) + (\sin x)(1)$$
$$= x\cos x + \sin x$$
$$f'\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}$$
$$= \frac{\pi}{2}(0) + 1$$
$$= 1$$

So an equation for the tangent at the point $x = \frac{\pi}{2}$ is

$$y - \frac{\pi}{2} = 1\left(x - \frac{\pi}{2}\right)$$
$$y - \frac{\pi}{2} = x - \frac{\pi}{2}$$
$$y = x$$
$$-x + y = 0$$

**10.** If $s(t) = \frac{\sin t}{3 + \cos 2t}$ is the function describing an object's position at time $t$, then $v(t) = s'(t)$ is the function describing the object's velocity at time $t$. So

$$v(t) = s'(t)$$
$$= \frac{(3 + \cos 2t)(\cos t) - (\sin t)(-2\sin 2t)}{(3 + \cos 2t)^2}$$
$$s'\left(\frac{\pi}{4}\right) = \frac{\left(3 + \cos 2 \cdot \frac{\pi}{4}\right)\left(\cos\frac{\pi}{4}\right)}{\left(3 + \cos 2 \cdot \frac{\pi}{4}\right)^2} - \frac{\left(\sin\frac{\pi}{4}\right)\left(-2\sin 2 \cdot \frac{\pi}{4}\right)}{\left(3 + \cos 2 \cdot \frac{\pi}{4}\right)^2}$$
$$= \frac{\left(3 + \cos\frac{\pi}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(-2\sin\frac{\pi}{2}\right)}{\left(3 + \cos\frac{\pi}{2}\right)^2}$$
$$= \frac{(3 + 0)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)(-2 \cdot 1)}{(3 + 0)^2}$$
$$= \frac{\frac{3\sqrt{2}}{2} + \sqrt{2}}{9}$$
$$= \frac{3\sqrt{2} + 2\sqrt{2}}{2} \cdot \frac{1}{9}$$
$$= \frac{5\sqrt{2}}{18}$$

So, the object's velocity at time $t = \frac{\pi}{4}$ is $\frac{5\sqrt{2}}{18} \doteq 0.3928$ metres per unit of time.

**11. a.** The question asks for the time $t$ when $N'(t) = 0$.

$N(t) = 60\,000 + 2000te^{-\frac{t}{20}}$

$N'(t) = 0 + (2000t)\left(-\frac{1}{20}e^{-\frac{t}{20}}\right) + (e^{-\frac{t}{20}})(2000)$

$= -100te^{-\frac{t}{20}} + 2000e^{-\frac{t}{20}}$

$= 100e^{-\frac{t}{20}}(-t + 20)$

Set $N'(t) = 0$ and solve for $t$.

$0 = 100e^{-\frac{t}{20}}(-t + 20)$

$100e^{-\frac{t}{20}}$ is never equal to zero.

$-t + 20 = 0$

$20 = t$

Therefore, the rate of change of the number of bacteria is equal to zero when time $t = 20$.

**b.** The question asks for $\frac{dM}{dt} = M'(t)$ when $t = 10$. That is, it asks for $M'(10)$.

$M(t) = (N + 1000)^{\frac{1}{3}}$

$M'(t) = \frac{1}{3}(N + 1000)^{-\frac{2}{3}} \cdot \frac{d(N + 1000)}{dt}$

$= \frac{1}{3(N + 1000)^{\frac{2}{3}}} \cdot \frac{dN}{dt}$

From part a., $\frac{dN}{dt} = N'(t) = 100e^{-\frac{t}{20}}(-t + 20)$ and $N(t) = 60000 + 2000te^{-\frac{t}{20}}$

So $M'(t) = \frac{100e^{-\frac{1}{20}}(-t + 20)}{3(N + 1000)^{\frac{2}{3}}}$

First calculate $N(10)$.

$N(10) = 60000 + 2000(10)e^{-\frac{10}{20}}$

$= 60000 + 20000e^{-\frac{1}{2}}$

$\doteq 72131$

So $M'(10) = \frac{100e^{-\frac{10}{20}}(-10 + 20)}{3(N(10) + 1000)^{\frac{2}{3}}}$

$= \frac{100e^{-\frac{1}{2}}(10)}{3(72131 + 1000)^{\frac{2}{3}}}$

$\doteq \frac{606.53}{5246.33}$

$\doteq 0.1156$

So, after 10 days, about 0.1156 mice are infected per day. Essentially, almost 0 mice are infected per day when $t = 10$.

**12. a.** $c_1(t) = te^{-t}$; $c_1(0) = 0$

$c_1'(t) = e^{-t} - te^{-t}$

$= e^{-t}(1 - t)$

Since $e^{-t} > 0$ for all $t$, $c_1'(t) = 0$ when $t = 1$.

Since $c_1'(t) > 0$ for $0 \le t < 1$, and $c_1'(t) < 0$ for all $t > 1$, $c_1(t)$ has a maximum value of $\frac{1}{e} \doteq 0.368$ at $t = 1$ h.

$c_2(t) = t^2e^{-t}$; $c_2(0) = 0$

$c_2'(t) = 2te^{-t} - t^2e^{-t}$

$= te^{-t}(2 - t)$

$c_2'(t) = 0$ when $t = 0$ or $t = 2$.

Since $c_2'(t) > 0$ for $0 < t < 2$ and $c_2'(t) < 0$ for all $t > 2$, $c_2(t)$ has a maximum value of $\frac{4}{e^2} \doteq 0.541$ at $t = 2$ h. The larger concentration occurs for medicine $c_2$.

**b.** $c_1(0.5) = 0.303$

$c_2(0.5) = 0.152$

In the first half-hour, the concentration of $c_1$ increases from 0 to 0.303, and that of $c_2$ increases from 0 to 0.152. Thus, $c_1$ has the larger concentration over this interval.

**13. a.** $y = (2 + 3e^{-x})^3$

$y' = 3(2 + 3e^{-x})^2[0 + 3e^{-x}(-1)]$

$= 3(2 + 3e^{-x})^2(-3e^{-x})$

$= -9e^{-x}(2 + 3e^{-x})^2$

**b.** $y = x^e$

$y' = ex^{e-1}$

**c.** $y = e^{e^x}$

$y' = e^{e^x}(e^x)(1)$

$= e^{x+e^x}$

**d.** $y = (1 - e^{5x})^5$

$y' = 5(1 - e^{5x})^4[0 - e^{5x}(5)]$

$= -25e^{5x}(1 - e^{5x})^4$

**14. a.** $y = 5^x$

$y' = 5^x \ln 5$

**b.** $y = (0.47)^x$

$y' = (0.47)^x \ln(0.47)$

**c.** $y = (52)^{2x}$

$y' = (52)^{2x}(2) \ln 52$

$= 2(52)^{2x} \ln 52$

**d.** $y = 5(2)^x$

$y' = 5(2)^x \ln 2$

**e.** $y = 4e^x$

$y' = 4e^x(1) \ln e$

$= 4e^x$

**f.** $y = -2(10)^{3x}$

$y' = -2(3)10^{3x} \ln 10$

$= -6(10)^{3x} \ln 10$

**15. a.** $y' = \cos 2^x \cdot \frac{d(2^x)}{dx}$

$= 2^x \ln 2 \cos 2^x$

**b.** $y' = (x^2)(\cos x) + (\sin x)(2x)$
$= x^2 \cos x + 2x \sin x$

**c.** $y' = \cos\left(\frac{\pi}{2} - x\right) \cdot \frac{d(\frac{\pi}{2} - x)}{dx}$
$= -\cos\left(\frac{\pi}{2} - x\right)$

**d.** $y' = (\cos x)(\cos x) + (\sin x)(-\sin x)$
$= \cos^2 x - \sin^2 x$

**e.** $y = (\cos x)^2$
$y' = 2(\cos x) \cdot \frac{d(\cos x)}{dx}$
$= -2 \cos x \sin x$

**f.** $y = \cos x\ (\sin x)^2$
$y' = (\cos x)(2\ (\sin x)(\cos x)) + (\sin x)^2(-\sin x)$
$= 2 \sin x \cos^2 x - \sin^3 x$

**16.** Compute $\frac{dy}{dx}$ when $x = \frac{\pi}{2}$ to find the slope of the line at the given point.
$y' = -\sin x$
So, at the point $x = \frac{\pi}{2}$, $y' = f'(x)$ is
$f'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1.$
Therefore, an equation of the line tangent to the curve at the given point is

$$y - 0 = -1\left(x - \frac{\pi}{2}\right)$$

$$y = -x + \frac{\pi}{2}$$

$$x + y - \frac{\pi}{2} = 0$$

**17.** The velocity of the object at any time $t$ is $v = \frac{ds}{dt}$.
Thus, $v = 8\ (\cos(10\pi t))(10\pi)$
$= 80\pi \cos(10\pi t)$.
The acceleration at any time $t$ is $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.
Hence, $a = 80\pi(-\sin(10\pi t))(10\pi) = -800\pi^2 \sin(10\pi t)$.
Now, $\frac{d^2s}{dt^2} + 100\pi^2 s$
$= -800\pi^2 \sin(10\pi t)$
$+ 100\pi^2(8 \sin(10\pi t)) = 0.$

**18.** Since $s = 5 \cos\left(2t + \frac{\pi}{4}\right)$,

$$v = \frac{ds}{dt} = 5\left(-\sin\left(2t + \frac{\pi}{4}\right)\right)$$

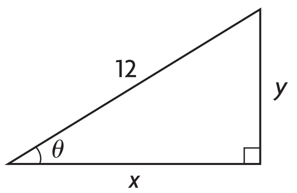
$$= -10 \sin\left(2t + \frac{\pi}{4}\right)$$

and $a = \frac{dv}{dt} = -10\left(\cos\left(2t + \frac{\pi}{4}\right)\right)$

$$= -20 \cos\left(2t + \frac{\pi}{4}\right).$$

The maximum values of the displacement, velocity, and acceleration are 5, 10, and 20, respectively.

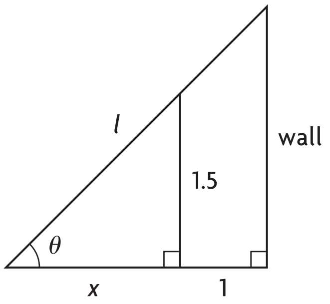
**19.** Let the base angle be $\theta$, $0 < \theta < \frac{\pi}{2}$, and let the sides of the triangle have lengths $x$ and $y$, as shown. Let the perimeter of the triangle be $P$ cm.




