CHAPTER 7
Trigonometric Identities and Equations

Getting Started, p. 386

1. a) Do the following to isolate and solve for x:

$$3x - 7 = 5 - 9x$$
$$3x - 7 + 9x = 5 - 9x + 9x$$
$$12x - 7 = 5$$
$$12x - 7 + 7 = 5 + 7$$
$$12x = 12$$
$$\frac{12x}{12} = \frac{12}{12}$$
$$x = 1$$

b) Do the following to isolate and solve for x:

$$2(x + 3) - \frac{x}{4} = \frac{1}{2}$$
$$2x + 6 - \frac{x}{4} = \frac{1}{2}$$
$$2x + 6 - \frac{x}{4} - 6 = \frac{1}{2} - 6$$
$$2x - \frac{x}{4} = -\frac{11}{2}$$
$$\frac{7x}{4} = -\frac{11}{2}$$
$$\frac{7x}{4} \times \frac{4}{7} = -\frac{11}{2} \times \frac{4}{7}$$
$$x = -\frac{44}{14}$$
$$x = -\frac{22}{7}$$

c) Factor the left side of the equation and solve for x as follows:

$$x^2 - 5x - 24 = 0$$
$$(x - 8)(x + 3) = 0$$
$$x - 8 = 0 \text{ or } x + 3 = 0$$
$$x - 8 + 8 = 0 + 8 \text{ or } x + 3 - 3 = 0 - 3$$
$$x = 8 \text{ or } x = -3$$

d) Move all terms to the left side of the equation, factor, and solve for x as follows:

$$6x^2 + 11x = 10$$
$$6x^2 + 11x - 10 = 10 - 10$$
$$6x^2 + 11x - 10 = 0$$
$$(3x - 2)(2x + 5) = 0$$
$$3x - 2 = 0 \quad \text{or} \quad 2x + 5 = 0$$
$$3x - 2 + 2 = 0 + 2 \text{ or } 2x + 5 - 5 = 0 - 5$$
$$3x = 2 \text{ or } 2x = -5$$
$$\frac{3x}{3} = \frac{2}{3} \text{ or } \frac{2x}{2} = -\frac{5}{2}$$
$$x = \frac{2}{3} \text{ or } x = -\frac{5}{2}$$

e) Use the quadratic formula to solve for x as follows:

$$x^2 + 2x - 1 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-1)}}{2(1)}$$
$$x = \frac{-2 \pm \sqrt{4 + 4}}{2}$$
$$x = \frac{-2 \pm \sqrt{8}}{2}$$
$$x = \frac{-2 \pm 2\sqrt{2}}{2}$$
$$x = -1 \pm \sqrt{2}$$

f) Move all terms to the left side of the equation and use the quadratic formula to solve for x as follows:

$$3x^2 = 3x + 1$$
$$3x^2 - 3x - 1 = 3x + 1 - 3x - 1$$
$$3x^2 - 3x - 1 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{3 \pm \sqrt{(-3)^2 - 4(3)(-1)}}{2(3)}$$
$$x = \frac{3 \pm \sqrt{9 + 12}}{6}$$
$$x = \frac{3 \pm \sqrt{21}}{6}$$

2. To show that $AB = CD$, the distance formula should be used as follows:
First for AB:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$d = \sqrt{(2 - 1)^2 + \left(\frac{1}{2} - 0\right)^2}$$

$$d = \sqrt{(1)^2 + \left(\frac{1}{2}\right)^2}$$

$$d = \sqrt{1 + \frac{1}{4}}$$

$$d = \sqrt{\frac{5}{4}}$$

$$d = \frac{\sqrt{5}}{2}$$

Now for CD:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{\left(0 - \left(-\frac{1}{2}\right)\right)^2 + (6 - 5)^2}$$

$$d = \sqrt{\left(\frac{1}{2}\right)^2 + (1)^2}$$

$$d = \sqrt{\frac{1}{4} + 1}$$

$$d = \sqrt{\frac{5}{4}}$$

$$d = \frac{\sqrt{5}}{2}$$

So, $AB = CD$.