Now, $\frac{x}{12} = \cos\theta$ and $\frac{y}{12} = \sin\theta$
so $x = 12\cos\theta$ and $y = 12\sin\theta$.
Therefore, $P = 12 + 12\cos\theta + 12\sin\theta$ and
$\frac{dP}{d\theta} = -12\sin\theta + 12\cos\theta$.
For critical values, $-12\sin\theta + 12\cos\theta = 0$
$\sin\theta = \cos\theta$
$\tan\theta = 1$

$\theta = \frac{\pi}{4}$, since $0 < \theta < \frac{\pi}{2}$.
When $\theta = \frac{\pi}{4}$, $P = 12 + \frac{12}{\sqrt{2}} + \frac{12}{\sqrt{2}}$
$= 12 + \frac{24}{\sqrt{2}}$
$= 12 + 12\sqrt{2}$.
As $\theta \to 0^+$, $\cos\theta \to 1$, $\sin\theta \to 0$, and
$P \to 12 + 12 + 0 = 24$.
As $\theta \to \frac{\pi}{2}$, $\cos\theta \to 0$, $\sin\theta \to 1$ and
$P \to 12 + 0 + 12 = 24$.
Therefore, the maximum value of the perimeter is $12 + 12\sqrt{2}$ cm, and occurs when the other two angles are each $\frac{\pi}{4}$ rad, or 45°.

**20.** Let $l$ be the length of the ladder, $\theta$ be the angle between the foot of the ladder and the ground, and $x$ be the distance of the foot of the ladder from the fence, as shown.
Thus, $\frac{x + 1}{l} = \cos\theta$ and $\frac{1.5}{x} = \tan\theta$
$x + 1 = l\cos\theta$ where $x = \frac{1.5}{\tan\theta}$.




Replacing $x$, $\frac{1.5}{\tan\theta} + 1 = l\cos\theta$

$$l = \frac{1.5}{\sin\theta} + \frac{1}{\cos\theta},\ 0 < \theta < \frac{\pi}{2}$$

$$\frac{dl}{d\theta} = -\frac{1.5\cos\theta}{\sin^2\theta} + \frac{\sin\theta}{\cos^2\theta}$$

$$= \frac{-1.5\cos^3\theta + \sin^3\theta}{\sin^2\theta\cos^2\theta}.$$

Solving $\frac{dl}{d\theta} = 0$ yields:

$$\sin^3\theta - 1.5\cos^3\theta = 0$$

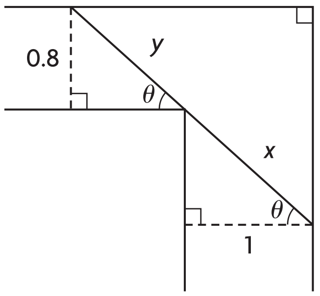
$$\tan^3\theta = 1.5$$

$$\tan\theta = \sqrt[3]{1.5}$$

$$\theta \doteq 0.46365.$$

The length of the ladder corresponding to this value of $\theta$ is $l \doteq 4.5$ m. As $\theta \to 0^+$ and $\frac{\pi^-}{2}$, $l$ increases without bound. Therefore, the shortest ladder that goes over the fence and reaches the wall has a length of 4.5 m.

**21.** The longest pole that can fit around the corner is determined by the minimum value of $x + y$. Thus, we need to find the minimum value of $l = x + y$.




From the diagram, $\frac{0.8}{y} = \sin\theta$ and $\frac{1}{x} = \cos\theta$.

Thus, $l = \frac{1}{\cos\theta} + \frac{0.8}{\sin\theta},\ 0 \le \theta \le \frac{\pi}{2}$:

$$\frac{dl}{d\theta} = \frac{1\sin\theta}{\cos^2\theta} - \frac{0.8\cos\theta}{\sin^2\theta}$$

$$= \frac{0.8\sin^3\theta - \cos^3\theta}{\cos^2\theta\sin^2\theta}.$$

Solving $\frac{dl}{d\theta} = 0$ yields:

$$0.8\sin^3\theta - \cos^3\theta = 0$$

$$\tan^3\theta = 1.25$$

$$\tan\theta = \sqrt[3]{1.25}$$

$$\tan\theta \doteq 1.077$$

$$\theta \doteq 0.822.$$

Now, $l = \frac{0.8}{\cos(0.822)} + \frac{1}{\sin(0.822)} \doteq 2.5.$

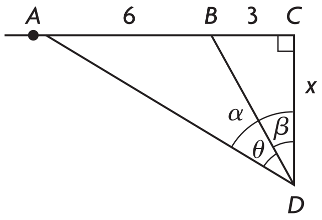
When $\theta = 0$, the longest possible pole would have a length of 0.8 m. When $\theta = \frac{\pi}{2}$, the longest possible pole would have a length of 1 m. Therefore, the longest pole that can be carried horizontally around the corner is one of length 2.5 m.

**22.** We want to find the value of $x$ that maximizes $\theta$. Let $\angle ADC = \alpha$ and $\angle BDC = \beta$.

Thus, $\theta = \alpha - \beta$:

$$\tan\theta = \tan(\alpha - \beta)$$

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$




From the diagram, $\tan\alpha = \frac{9}{x}$ and $\tan\beta = \frac{3}{x}$.

Hence, $\tan\theta = \dfrac{\frac{9}{x} - \frac{3}{x}}{1 + \frac{27}{x^2}}$

$$= \frac{9x - 3x}{x^2 + 27}$$

$$= \frac{6x}{x^2 + 27}.$$

We differentiate implicitly with respect to $x$:

$$\sec^2\theta\frac{d\theta}{dx} = \frac{6(x^2 + 27) - 6x(2x)}{(x^2 + 27)^2}$$

$$\frac{d\theta}{dx} = \frac{162 - 6x^2}{\sec^2\theta(x^2 + 27)^2}$$

Solving $\frac{d\theta}{dx} = 0$ yields:

$$162 - 6x^2 = 0$$

$$x^2 = 27$$

$$x = 3\sqrt{3}.$$

**23. a.** $f(x) = 4(\sin(x-2))^2$

$f'(x) = 8\sin(x-2)\cos(x-2)$

$f''(x) = (8\sin(x-2))(-\sin(x-2)) + (\cos(x-2))(8\cos(x-2))$

$= -8\sin^2(x-2) + 8\cos^2(x-2)$

**b.** $f(x) = (2\cos x)(\sec x)^2$

$f'(x) = (2\cos x)(2\sec x \cdot \sec x \tan x) + (\sec x)^2(-2\sin x)$

$= (4\cos x)(\sec^2 x \tan x) - 2\sin x(\sec x)^2$

Using the product rule multiple times,

$f''(x) = (4\cos x)\left[\sec^2 x \cdot \sec^2 x + \tan x(2\sec x \cdot \sec x \tan x)\right]$
$+ (\sec^2 x \tan x)(-4\sin x)$
$+ (-2\sin x)(2\sec x \cdot \sec x \tan x)$
$+ (\sec x)^2(-2\cos x)$

$= 4\cos x \sec^4 x + 8\cos x \tan^2 x \sec^2 x - 4\sin x \tan x \sec^2 x - 4\sin x \tan x \sec^2 x - 2\cos x \sec^2 x$

$= 4\cos x \sec^4 x + 8\cos x \tan^2 x \sec^2 x - 8\sin x \tan x \sec^2 x - 2\cos x \sec^2 x$

## Chapter 5 Test, p. 266

**1. a.** $y = e^{-2x^2}$

$\frac{dy}{dx} = -4xe^{-2x^2}$

**b.** $y = 3^{x^2+3x}$

$\frac{dy}{dx} = 3^{x^2+3x} \cdot \ln 3 \cdot (2x+3)$

**c.** $y = \frac{e^{3x} + e^{-3x}}{2}$

$\frac{dy}{dx} = \frac{1}{2}[3e^{3x} - 3e^{-3x}]$

$= \frac{3}{2}[e^{3x} - e^{-3x}]$

**d.** $y = 2\sin x - 3\cos 5x$

$\frac{dy}{dx} = 2\cos x - 3(-\sin 5x)(5)$

$= 2\cos x + 15\sin 5x$

**e.** $y = \sin^3(x^2)$

$\frac{dy}{dx} = 3\sin^2(x^2)(\cos(x^2)(2x))$

$= 6x\sin^2(x^2)\cos(x^2)$

**f.** $y = \tan\sqrt{1-x}$

$\frac{dy}{dx} = \sec^2\sqrt{1-x}\left(\frac{1}{2} \times \frac{1}{\sqrt{1-x}}\right)(-1)$

$= -\frac{\sec^2\sqrt{1-x}}{2\sqrt{1-x}}$

**2.** The given line is $-6x + y = 2$ or $y = 6x + 2$, so the slope is 6.

$y = 2e^{3x}$

$\frac{dy}{dx} = 2e^{3x}(3)$

$= 6e^{3x}$

In order for the tangent line to be parallel to the given line, the derivative has to equal 6 at the tangent point.

$6e^{3x} = 6$

$e^{3x} = 1$

$x = 0$

When $x = 0$, $y = 2$.

The equation of the tangent line is $y - 2 = 6(x - 0)$ or $-6x + y = 2$. The tangent line is the given line.

**3.** $y = e^x + \sin x$

$\frac{dy}{dx} = e^x + \cos x$

When $x = 0$, $\frac{dy}{dx} = 1 + 1$ or 2, so the slope of the tangent line at $(0, 1)$ is 2.

The equation of the tangent line at $(0, 1)$ is $y - 1 = 2(x - 0)$ or $-2x + y = 1$.

**4.** $v(t) = 10e^{-kt}$

**a.** $a(t) = v'(t) = -10ke^{-kt}$

$= -k(10e^{-kt})$

$= -kv(t)$

Thus, the acceleration is a constant multiple of the velocity. As the velocity of the particle decreases, the acceleration increases by a factor of $k$.

**b.** At time $t = 0$, $v = 10$ cm/s.

**c.** When $v = 5$, we have $10e^{-kt} = 5$

$e^{-kt} = \frac{1}{2}$

$-kt = \ln\left(\frac{1}{2}\right) = -\ln 2$

$t = \frac{\ln 2}{k}.$

After $\frac{\ln 2}{k}$ s have elapsed, the velocity of the particle is 5 cm/s. The acceleration of the particle is $-5k$ at this time.

**5. a.** $f(x) = (\cos x)^2$

$f'(x) = 2(\cos x) \cdot \frac{d(\cos x)}{dx}$

$= 2(\cos x) \cdot (-\sin x)$

$= -2\sin x \cos x$

$f''(x) = (-2\sin x)(-\sin x) + (\cos x)(-2\cos x)$

$= 2\sin^2 x - 2\cos^2 x$

$= 2(\sin^2 x - \cos^2 x)$

**23. a.** $f(x) = 4(\sin(x-2))^2$

$f'(x) = 8\sin(x-2)\cos(x-2)$

$f''(x) = (8\sin(x-2))(-\sin(x-2)) + (\cos(x-2))(8\cos(x-2))$

$= -8\sin^2(x-2) + 8\cos^2(x-2)$

**b.** $f(x) = (2\cos x)(\sec x)^2$

$f'(x) = (2\cos x)(2\sec x \cdot \sec x \tan x) + (\sec x)^2(-2\sin x)$

$= (4\cos x)(\sec^2 x \tan x) - 2\sin x(\sec x)^2$

Using the product rule multiple times,

$f''(x) = (4\cos x)\left[\sec^2 x \cdot \sec^2 x + \tan x(2\sec x \cdot \sec x \tan x)\right]$

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$+ (-2\sin x)(2\sec x \cdot \sec x \tan x)$

$+ (\sec x)^2(-2\cos x)$

$= 4\cos x \sec^4 x + 8\cos x \tan^2 x \sec^2 x - 4\sin x \tan x \sec^2 x - 4\sin x \tan x \sec^2 x - 2\cos x \sec^2 x$

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## Chapter 5 Test, p. 266

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$\frac{dy}{dx} = 3^{x^2+3x} \cdot \ln 3 \cdot (2x+3)$

**c.** $y = \frac{e^{3x} + e^{-3x}}{2}$

$\frac{dy}{dx} = \frac{1}{2}[3e^{3x} - 3e^{-3x}]$

$= \frac{3}{2}[e^{3x} - e^{-3x}]$

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$= 6x\sin^2(x^2)\cos(x^2)$

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$= -\frac{\sec^2\sqrt{1-x}}{2\sqrt{1-x}}$

**2.** The given line is $-6x + y = 2$ or $y = 6x + 2$, so the slope is 6.

$y = 2e^{3x}$

$\frac{dy}{dx} = 2e^{3x}(3)$

$= 6e^{3x}$

In order for the tangent line to be parallel to the given line, the derivative has to equal 6 at the tangent point.

$6e^{3x} = 6$

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$x = 0$

When $x = 0$, $y = 2$.

The equation of the tangent line is $y - 2 = 6(x - 0)$ or $-6x + y = 2$. The tangent line is the given line.

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Thus, the acceleration is a constant multiple of the velocity. As the velocity of the particle decreases, the acceleration increases by a factor of $k$.

**b.** At time $t = 0$, $v = 10$ cm/s.

**c.** When $v = 5$, we have $10e^{-kt} = 5$

$e^{-kt} = \frac{1}{2}$

$-kt = \ln\left(\frac{1}{2}\right) = -\ln 2$

$t = \frac{\ln 2}{k}.$

After $\frac{\ln 2}{k}$ s have elapsed, the velocity of the particle is 5 cm/s. The acceleration of the particle is $-5k$ at this time.

**5. a.** $f(x) = (\cos x)^2$

$f'(x) = 2(\cos x) \cdot \frac{d(\cos x)}{dx}$

$= 2(\cos x) \cdot (-\sin x)$

$= -2\sin x \cos x$

$f''(x) = (-2\sin x)(-\sin x) + (\cos x)(-2\cos x)$

$= 2\sin^2 x - 2\cos^2 x$

$= 2(\sin^2 x - \cos^2 x)$

**b.** $f(x) = \cos x \cot x$

$$f'(x) = (\cos x)(-\csc^2 x) + (\cot x)(-\sin x)$$
$$= -\cos x \cdot \frac{1}{\sin^2 x} - \frac{\cos x}{\sin x} \cdot \sin x$$
$$= -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} - \cos x$$
$$= -\cot x \csc x - \cos x$$

$f''(x) =$

$(-\cot x)(-\csc x \cot x) + (\csc x)(\csc^2 x) + \sin x$

$= \csc x \cot^2 x + \csc^3 x + \sin x$

**6.** $f(x) = (\sin x)^2$

To find the absolute extreme values, first find the derivative, set it equal to zero, and solve for $x$.

$$f'(x) = 2(\sin x) \cdot \frac{d(\sin x)}{dx}$$
$$= 2 \sin x \cos x$$
$$= \sin 2x$$

Now set $f'(x) = 0$ and solve for $x$.

$0 = \sin 2x$

$2x = 0, \pi, 2\pi$

$x = 0, \frac{\pi}{2}, \pi$ in the interval $0 \le x \le \pi$.

Evaluate $f(x)$ at the critical numbers, including the endpoints of the interval.

| $x$ | 0 | $\frac{\pi}{2}$ | $\pi$ |
|---|---|---|---|
| $f(x) = (\sin^2 x)$ | 0 | 1 | 0 |

So, the absolute maximum value on the interval is 1 when $x = \frac{\pi}{2}$ and the absolute minimum value on the interval is 0 when $x = 0$ and $x = \pi$.

**7.** $y = f(x) = 5^x$

Find the derivative, $f'(x)$, and evaluate the derivative at $x = 2$ to find the slope of the tangent when $x = 2$.

$f'(x) = 5^x \ln 5$

$f'(2) = 5^2 \ln 5$

$= 25 \ln 5$

$\doteq 40.24$

**8.** $y = xe^x + 3e^x$

To find the maximum and minimum values, first find the derivative, set it equal to zero, and solve for $x$.

$y' = (x)(e^x) + (e^x)(1) + 3e^x$

$= xe^x + e^x + 3e^x$

$= xe^x + 4e^x$

$= e^x(x + 4)$

Now set $y' = 0$ and solve for $x$.

$0 = e^x(x + 4)$

$e^x$ is never equal to zero.

$(x + 4) = 0$

So $x = -4$.

Therefore, the critical value is $-4$.

| Interval | $e^x(x + 4)$ |
|---|---|
| $x < -4$ | − |
| $-4 < x$ | + |

So $f(x)$ is decreasing on the left of $x = -4$ and increasing on the right of $x = -4$. Therefore, the function has a minimum value at $\left(-4, -\frac{1}{e^4}\right)$. There is no maximum value.

**9.** $f(x) = 2 \cos x - \sin 2x$

So, $f(x) = 2 \cos x - 2 \sin x \cos x$.

**a.** $f'(x) = -2 \sin x - (2 \sin x)(-\sin x)$
$- (\cos x)(2 \cos x)$
$= -2 \sin x + 2 \sin^2 x - 2 \cos^2 x$

Set $f'(x) = 0$ to solve for the critical values.

$$-2 \sin x + 2 \sin^2 x - 2 \cos^2 x = 0$$
$$-2 \sin x + 2 \sin^2 x - 2(1 - \sin^2 x) = 0$$
$$-2 \sin x + 2 \sin^2 x - 2 + 2 \sin^2 x = 0$$
$$4 \sin^2 x - 2 \sin x - 2 = 0$$
$$(2 \sin x + 1)(2 \sin x - 2) = 0$$
$$2 \sin x + 1 = 0 \text{ and } 2 \sin x - 2 = 0$$

So, $\sin x = -\frac{1}{2}$.

In the given interval, this occurs when $x = -\frac{\pi}{6}, -\frac{5\pi}{6}$.

Also, $\sin x = 1$.

In the given interval, this occurs when $x = \frac{\pi}{2}$.

Therefore, on the given interval, the critical numbers for $f(x)$ are $x = -\frac{\pi}{6}, -\frac{5\pi}{6}, \frac{\pi}{2}$.

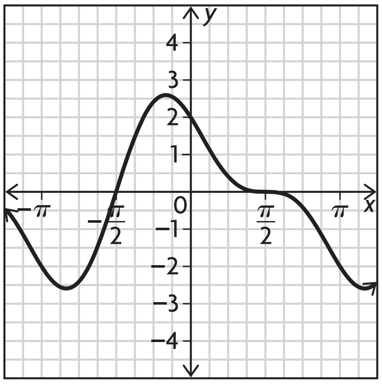
**b.** To determine the intervals where $f(x)$ is increasing and where $f(x)$ is decreasing, find the slope of $f(x)$ in the intervals between the endpoints and the critical numbers. To do this, it helps to make a table.

| $x$ | slope of $f(x)$ |
|---|---|
| $-\pi \le x < -\frac{5\pi}{6}$ | − |
| $-\frac{5\pi}{6} < x < -\frac{\pi}{6}$ | + |
| $-\frac{\pi}{6} < x < \frac{\pi}{2}$ | − |
| $\frac{\pi}{2} < x \le \pi$ | − |

So, $f(x)$ is increasing on the interval $-\frac{5\pi}{6} < x < -\frac{\pi}{6}$ and $f(x)$ is decreasing on the intervals $-\pi \le x < -\frac{5\pi}{6}$ and $-\frac{\pi}{6} < x < \pi$.

**c.** From the table in part b., it can be seen that there is a local maximum at the point where $x = -\frac{\pi}{6}$ and there is a local minimum at the point where $x = -\frac{5\pi}{6}$.

**d.**




## Cumulative Review of Calculus

**1. a.** $f(x) = 3x^2 + 4x - 5$

$$m = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$
$$= \lim_{h\to 0}\frac{f(2+h)-15}{h}$$
$$= \lim_{h\to 0}\frac{3(2+h)^2+4(2+h)-5-15}{h}$$
$$= \lim_{h\to 0}\frac{12+12h+3h^2+8+4h-20}{h}$$
$$= \lim_{h\to 0}\frac{3h^2+16h}{h}$$
$$= \lim_{h\to 0} 3h+16$$
$$= 16$$

**b.** $f(x) = \frac{2}{x-1}$

$$m = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$
$$= \lim_{h\to 0}\frac{f(2+h)-2}{h}$$
$$= \lim_{h\to 0}\frac{\frac{2}{2+h-1}-2}{h}$$
$$= \lim_{h\to 0}\frac{\frac{2}{1+h}-\frac{2(1+h)}{1+h}}{h}$$
$$= \lim_{h\to 0}\frac{2-2(1+h)}{h(1+h)}$$
$$= \lim_{h\to 0}\frac{-2h}{h(1+h)}$$
$$= \lim_{h\to 0}\frac{-2}{1+h}$$
$$= -2$$

**c.** $f(x) = \sqrt{x+3}$

$$m = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$
$$= \lim_{h\to 0}\frac{f(6+h)-3}{h}$$
$$= \lim_{h\to 0}\frac{\sqrt{h+9}-3}{h}$$
$$= \lim_{h\to 0}\frac{(\sqrt{h+9}-3)(\sqrt{h+9}+3)}{h(\sqrt{h+9}+3)}$$
$$= \lim_{h\to 0}\frac{h+9-9}{h(\sqrt{h+9}+3)}$$
$$= \lim_{h\to 0}\frac{h}{h(\sqrt{h+9}+3)}$$
$$= \lim_{h\to 0}\frac{1}{(\sqrt{h+9}+3)}$$
$$= \frac{1}{6}$$