3. a) $\sin A = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{8}{17}$;

$\cos A = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{15}{17}$;

$\tan A = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{8}{15}$;

$\csc A = \dfrac{\text{hypotenuse}}{\text{opposite side}} = \dfrac{17}{8}$;

$\sec A = \dfrac{\text{hypotenuse}}{\text{adjacent side}} = \dfrac{17}{15}$;

$\cot A = \dfrac{\text{adjacent side}}{\text{opposite side}} = \dfrac{15}{8}$

b) To determine the measure of $\angle A$ in radians, any of the ratios found in part (a) can be used. In this case, sin A will be used:

$$\sin A = \frac{8}{17}$$

$$\sin^{-1}(\sin A) = \sin^{-1}\frac{8}{17}$$

$$m\angle A = \sin^{-1}\frac{8}{17}$$

$$m\angle A = 0.5 \text{ radians}$$

c) From the figure:

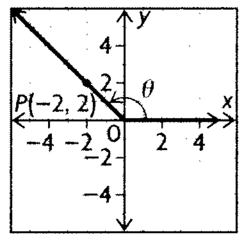
$$\sin B = \frac{15}{17}$$

$$\sin^{-1}(\sin B) = \sin^{-1}\left(\frac{15}{17}\right)$$

$$m\angle B = \sin^{-1}\left(\frac{15}{17}\right)$$

$$m\angle B = 61.9°$$

4. a)



b) Drawing a segment from $(-2, 2)$ down to the negative x-axis forms a right triangle with two legs of length 2. The tangent of the related acute angle is $\frac{2}{2}$ or 1.

The measure in radians of the acute angle that has a tangent equal to 1 is $\frac{\pi}{4}$, so the value of the related acute angle is $\frac{\pi}{4}$ radians.

c) Since the point at $(-2, 2)$ is in the second quadrant, the terminal arm of the angle θ must also be in the second quadrant. Since the angle with a terminal arm in the second quadrant and a tangent of -1 measures $\frac{3\pi}{4}$ radians, $\theta = \frac{3\pi}{4}$ radians.

5. a) The x-coordinate of point A is $\cos\frac{\pi}{4}$, while the y-coordinate is $\sin\frac{\pi}{4}$. Therefore, the coordinates of point A are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. The x-coordinate of point B is $\cos\frac{\pi}{3}$, while the y-coordinate is $\sin\frac{\pi}{3}$. Therefore, the coordinates of point B are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The x-coordinate of point C is $\cos\frac{2\pi}{3}$, while the y-coordinate is $\sin\frac{2\pi}{3}$. Therefore, the coordinates of point C are $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The x-coordinate of point D the is $\cos\frac{5\pi}{6}$, while the y-coordinate is $\sin\frac{5\pi}{6}$. Therefore, coordinates of point D are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. The x-coordinate of point E is $\cos\frac{7\pi}{6}$, while the y-coordinate is $\sin\frac{7\pi}{6}$. Therefore, the coordinates of point E are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. The x-coordinate of point F is $\cos\frac{5\pi}{4}$, while the

y-coordinate is $\sin \frac{5\pi}{4}$. Therefore, the coordinates of point F are $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. The x-coordinate of point G is $\cos \frac{4\pi}{3}$, while the y-coordinate is $\sin \frac{4\pi}{3}$. Therefore, the coordinates of point G are $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. The x-coordinate of point H is $\cos \frac{5\pi}{3}$, while the y-coordinate is $\sin \frac{5\pi}{3}$. Therefore, the coordinates of point H are $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. The x-coordinate of point I is $\cos\frac{7\pi}{4}$, while the y-coordinate is $\sin\frac{7\pi}{4}$. Therefore, the coordinates of point I are $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. The x-coordinate of point J is $\cos \frac{11\pi}{6}$, while the y-coordinate is $\sin \frac{11\pi}{6}$. Therefore, the coordinates of point J are $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

b) i) $\cos \frac{3\pi}{4}$ is equal to the x-coordinate of the point at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, so $\cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$.

ii) $\sin \frac{11\pi}{6}$ is equal to the y-coordinate of point J, so $\sin \frac{11\pi}{6} = -\frac{1}{2}$.

iii) $\cos \pi$ is equal to the x-coordinate of the point at $(-1, 0)$, so $\cos \pi = -1$.

iv) $\csc \frac{\pi}{6}$ is equal to the reciprocal of the y-coordinate of the point at $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, so $\csc \frac{\pi}{6} = \frac{1}{\frac{1}{2}} = 2$.