**d.** $f(x) = 2^{5x}$

$$m = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$
$$= \lim_{h\to 0}\frac{2^{5(1+h)}-32}{h}$$
$$= \lim_{h\to 0}\frac{2^5\cdot 2^{5h}-32}{h}$$
$$= \lim_{h\to 0}\frac{32(2^{5h}-1)}{h}$$
$$= 32\lim_{h\to 0}\frac{5(2^{5h}-1)}{5h}$$
$$= 160\lim_{h\to 0}\frac{(2^{5h}-1)}{5h}$$
$$= 160\ln 2$$

**2. a.** average velocity $= \frac{\text{change in distance}}{\text{change in time}}$

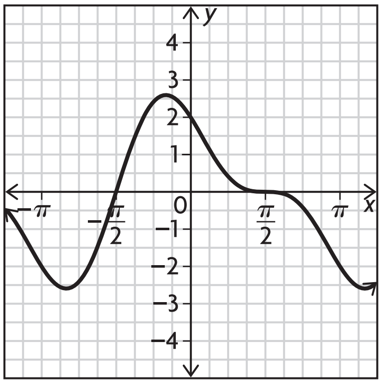
$$= \frac{s(t_2)-s(t_1)}{t_2-t_1}$$
$$= \frac{[2(4)^2+3(4)+1]-[(2(1)^2+3(1)+1)]}{4-1}$$
$$= \frac{45-6}{3}$$
$$= 13\text{ m/s}$$

**b.** instantaneous velocity = slope of the tangent

$$m = \lim_{h\to 0}\frac{s(a+h)-s(a)}{h}$$

**c.** From the table in part b., it can be seen that there is a local maximum at the point where $x = -\frac{\pi}{6}$ and there is a local minimum at the point where $x = -\frac{5\pi}{6}$.

**d.**



## Cumulative Review of Calculus

**1. a.** $f(x) = 3x^2 + 4x - 5$

$$m = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$
$$= \lim_{h\to 0}\frac{f(2+h) - 15}{h}$$
$$= \lim_{h\to 0}\frac{3(2+h)^2 + 4(2+h) - 5 - 15}{h}$$
$$= \lim_{h\to 0}\frac{12 + 12h + 3h^2 + 8 + 4h - 20}{h}$$
$$= \lim_{h\to 0}\frac{3h^2 + 16h}{h}$$
$$= \lim_{h\to 0} 3h + 16$$
$$= 16$$

**b.** $f(x) = \dfrac{2}{x-1}$

$$m = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$
$$= \lim_{h\to 0}\frac{f(2+h) - 2}{h}$$
$$= \lim_{h\to 0}\frac{\frac{2}{2+h-1} - 2}{h}$$
$$= \lim_{h\to 0}\frac{\frac{2}{1+h} - \frac{2(1+h)}{1+h}}{h}$$
$$= \lim_{h\to 0}\frac{2 - 2(1+h)}{h(1+h)}$$
$$= \lim_{h\to 0}\frac{-2h}{h(1+h)}$$
$$= \lim_{h\to 0}\frac{-2}{1+h}$$
$$= -2$$

**c.** $f(x) = \sqrt{x+3}$

$$m = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$
$$= \lim_{h\to 0}\frac{f(6+h) - 3}{h}$$
$$= \lim_{h\to 0}\frac{\sqrt{h+9} - 3}{h}$$
$$= \lim_{h\to 0}\frac{(\sqrt{h+9} - 3)(\sqrt{h+9} + 3)}{h(\sqrt{h+9} + 3)}$$
$$= \lim_{h\to 0}\frac{h + 9 - 9}{h(\sqrt{h+9} + 3)}$$
$$= \lim_{h\to 0}\frac{h}{h(\sqrt{h+9} + 3)}$$
$$= \lim_{h\to 0}\frac{1}{(\sqrt{h+9} + 3)}$$
$$= \frac{1}{6}$$

**d.** $f(x) = 2^{5x}$

$$m = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$
$$= \lim_{h\to 0}\frac{2^{5(1+h)} - 32}{h}$$
$$= \lim_{h\to 0}\frac{2^5 \cdot 2^{5h} - 32}{h}$$
$$= \lim_{h\to 0}\frac{32(2^{5h} - 1)}{h}$$
$$= 32\lim_{h\to 0}\frac{5(2^{5h} - 1)}{5h}$$
$$= 160\lim_{h\to 0}\frac{(2^{5h} - 1)}{5h}$$
$$= 160 \ln 2$$

**2. a.** average velocity $= \dfrac{\text{change in distance}}{\text{change in time}}$

$$= \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$
$$= \frac{[2(4)^2 + 3(4) + 1] - [(2(1)^2 + 3(1) + 1)]}{4 - 1}$$
$$= \frac{45 - 6}{3}$$
$$= 13 \text{ m/s}$$

**b.** instantaneous velocity = slope of the tangent

$$m = \lim_{h\to 0}\frac{s(a+h) - s(a)}{h}$$

$$= \lim_{h \to 0} \frac{s(3+h) - s(3)}{h}$$

$$= \lim_{h \to 0} \left[ \frac{2(3+h)^2 + 3(3+h) + 1}{h} - \frac{(2(3)^2 + 3(3) + 1)}{h} \right]$$

$$= \lim_{h \to 0} \frac{18 + 12h + 2h^2 + 9 + 3h + 1 - 28}{h}$$

$$= \lim_{h \to 0} \frac{15h + 2h^2}{h}$$

$$= \lim_{h \to 0} (15 + 2h)$$

$= 15$ m/s

**3.** $$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$\lim_{h \to 0} \frac{(4+h)^3 - 64}{h} = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$

$$(4+h)^3 - 64 = f(4+h) - f(4)$$

Therefore, $f(x) = x^3$.

**4. a.** Average rate of change in distance with respect to time is average velocity, so

$$\text{average velocity} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

$$= \frac{s(3) - s(1)}{3 - 1}$$

$$= \frac{4.9(3)^2 - 4.9(1)}{3 - 1}$$

$= 19.6$ m/s

**b.** Instantaneous rate of change in distance with respect to time = slope of the tangent.

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{4.9(2+h)^2 - 4.9(2)^2}{h}$$

$$= \lim_{h \to 0} \frac{19.6 + 19.6h + 4.9h^2 - 19.6}{h}$$

$$= \lim_{h \to 0} \frac{19.6h + 4.9h^2}{h}$$

$$= \lim_{h \to 0} 19.6 + 4.9h$$

$= 19.6$ m/s

**c.** First, we need to determine $t$ for the given distance:

$146.9 = 4.9t^2$

$29.98 = t^2$

$5.475 = t$

Now use the slope of the tangent to determine the instantaneous velocity for $t = 5.475$:

$$m = \lim_{h \to 0} \frac{f(5.475 + h) - f(5.475)}{h}$$

$$= \lim_{h \to 0} \frac{4.9(5.475 + h)^2 - 4.9(5.475)^2}{h}$$

$$= \lim_{h \to 0} \frac{146.9 + 53.655h + 4.9h^2 - 146.9}{h}$$

$$= \lim_{h \to 0} \frac{53.655h + 4.9h^2}{h}$$

$$= \lim_{h \to 0} [53.655 + 4.9h]$$

$= 53.655$ m/s

**5. a.** Average rate of population change

$$= \frac{p(t_2) - p(t_1)}{t_2 - t_1}$$

$$= \frac{2(8)^2 + 3(8) + 1 - (2(0) + 3(0) + 1)}{8 - 0}$$

$$= \frac{128 + 24 + 1 - 1}{8 - 0}$$

$= 19$ thousand fish/year

**b.** Instantaneous rate of population change

$$= \lim_{h \to 0} \frac{p(t+h) - p(t)}{h}$$

$$= \lim_{h \to 0} \frac{p(5+h) - p(5)}{h}$$

$$= \lim_{h \to 0} \left[ \frac{2(5+h)^2 + 3(5+h) + 1}{h} - \frac{(2(5)^2 + 3(5) + 1)}{h} \right]$$

$$= \lim_{h \to 0} \frac{50 + 20h + 2h^2 + 15 + 3h + 1 - 66}{h}$$

$$= \lim_{h \to 0} \frac{2h^2 + 23h}{h}$$

$$= \lim_{h \to 0} 2h + 23$$

$= 23$ thousand fish/year

**6. a. i.** $f(2) = 3$

**ii.** $\lim_{x \to 2^-} f(x) = 1$

**iii.** $\lim_{x \to 2^+} f(x) = 3$

**iv.** $\lim_{x \to 6} f(x) = 2$

**b.** No, $\lim_{x \to 4} f(x)$ does not exist. In order for the limit to exist, $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$ must exist and they must be equal. In this case, $\lim_{x \to 4^-} f(x) = \infty$, but $\lim_{x \to 4^+} f(x) = -\infty$, so $\lim_{x \to 4} f(x)$ does not exist.

**7.** $f(x)$ is discontinuous at $x = 2$. $\lim_{x\to 2^-} f(x) = 5$, but $\lim_{x\to 2^+} f(x) = 3$.

**8. a.** $\lim_{x\to 0} \frac{2x^2+1}{x-5} = \frac{2(0)^2+1}{0-5}$

$= -\frac{1}{5}$

**b.** $\lim_{x\to 3} \frac{x-3}{\sqrt{x+6}-3}$

$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x+6}+3)}{(\sqrt{x+6}-3)(\sqrt{x+6}+3)}$

$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x+6}+3)}{x+6-9}$

$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x+6}+3)}{x-3}$

$= \lim_{x\to 3} \sqrt{x+6}+3$

$= 6$

**c.** $\lim_{x\to -3} \frac{\frac{1}{x}+\frac{1}{3}}{x+3}$

$= \lim_{x\to -3} \frac{\frac{x+3}{3x}}{x+3}$

$= \lim_{x\to -3} \frac{x+3}{3x(x+3)}$

$= \lim_{x\to -3} \frac{1}{3x}$

$= -\frac{1}{9}$

**d.** $\lim_{x\to 2} \frac{x^2-4}{x^2-x-2}$

$= \lim_{x\to 2} \frac{(x+2)(x-2)}{(x+1)(x-2)}$

$= \lim_{x\to 2} \frac{x+2}{x+1}$

$= \frac{4}{3}$

**e.** $\lim_{x\to 2} \frac{x-2}{x^3-8}$

$= \lim_{x\to 2} \frac{x-2}{(x-2)(x^2+2x+4)}$

$= \lim_{x\to 2} \frac{1}{x^2+2x+4}$

$= \frac{1}{12}$

**f.** $\lim_{x\to 0} \frac{\sqrt{x+4}-\sqrt{4-x}}{x}$

$= \lim_{x\to 0} \frac{(\sqrt{x+4}-\sqrt{4-x})(\sqrt{x+4}+\sqrt{4-x})}{x(\sqrt{x+4}+\sqrt{4-x})}$

$= \lim_{x\to 0} \frac{x+4-(4-x)}{x(\sqrt{x+4}+\sqrt{4-x})}$

$= \lim_{x\to 0} \frac{2x}{x(\sqrt{x+4}+\sqrt{4-x})}$

$= \lim_{x\to 0} \frac{2}{(\sqrt{x+4}+\sqrt{4-x})}$

$= \frac{1}{2}$

**9. a.** $f(x) = 3x^2 + x + 1$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \left[\frac{3(x+h)^2+(x+h)+1}{h} - \frac{(3x^2+x+1)}{h}\right]$

$= \lim_{h\to 0} \left[\frac{3x^2+6hx+6h^2+x+h}{h} + \frac{1-3x^2-x-1}{h}\right]$

$= \lim_{h\to 0} \frac{6hx+6h^2+h}{h}$

$= \lim_{h\to 0} 6x+6h+1$

$= 6x+1$

**b.** $f(x) = \frac{1}{x}$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h}$

$= \lim_{h\to 0} \frac{x-(x+h)}{h(x)(x+h)}$

$= \lim_{h\to 0} \frac{-h}{h(x)(x+h)}$

$= \lim_{h\to 0} \frac{-1}{x(x+h)}$

$= -\frac{1}{x^2}$

**10. a.** To determine the derivative, use the power rule:

$y = x^3 - 4x^2 + 5x + 2$

$\frac{dy}{dx} = 3x^2 - 8x + 5$

**b.** To determine the derivative, use the chain rule:

$$y = \sqrt{2x^3 + 1}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{2x^3 + 1}}(6x^2)$$

$$= \frac{3x^2}{\sqrt{2x^3 + 1}}$$

**c.** To determine the derivative, use the quotient rule:

$$y = \frac{2x}{x + 3}$$

$$\frac{dy}{dx} = \frac{2(x + 3) - 2x}{(x + 3)^2}$$

$$= \frac{6}{(x + 3)^2}$$

**d.** To determine the derivative, use the product rule:

$$y = (x^2 + 3)^2(4x^5 + 5x + 1)$$

$$\frac{dy}{dx} = 2(x^2 + 3)(2x)(4x^5 + 5x + 1) + (x^2 + 3)^2(20x^4 + 5)$$

$$= 4x(x^2 + 3)(4x^5 + 5x + 1) + (x^2 + 3)^2(20x^4 + 5)$$

**e.** To determine the derivative, use the quotient rule:

$$y = \frac{(4x^2 + 1)^5}{(3x - 2)^3}$$

$$\frac{dy}{dx} = \frac{5(4x^2 + 1)^4(8x)(3x - 2)^3}{(3x - 2)^6} - \frac{3(3x - 2)^2(3)(4x^2 + 1)^5}{(3x - 2)^6}$$

$$= (4x^2 + 1)^4(3x - 2)^2 \times \frac{40x(3x - 2) - 9(4x^2 + 1)}{(3x - 2)^6}$$

$$= \frac{(4x^2 + 1)^4(120x^2 - 80x - 36x^2 - 9)}{(3x - 2)^4}$$

$$= \frac{(4x^2 + 1)^4(84x^2 - 80x - 9)}{(3x - 2)^4}$$

**f.** $y = [x^2 + (2x + 1)^3]^5$

Use the chain rule

$$\frac{dy}{dx} = 5[x^2 + (2x + 1)^3]^4[2x + 6(2x + 1)^2]$$

**11.** To determine the equation of the tangent line, we need to determine its slope at the point $(1, 2)$. To do this, determine the derivative of $y$ and evaluate for $x = 1$:

$$y = \frac{18}{(x + 2)^2}$$

$$= 18(x + 2)^{-2}$$

$$\frac{dy}{dx} = -36(x + 2)^{-3}$$

$$= \frac{-36}{(x + 2)^3}$$

$$m = \frac{-36}{(x + 2)^3}$$

$$= \frac{-36}{27} = \frac{-4}{3}$$

Since we have a given point and we know the slope, use point-slope form to write the equation of the tangent line:

$$y - 2 = \frac{-4}{3}(x - 1)$$

$$3y - 6 = -4x + 4$$

$$4x + 3y - 10 = 0$$

**12.** The intersection point of the two curves occurs when

$$x^2 + 9x + 9 = 3x$$

$$x^2 + 6x + 9 = 0$$

$$(x + 3)^2 = 0$$

$$x = -3.$$

At a point $x$, the slope of the line tangent to the curve $y = x^2 + 9x + 9$ is given by

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 9x + 9)$$

$$= 2x + 9.$$

At $x = -3$, this slope is $2(-3) + 9 = 3$.

**13. a.** $p'(t) = \frac{d}{dt}(2t^2 + 6t + 1100)$

$$= 4t + 6$$

**b.** 1990 is 10 years after 1980, so the rate of change of population in 1990 corresponds to the value

$$p'(10) = 4(10) + 6$$

$$= 46 \text{ people per year.}$$

**c.** The rate of change of the population will be 110 people per year when

$$4t + 6 = 110$$

$$t = 26.$$

This corresponds to 26 years after 1980, which is the year 2006.

**14. a.** $f'(x) = \frac{d}{dx}(x^5 - 5x^3 + x + 12)$

$$= 5x^4 - 15x^2 + 1$$

$$f''(x) = \frac{d}{dx}(5x^4 - 15x^2 + 1)$$

$$= 20x^3 - 30x$$

**b.** $f(x)$ can be rewritten as $f(x) = -2x^{-2}$

$$f'(x) = \frac{d}{dx}(-2x^{-2})$$
$$= 4x^{-3}$$
$$= \frac{4}{x^3}$$
$$f''(x) = \frac{d}{dx}(4x^{-3})$$
$$= -12x^{-4}$$
$$= -\frac{12}{x^4}$$

**c.** $f(x)$ can be rewritten as $f(x) = 4x^{-\frac{1}{2}}$

$$f'(x) = \frac{d}{dx}(4x^{-\frac{1}{2}})$$
$$= -2x^{-\frac{3}{2}}$$
$$= -\frac{2}{\sqrt{x^3}}$$
$$f''(x) = \frac{d}{dx}(-2x^{-\frac{3}{2}})$$
$$= 3x^{-\frac{5}{2}}$$
$$= \frac{3}{\sqrt{x^5}}$$

**d.** $f(x)$ can be rewritten as $f(x) = x^4 - x^{-4}$

$$f'(x) = \frac{d}{dx}(x^4 - x^{-4})$$
$$= 4x^3 + 4x^{-5}$$
$$= 4x^3 + \frac{4}{x^5}$$
$$f''(x) = \frac{d}{dx}(4x^3 + 4x^{-5})$$
$$= 12x^2 - 20x^{-6}$$
$$= 12x^2 - \frac{20}{x^6}$$

**15.** Extreme values of a function on an interval will only occur at the endpoints of the interval or at a critical point of the function.

**a.** $f'(x) = \frac{d}{dx}(1 + (x+3)^2)$

$$= 2(x+3)$$

The only place where $f'(x) = 0$ is at $x = -3$, but that point is outside of the interval in question. The extreme values therefore occur at the endpoints of the interval:

$f(-2) = 1 + (-2+3)^2 = 2$

$f(6) = 1 + (6+3)^2 = 82$

The maximum value is 82, and the minimum value is 6

**b.** $f(x)$ can be rewritten as $f(x) = x + x^{-\frac{1}{2}}$

$$f'(x) = \frac{d}{dx}(x + x^{-\frac{1}{2}})$$
$$= 1 + -\frac{1}{2}x^{-\frac{3}{2}}$$
$$= 1 - \frac{1}{2\sqrt{x^3}}$$

On this interval, $x \geq 1$, so the fraction on the right is always less than or equal to $\frac{1}{2}$. This means that $f'(x) > 0$ on this interval and so the extreme values occur at the endpoints.

$$f(1) = 1 + \frac{1}{\sqrt{1}} = 2$$
$$f(9) = 9 + \frac{1}{\sqrt{9}} = 9\frac{1}{3}$$

The maximum value is $9\frac{1}{3}$, and the minimum value is 2.

**c.** $f'(x) = \frac{d}{dx}\left(\frac{e^x}{1+e^x}\right)$

$$= \frac{(1+e^x)(e^x) - (e^x)(e^x)}{(1+e^x)^2}$$
$$= \frac{e^x}{(1+e^x)^2}$$

Since $e^x$ is never equal to zero, $f'(x)$ is never zero, and so the extreme values occur at the endpoints of the interval.

$$f(0) = \frac{e^0}{1+e^0} = \frac{1}{2}$$
$$f(4) = \frac{e^4}{1+e^4}$$

The maximum value is $\frac{e^4}{1+e^4}$, and the minimum value is $\frac{1}{2}$.

**d.** $f'(x) = \frac{d}{dx}(2\sin(4x) + 3)$

$$= 8\cos(4x)$$

Cosine is 0 when its argument is a multiple of $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

$$4x = \frac{\pi}{2} + 2k\pi \text{ or } 4x = \frac{3\pi}{2} + 2k\pi$$
$$x = \frac{\pi}{8} + \frac{\pi}{2}k \qquad x = \frac{3\pi}{8} + \frac{\pi}{2}k$$

Since $x \in [0, \pi]$, $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$.

Also test the function at the endpoints of the interval.

$$f(0) = 2\sin 0 + 3 = 3$$
$$f\left(\frac{\pi}{8}\right) = 2\sin\frac{\pi}{2} + 3 = 5$$

$$f\left(\frac{3\pi}{8}\right) = 2\sin\frac{3\pi}{2} + 3 = 1$$
$$f\left(\frac{5\pi}{8}\right) = 2\sin\frac{5\pi}{2} + 3 = 5$$
$$f\left(\frac{7\pi}{8}\right) = 2\sin\frac{7\pi}{2} + 3 = 1$$
$$f(\pi) = 2\sin(4\pi) + 3 = 3$$

The maximum value is 5, and the minimum value is 1.

**16. a.** The velocity of the particle is given by

$$v(t) = s'(t) = \frac{d}{dt}(3t^3 - 40.5t^2 + 162t) = 9t^2 - 81t + 162.$$

The acceleration is

$$a(t) = v'(t) = \frac{d}{dt}(9t^2 - 81t + 162) = 18t - 81$$

**b.** The object is stationary when $v(t) = 0$:

$$9t^2 - 81t + 162 = 0$$
$$9(t - 6)(t - 3) = 0$$
$$t = 6 \text{ or } t = 3$$

The object is advancing when $v(t) > 0$ and retreating when $v(t) < 0$. Since $v(t)$ is the product of two linear factors, its sign can be determined using the signs of the factors:

| $t$-values | $t - 3$ | $t - 6$ | $v(t)$ | Object |
|---|---|---|---|---|
| $0 < t < 3$ | $< 0$ | $< 0$ | $> 0$ | Advancing |
| $3 < t < 6$ | $> 0$ | $< 0$ | $< 0$ | Retreating |
| $6 < t < 8$ | $> 0$ | $> 0$ | $> 0$ | Advancing |

**c.** The velocity of the object is unchanging when the acceleration is 0; that is, when

$$a(t) = 18t - 81 = 0$$
$$t = 4.5$$

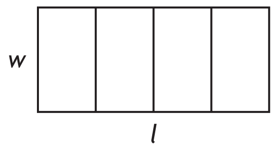
**d.** The object is decelerating when $a(t) < 0$, which occurs when

$$18t - 81 < 0$$
$$0 \le t < 4.5$$

**e.** The object is accelerating when $a(t) > 0$, which occurs when

$$18t - 81 > 0$$
$$4.5 < t \le 8$$

**17.**




Let the length and width of the field be $l$ and $w$, as shown. The total amount of fencing used is then $2l + 5w$. Since there is 750 m of fencing available, this gives

$$2l + 5w = 750$$
$$l = 375 - \frac{5}{2}w$$

The total area of the pens is

$$A = lw = 375w - \frac{5}{2}w^2$$

The maximum value of this area can be found by expressing $A$ as a function of $w$ and examining its derivative to determine critical points.

$A(w) = 375w - \frac{5}{2}w^2$, which is defined for $0 \le w$ and $0 \le l$. Since $l = 375 - \frac{5}{2}w$, $0 \le l$ gives the restriction $w \le 150$. The maximum area is therefore the maximum value of the function $A(w)$ on the interval $0 \le w \le 150$.

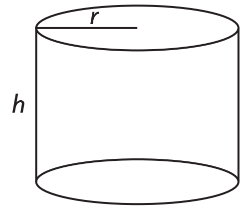
$$A'(w) = \frac{d}{dw}\left(375w - \frac{5}{2}w^2\right) = 375 - 5w$$

Setting $A'(w) = 0$ shows that $w = 75$ is the only critical point of the function. The only values of interest are therefore:

$$A(0) = 375(0) - \frac{5}{2}(0)^2 = 0$$
$$A(75) = 375(75) - \frac{5}{2}(75)^2 = 14\,062.5$$
$$A(150) = 375(150) - \frac{5}{2}(150)^2 = 0$$

The maximum area is 14 062.5 $m^2$

**18.**




Let the height and radius of the can be $h$ and $r$, as shown. The total volume of the can is then $\pi r^2 h$. The volume of the can is also give at 500 mL, so

$$\pi r^2 h = 500$$
$$h = \frac{500}{\pi r^2}$$

The total surface area of the can is

$A = 2\pi rh + 2\pi r^2$

$= \frac{1000}{r} + 2\pi r^2$

The minimum value of this surface area can be found by expressing $A$ as a function of $r$ and examining its derivative to determine critical points. $A(r) = \frac{1000}{r} + 2\pi r^2$, which is defined for $0 < r$ and $0 < h$. Since $h = \frac{500}{\pi r^2}$, $0 < h$ gives no additional restriction on $r$. The maximum area is therefore the maximum value of the function $A(r)$ on the interval $0 < r$.

$$A'(r) = \frac{d}{dr}\left(\frac{1000}{r} + 2\pi r^2\right)$$
$$= -\frac{1000}{r^2} + 4\pi r$$

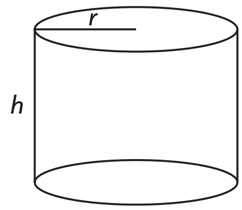
The critical points of $A(r)$ can be found by setting $A'(r) = 0$:

$$-\frac{1000}{r^2} + 4\pi r = 0$$
$$4\pi r^3 = 1000$$
$$r = \sqrt[3]{\frac{1000}{4\pi}} \doteq 4.3 \text{ cm}$$

So $r = 4.3$ cm is the only critical point of the function. This gives the value

$$h = \frac{500}{\pi(4.3)^2} \doteq 8.6 \text{ cm.}$$

**19.**




Let the radius be $r$ and the height $h$.

Minimize the cost:

$C = 2\pi r^2(0.005) + 2\pi rh(0.0025)$

$V = \pi r^2 h = 4000$

$$h = \frac{4000}{\pi r^2}$$

$$C(r) = 2\pi r^2(0.005) + 2\pi r\left(\frac{4000}{\pi r^2}\right)(0.0025)$$
$$= 0.01\pi r^2 + \frac{20}{r}, 1 \le r \le 36$$

$$C'(r) = 0.02\pi r - \frac{20}{r^2}.$$

For a maximum or minimum value, let $C'(r) = 0$.

$$0.02\pi r^2 - \frac{20}{r^2} = 0$$

$$r^3 = \frac{20}{0.02\pi}$$
$$r \doteq 6.8$$

Using the max min algorithm:

$C(1) = 20.03$, $C(6.8) = 4.39$, $C(36) = 41.27$.

The dimensions for the cheapest container are a radius of 6.8 cm and a height of 27.5 cm.

**20. a.** Let the length, width, and depth be $l$, $w$, and $d$, respectively. Then, the given information is that $l = x$, $w = x$, and

$l + w + d = 140$. Substituting gives

$$2x + d = 140$$
$$d = 140 - 2x$$

**b.** The volume of the box is $V = lwh$. Substituting in the values from part **a**. gives

$V = (x)(x)(140 - 2x)$

$= 140x^2 - 2x^3$

In order for the dimensions of the box to make sense, the inequalities $l \ge 0$, $w \ge 0$, and $h \ge 0$ must be satisfied. The first two give $x \ge 0$, the third requires $x \le 70$. The maximum volume is therefore the maximum value of $V(x) = 140x^2 - 2x^3$ on the interval $0 \le x \le 70$, which can be found by determining the critical points of the derivative $V'(x)$.

$$V'(x) = \frac{d}{dx}(140x^2 - 2x^3)$$
$$= 280x - 6x^2$$
$$= 2x(140 - 3x)$$

Setting $V'(x) = 0$ shows that $x = 0$ and $x = \frac{140}{3} \doteq 46.7$ are the critical points of the function.

The maximum value therefore occurs at one of these points or at one of the endpoints of the interval:

$V(0) = 140(0)^2 - 2(0)^3 = 0$

$V(46.7) = 140(46.7)^2 - 2(46.7)^3 = 101\,629.5$

$V(0) = 140(70)^2 - 2(70)^3 = 0$

So the maximum volume is 101 629.5 $cm^3$, from a box with length and width 46.7 cm and depth $140 - 2(46.7) = 46.6$ cm.

**21.** The revenue function is

$R(x) = x(50 - x^2)$

$= 50x - x^3$. Its maximum for $x \ge 0$ can be found by examining its derivative to determine critical points.

$$R'(x) = \frac{d}{dx}(50x - x^3)$$
$$= 50 - 3x^2$$

The critical points can be found by setting $R'(x) = 0$:

$$50 - 3x^2 = 0$$
$$x = \pm\sqrt{\frac{50}{3}} \doteq \pm 4.1$$

Only the positive root is of interest since the number of MP3 players sold must be positive. The number must also be an integer, so both $x = 4$ and $x = 5$ must be tested to see which is larger.

$R(4) = 50(4) - 4^3 = 136$
$R(4) = 50(5) - 5^3 = 125$

So the maximum possible revenue is \$136, coming from a sale of 4 MP3 players.

**22.** Let $x$ be the fare, and $p(x)$ be the number of passengers per year. The given information shows that $p$ is a linear function of $x$ such that an increase of 10 in $x$ results in a decrease of 1000 in $p$. This means that the slope of the line described by $p(x)$ is $\frac{-1000}{10} = -100$. Using the initial point given,

$$\begin{aligned} p(x) &= -100(x - 50) + 10\,000 \\ &= -100x + 15\,000 \end{aligned}$$

The revenue function can now be written:

$$\begin{aligned} R(x) &= xp(x) \\ &= x(-100x + 15\,000) \\ &= 15\,000x - 100x^2 \end{aligned}$$

Its maximum for $x \geq 0$ can be found by examining its derivative to determine critical points.

$$\begin{aligned} R'(x) &= \frac{d}{dx}(15\,000x - 100x^2) \\ &= 15\,000 - 200x \end{aligned}$$

Setting $R'(x) = 0$ shows that $x = 75$ is the only critical point of the function. The problem states that only \$10 increases in fare are possible, however, so the two nearest must be tried to determine the maximum possible revenue:

$R(70) = 15\,000(70) - 100(70)^2 = 560\,000$
$R(80) = 15\,000(80) - 100(80)^2 = 560\,000$

So the maximum possible revenue is \$560 000, which can be achieved by a fare of either \$70 or \$80.

**23.** Let the number of \$30 price reductions be $n$. The resulting number of tourists will be $80 + n$ where $0 \leq n \leq 70$. The price per tourist will be $5000 - 30n$ dollars. The revenue to the travel agency will be $(5000 - 30n)(80 + n)$ dollars. The cost to the agency will be $250\,000 + 300(80 + n)$ dollars.

Profit = Revenue − Cost

$$\begin{aligned} P(n) &= (5000 - 30n)(80 + n) \\ &\quad - 250\,000 - 300(80 + n), 0 \leq n \leq 70 \\ P'(n) &= -30(80 + n) + (5000 - 30n)(1) - 300 \\ &= 2300 - 60n \end{aligned}$$

$P'(n) = 0$ when $n = 38\frac{1}{3}$

Since $n$ must be an integer, we now evaluate $P(n)$ for $n = 0, 38, 39$, and 70. (Since $P(n)$ is a quadratic function whose graph opens downward with vertex at $38\frac{1}{3}$, we know $P(38) > P(39)$.)

$$\begin{aligned} P(0) &= 126\,000 \\ P(38) &= (3860)(118) - 250\,000 - 300(118) \\ &= 170\,080 \\ P(39) &= (3830)(119) - 250\,000 - 300(119) \\ &= 170\,070 \\ P(70) &= (2900)(150) - 250\,000 - 300(150) \\ &= 140\,000 \end{aligned}$$

The price per person should be lowered by \$1140 (38 decrements of \$30) to realize a maximum profit of \$170 080.

**24. a.** $$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(-5x^2 + 20x + 2) \\ &= -10x + 20 \end{aligned}$$

Setting $\frac{dy}{dx} = 0$ shows that $x = 2$ is the only critical number of the function.

| **x** | $x < 2$ | $x = 2$ | $x > 2$ |
|---|---|---|---|
| **y′** | + | 0 | − |
| **Graph** | Inc. | Local Max | Dec. |

**b.** $$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(6x^2 + 16x - 40) \\ &= 12x + 16 \end{aligned}$$

Setting $\frac{dy}{dx} = 0$ shows that $x = -\frac{4}{3}$ is the only critical number of the function.

| **x** | $x < -\frac{4}{3}$ | $x = -\frac{4}{3}$ | $x > -\frac{4}{3}$ |
|---|---|---|---|
| **y′** | − | 0 | + |
| **Graph** | Dec. | Local Min | Inc. |

**c.** $$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(2x^3 - 24x) \\ &= 6x^2 - 24 \end{aligned}$$

The critical numbers are found by setting $\frac{dy}{dx} = 0$:

$$\begin{aligned} 6x^2 - 24 &= 0 \\ 6x^2 &= 24 \\ x &= \pm 2 \end{aligned}$$

| **x** | $x < -2$ | $x = -2$ | $-2 < x < 2$ | $x = 2$ | $x > 2$ |
|---|---|---|---|---|---|
| **y′** | + | 0 | − | 0 | + |
| **Graph** | Inc. | Local Max | Dec. | Local Min | Inc. |

**d.** $$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x}{x - 2}\right) \\ &= \frac{(x - 2)(1) - x(1)}{(x - 2)^2} \\ &= \frac{-2}{(x - 2)^2} \end{aligned}$$

This derivative is never equal to zero, so the function has no critical numbers. Since the numerator is always negative and the denominator is never negative, the derivative is always negative. This means that the function is decreasing everywhere it is defined, that is, $x \neq 2$.

**25. a.** This function is discontinuous when

$x^2 - 9 = 0$

$x = \pm 3$. The numerator is non-zero at these points, so these are the equations of the vertical asymptotes.

To check for a horizontal asymptote:

$$\lim_{x\to\infty} \frac{8}{x^2 - 9} = \lim_{x\to\infty} \frac{8}{x^2\left(1 - \frac{9}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty} (8)}{\lim_{x\to\infty} x^2\left(1 - \frac{9}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty} (8)}{\lim_{x\to\infty} (x)^2 \times \lim_{x\to\infty}\left(1 - \frac{9}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{1}{x^2} \times \frac{8}{1 - 0}$$

$$= 0$$

Similarly, $\lim_{x\to-\infty} \frac{8}{x^2 - 9} = 0$, so $y = 0$ is a horizontal asymptote of the function.

There is no oblique asymptote because the degree of the numerator does not exceed the degree of the denominator by 1.

Local extrema can be found by examining the derivative to determine critical points:

$$y' = \frac{(x^2 - 9)(0) - (8)(2x)}{(x^2 - 9)^2}$$

$$= \frac{-16x}{(x^2 - 9)^2}$$

Setting $y' = 0$ shows that $x = 0$ is the only critical point of the function.

| $x$ | $x < 0$ | $x = 0$ | $x > 0$ |
|---|---|---|---|
| $y'$ | + | 0 | + |
| Graph | Inc. | Local Max | Dec. |

So $(0, -\frac{8}{9})$ is a local maximum.

**b.** This function is discontinuous when

$x^2 - 1 = 0$

$x = \pm 1$. The numerator is non-zero at these points, so these are the equations of the vertical asymptotes.

To check for a horizontal asymptote:

$$\lim_{x\to\infty} \frac{4x^3}{x^2 - 1} = \lim_{x\to\infty} \frac{x^3(4)}{x^2\left(1 - \frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{x(4)}{1 - \frac{1}{x^2}}$$

$$= \frac{\lim_{x\to\infty} (x(4))}{\lim_{x\to\infty}\left(1 - \frac{1}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty} (x) \times \lim_{x\to\infty} (4)}{\lim_{x\to\infty}\left(1 - \frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} (x) \times \frac{4}{1 - 0}$$

$$= \infty$$

Similarly, $\lim_{x\to-\infty} \frac{4x^3}{x^2 - 1} = \lim_{x\to-\infty} (x) = -\infty$, so this function has no horizontal asymptote.

To check for an oblique asymptote:

$$\begin{array}{r} 4x \phantom{{}+0x+0} \\ x^2 - 1 \overline{)\, 4x^3 + 0x^2 + 0x + 0} \\ 4x^3 + 0x^2 - 4x \phantom{{}+0} \\ \hline 0 \quad + 0 \quad + 4x + 0 \end{array}$$

So $y$ can be written in the form

$y = 4x + \frac{4x}{x^2 - 1}$. Since

$$\lim_{x\to\infty} \frac{4x}{x^2 - 1} = \lim_{x\to\infty} \frac{x(4)}{x^2\left(1 - \frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{4}{x\left(1 - \frac{1}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty} (4)}{\lim_{x\to\infty}\left(x\left(1 - \frac{1}{x^2}\right)\right)}$$

$$= \frac{\lim_{x\to\infty} (4)}{\lim_{x\to\infty} (x) \times \lim_{x\to\infty}\left(1 - \frac{1}{x^2}\right)}$$

$$= \lim\left(\frac{1}{x}\right) \times \frac{4}{1 - 0}$$

$$= 0,$$

and similarly $\lim_{x\to-\infty} \frac{4x}{x^2 - 1} = 0$, the line $y = 4x$ is an asymptote to the function $y$.

Local extrema can be found by examining the derivative to determine critical points:

$$y' = \frac{(x^2 - 1)(12x^2) - (4x^3)(2x)}{(x^2 - 1)^2}$$
$$= \frac{12x^4 - 12x^2 - 8x^4}{(x^2 - 1)^2}$$
$$= \frac{4x^4 - 12x^2}{(x^2 - 1)^2}$$

Setting $y' = 0$:
$4x^4 - 12x^2 = 0$
$x^2(x^2 - 3) = 0$
so $x = 0, x = \pm\sqrt{3}$ are the critical points of the function
$(-\sqrt{3}, -6\sqrt{3})$ is a local maximum, $(\sqrt{3}, 6\sqrt{3})$ is a local minimum, and $(0, 0)$ is neither.

| **x** | $x < -\sqrt{3}$ | $x = -\sqrt{3}$ | $-\sqrt{3} < x < 0$ | $x = 0$ |
|---|---|---|---|---|
| **y′** | + | 0 | − | 0 |
| **Graph** | Inc. | Local Max | Dec. | Horiz. |

| **x** | $0 < x < \sqrt{3}$ | $x = \sqrt{3}$ | $x > \sqrt{3}$ |
|---|---|---|---|
| **y′** | − | 0 | − |
| **Graph** | Dec. | Local Min | Inc. |

**26. a.** This function is continuous everywhere, so it has no vertical asymptotes. To check for a horizontal asymptote:

$$\lim_{x\to\infty}(4x^3 + 6x^2 - 24x - 2)$$
$$= \lim_{x\to\infty} x^3\left(4 + \frac{6}{x} - \frac{24}{x^2} - \frac{2}{x^3}\right)$$
$$= \lim_{x\to\infty}(x^3) \times \lim_{x\to\infty}\left(4 + \frac{6}{x} - \frac{24}{x^2} - \frac{2}{x^3}\right)$$
$$= \lim_{x\to\infty}(x^3) \times (4 + 0 - 0 - 0)$$
$$= \infty$$

Similarly,
$\lim_{x\to-\infty}(4x^3 + 6x^2 - 24x - 2) = \lim_{x\to-\infty}(x^3) = -\infty$,
so this function has no horizontal asymptote.

The $y$-intercept can be found by letting $x = 0$, which gives $y = 4(0)^3 + 6(0)^2 - 24(0) - 2$
$= -2$

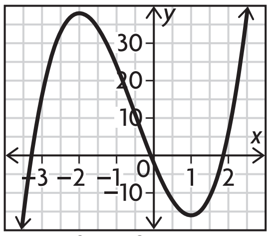
The derivative is of the function is

$$y' = \frac{d}{dx}(4x^3 + 6x^2 - 24x - 2)$$
$$= 12x^2 + 12x - 24$$
$= 12(x + 2)(x - 1)$, and the second derivative is
$$y'' = \frac{d}{dx}(12x^2 + 12x - 24)$$
$$= 24x + 12$$

Letting $f'(x) = 0$ shows that $x = -2$ and $x = 1$ are critical points of the function. Letting $y'' = 0$ shows that $x = -\frac{1}{2}$ is an inflection point of the function.

| **x** | $x < -2$ | $x = -2$ | $-2 < x$ | $x = -\frac{1}{2}$ |
|---|---|---|---|---|
| **y′** | + | 0 | − | − |
| **Graph** | Inc. | Local Max | Dec. | Dec. |
| **y″** | − | − | − | 0 |
| **Concavity** | Down | Down | Down | Infl. |

| **x** | $-\frac{1}{2} < x < 1$ | $x = 1$ | $x > 1$ |
|---|---|---|---|
| **y′** | − | 0 | + |
| **Graph** | Dec. | Local Min | Inc. |
| **y″** | + | + | + |
| **Concavity** | Up | Up | Up |



$y = 4x^3 + 6x^2 - 24x - 2$

**b.** This function is discontinuous when
$x^2 - 4 = 0$
$(x + 2)(x - 2) = 0$
$x = 2$ or $x = -2$. The numerator is non-zero at these points, so the function has vertical asymptotes at both of them. The behaviour of the function near these asymptotes is:

| **x-values** | **3x** | **x + 2** | **x − 2** | **y** | $\lim_{x\to\infty} y$ |
|---|---|---|---|---|---|
| $x \to -2^-$ | $< 0$ | $< 0$ | $< 0$ | $< 0$ | $-\infty$ |
| $x \to -2^+$ | $< 0$ | $> 0$ | $< 0$ | $> 0$ | $+\infty$ |
| $x \to 2^-$ | $> 0$ | $> 0$ | $< 0$ | $< 0$ | $-\infty$ |
| $x \to 2^+$ | $> 0$ | $> 0$ | $> 0$ | $> 0$ | $+\infty$ |

To check for a horizontal asymptote:

$$\lim_{x\to\infty}\frac{3x}{x^2 - 4} = \lim_{x\to\infty}\frac{x(3)}{x^2\left(1 - \frac{4}{x^2}\right)}$$
$$= \lim_{x\to\infty}\frac{3}{x\left(1 - \frac{4}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty}(3)}{\lim_{x\to\infty}\left(x\left(1 - \frac{4}{x^2}\right)\right)}$$

$$= \frac{\lim_{x\to\infty}(3)}{\lim_{x\to\infty}(x) \times \lim_{x\to\infty}\left(1 - \frac{4}{x^2}\right)}$$

$$= \lim_{x\to\infty}\frac{1}{x} \times \frac{3}{1-0}$$

$$= 0$$

Similarly, $\lim_{x\to-\infty}\frac{3x}{x^2-4} = 0$, so $y = 0$ is a horizontal asymptote of the function.

This function has $y = 0$ when $x = 0$, so the origin is both the $x$- and $y$-intercept.

The derivative is

$$y' = \frac{(x^2-4)(3) - (3x)(2x)}{(x^2-4)^2}$$

$$= \frac{-3x^2-12}{(x^2-4)^2}$$, and the second derivative is

$$y'' = \frac{(x^2-4)^2(-6x)}{(x^2-4)^4} - \frac{(-3x^2-12)(2(x^2-4)(2x))}{(x^2-4)^4}$$

$$= \frac{-6x^3+24x+12x^3+48x}{(x^2-4)^3}$$

$$= \frac{6x^3+72x}{(x^2-4)^3}$$

The critical points of the function can be found by letting $y' = 0$, so

$-3x^2 - 12 = 0$

$x^2 + 4 = 0$. This has no real solutions, so the function $y$ has no critical points.

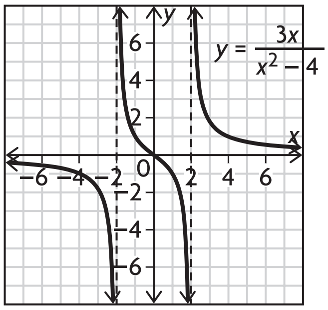
The inflection points can be found by letting $y'' = 0$, so

$6x^3 + 72x = 0$

$6x(x^2 + 12) = 0$

The only real solution to this equation is $x = 0$, so that is the only possible inflection point.

| $x$ | $x < -2$ | $-2 < x < 0$ | $x = 0$ | $0 < x < 2$ | $x > 2$ |
|---|---|---|---|---|---|
| $y'$ | − | − | − | − | − |
| Graph | Dec. | Dec. | Dec. | Dec. | Dec. |
| $y''$ | − | + | 0 | − | + |
| Concavity | Down | Up | Infl. | Down | Up |




**27. a.** $f'(x) = \frac{d}{dx}((-4)e^{5x+1})$

$$= (-4)e^{5x+1} \times \frac{d}{dx}(5x+1)$$

$$= (-20)e^{5x+1}$$

**b.** $f'(x) = \frac{d}{dx}(xe^{3x})$

$$= xe^{3x} \times \frac{d}{dx}(3x) + (1)e^{3x}$$

$$= e^{3x}(3x+1)$$

**c.** $y' = \frac{d}{dx}(6^{3x-8})$

$$= (\ln 6)6^{3x-8} \times \frac{d}{dx}(3x-8)$$

$$= (3\ln 6)6^{3x-8}$$

**d.** $y' = \frac{d}{dx}(e^{\sin x})$

$$= e^{\sin x} \times \frac{d}{dx}(\sin x)$$

$$= (\cos x)e^{\sin x}$$

**28.** The slope of the tangent line at $x = 1$ can be found by evaluating the derivative $\frac{dy}{dx}$ for $x = 1$:

$$\frac{dy}{dx} = \frac{d}{dx}(e^{2x-1})$$

$$= e^{2x-1} \times \frac{d}{dx}(2x-1)$$

$$= 2e^{2x-1}$$

Substituting $x = 1$ shows that the slope is $2e$. The value of the original function at $x = 1$ is $e$, so the equation of the tangent line at $x = 1$ is $y = 2e(x-1) + e$.

**29. a.** The maximum of the function modelling the number of bacteria infected can be found by examining its derivative.

$$N'(t) = \frac{d}{dt}((15t)e^{-\frac{t}{5}})$$

$$= 15te^{-\frac{t}{5}} \times \frac{d}{dt}\left(-\frac{t}{5}\right) + (15)e^{-\frac{t}{5}}$$

$$= e^{-\frac{t}{5}}(15 - 3t)$$

Setting $N'(t) = 0$ shows that $t = 5$ is the only critical point of the function (since the exponential function is never zero). The maximum number of infected bacteria therefore occurs after 5 days.

**b.** $N(5) = (15(5))e^{-\frac{5}{5}}$
$= 27$ bacteria

**30. a.** $\frac{dy}{dx} = \frac{d}{dx}(2\sin x - 3\cos 5x)$
$= 2\cos x - 3(-\sin 5x) \times \frac{d}{dx}(5x)$
$= 2\cos x + 15\sin 5x$

**b.** $\frac{dy}{dx} = \frac{d}{dx}(\sin 2x + 1)^4$
$= 4(\sin 2x + 1)^3 \times \frac{d}{dx}(\sin 2x + 1)$
$= 4(\sin 2x + 1)^3 \times (\cos 2x) \times \frac{d}{dx}(2x)$
$= 8\cos 2x(\sin 2x + 1)^3$

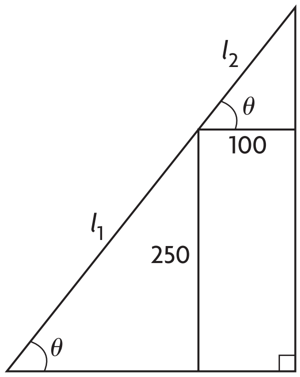
**c.** $y$ can be rewritten as $y = (x^2 + \sin 3x)^{\frac{1}{2}}$. Then,
$\frac{dy}{dx} = \frac{d}{dx}(x^2 + \sin 3x)^{\frac{1}{2}}$
$= \frac{1}{2}(x^2 + \sin 3x)^{-\frac{1}{2}} \times \frac{d}{dx}(x^2 + \sin 3x)$
$= \frac{1}{2}(x^2 + \sin 3x)^{-\frac{1}{2}}$
$\times \left(2x + \cos 3x \times \frac{d}{dx}(3x)\right)$
$= \frac{2x + 3\cos 3x}{2\sqrt{x^2 + \sin 3x}}$

**d.** $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{\cos x + 2}\right)$
$= \frac{(\cos x + 2)(\cos x) - (\sin x)(-\sin x)}{(\cos x + 2)^2}$
$= \frac{\cos^2 x + \sin^2 x + 2\cos x}{(\cos x + 2)^2}$
$= \frac{1 + 2\cos x}{(\cos x + 2)^2}$

**e.** $\frac{dy}{dx} = \frac{d}{dx}(\tan x^2 - \tan^2 x)$
$= \frac{d}{dx}\sec^2 x^2 \times \frac{d}{dx}(x^2)$
$- 2\tan x \times \frac{d}{dx}(\tan x)$
$= 2x\sec^2 x^2 - 2\tan x\sec^2 x$

**f.** $\frac{dy}{dx} = \frac{d}{dx}(\sin(\cos x^2))$
$= \cos(\cos x^2) \times \frac{d}{dx}(\cos x^2)$
$= \cos(\cos x^2) \times (-\sin x^2) \times \frac{d}{dx}(x^2)$
$= -2x\sin x^2\cos(\cos x^2)$

**31.**




As shown in the diagram, let $\theta$ be the angle between the ladder and the ground, and let the total length of the ladder be $l = l_1 + l_2$, where $l_1$ is the length from the ground to the top corner of the shed and $l_2$ is the length from the corner of the shed to the wall.

$\sin\theta = \frac{250}{l_1}$ $\quad\cos\theta = \frac{100}{l_2}$
$l_1 = 250\csc\theta$ $\quad l_2 = 100\sec\theta$
$l = 250\csc\theta + 100\sec\theta$

$\frac{dl}{d\theta} = -250\csc\theta\cot\theta + 100\sec\theta\tan\theta$
$= -\frac{250\cos\theta}{\sin^2\theta} + \frac{100\sin\theta}{\cos^2\theta}$

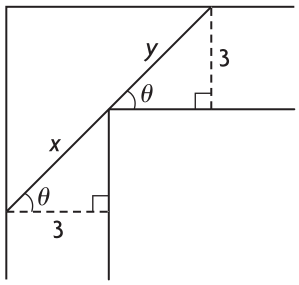
To determine the minimum, solve $\frac{dl}{d\theta} = 0$.

$\frac{250\cos\theta}{\sin^2\theta} = \frac{100\sin\theta}{\cos^2\theta}$
$250\cos^3\theta = 100\sin^3\theta$
$2.5 = \tan^3\theta$
$\tan\theta = \sqrt[3]{2.5}$
$\theta \doteq 0.94$

At $\theta = 0.94$, $l = 250\csc 0.94 + 100\sec 0.94$
$\doteq 479$ cm

The shortest ladder is about 4.8 m long.

**32.** The longest rod that can fit around the corner is determined by the minimum value of $x + y$. So, determine the minimum value of $l = x + y$.




From the diagram, $\sin\theta = \frac{3}{y}$ and $\cos\theta = \frac{3}{x}$. So,

$l = \frac{3}{\cos\theta} + \frac{3}{\sin\theta}$, for $0 \leq \theta \leq \frac{\pi}{2}$.

$$\frac{dl}{d\theta} = \frac{3\sin\theta}{\cos^2\theta} - \frac{3\cos\theta}{\sin^2\theta}$$
$$= \frac{3\sin^3\theta - 3\cos^3\theta}{\cos^2\theta\sin^2\theta}$$

Solving $\frac{dl}{d\theta} = 0$ yields:

$$3\sin^3\theta - 3\cos^3\theta = 0$$
$$\tan^3\theta = 1$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}$$

So $l = \frac{3}{\cos\frac{\pi}{4}} + \frac{3}{\sin\frac{\pi}{4}}$
$$= 3\sqrt{2} + 3\sqrt{2}$$
$$= 6\sqrt{2}$$

When $\theta = 0$ or $\theta = \frac{\pi}{2}$, the longest possible rod would have a length of 3 m. Therefore the longest rod that can be carried horizontally around the corner is one of length $6\sqrt{2}$, or about 8.5 m.