6. a) Since $\tan x = -\frac{3}{4}$, the leg opposite angle x in a right triangle has a length of 3, while the leg adjacent to angle x has a length of 4. For this reason, the length of the hypotenuse can be calculated as follows:

$$3^2 + 4^2 = z^2$$
$$9 + 16 = z^2$$
$$25 = z^2$$
$$z = 5$$

Therefore, if the angle x is in the second quadrant, the other five trigonometric ratios are as follows:

$$\sin x = \frac{\text{opposite leg}}{\text{hypotenuse}} = \frac{3}{5};$$
$$\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}} = -\frac{4}{5};$$
$$\csc x = \frac{\text{hypotenuse}}{\text{opposite leg}} = \frac{5}{3};$$
$$\sec x = \frac{\text{hypotenuse}}{\text{adjacent leg}} = -\frac{5}{4};$$
$$\cot x = \frac{\text{adjacent leg}}{\text{opposite leg}} = -\frac{4}{3}$$

If the angle x is in the fourth quadrant, the other five trigonometric ratios are as follows:

$$\sin x = \frac{\text{opposite leg}}{\text{hypotenuse}} = -\frac{3}{5};$$
$$\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}} = \frac{4}{5};$$
$$\csc x = \frac{\text{hypotenuse}}{\text{opposite leg}} = -\frac{5}{3};$$
$$\sec x = \frac{\text{hypotenuse}}{\text{adjacent leg}} = \frac{5}{4};$$
$$\cot x = \frac{\text{adjacent leg}}{\text{opposite leg}} = -\frac{4}{3}$$

b) To determine the value of the angle x, any of the ratios found in part (a) can be used. If x is in the second quadrant, $\sin x = \frac{3}{5}$, so x can be solved for as follows:

$$\sin x = \frac{3}{5}$$
$$\sin^{-1}(\sin x) = \sin^{-1}\left(\frac{3}{5}\right)$$
$$x = \sin^{-1}\left(\frac{3}{5}\right)$$
$$x = 2.5$$

If x is in the fourth quadrant, $\sin x = -\frac{3}{5}$, so x can be solved for as follows:

$$\sin x = -\frac{3}{5}$$
$$\sin^{-1}(\sin x) = \sin^{-1}\left(-\frac{3}{5}\right)$$
$$x = \sin^{-1}\left(-\frac{3}{5}\right)$$
$$x = 5.6$$

7. a) $\tan \theta = \frac{\sin \theta}{\cos \theta}$, $\cos \theta \neq 0$ is true.

b) $\sin^2 \theta + \cos^2 \theta = 1$ is true.

c) $\sec \theta = \frac{1}{\sin \theta}$, $\sin \theta \neq 0$ is false, since $\sec \theta$ is the reciprocal of $\cos \theta$, not $\sin \theta$.

d) $\cos^2 \theta = \sin^2 \theta - 1$ is false. This can be shown by performing the following operations:

$$\cos^2 \theta = \sin^2 \theta - 1$$
$$\cos^2 \theta + 1 = \sin^2 \theta - 1 + 1$$
$$\cos^2 \theta + 1 = \sin^2 \theta$$
$$\cos^2 \theta + 1 - \cos^2 \theta = \sin^2 \theta - \cos^2 \theta$$
$$\sin^2 \theta - \cos^2 \theta = 1$$

Since $\sin^2\theta + \cos^2\theta = 1$ is true, $\sin^2\theta - \cos^2\theta = 1$ must be false.

e) $1 + \tan^2\theta = \sec^2\theta$ is true

f) $\cot\theta = \frac{\cos\theta}{\sin\theta}$, $\sin\theta \neq 0$ is true.

8. The sinusoidal function $y = \sin x$ is transformed into the function $y = a \sin k(x - d) + c$ by vertically stretching or compressing the function $y = \sin x$ by a factor of a, horizontally stretching or compressing it by a factor of $\frac{1}{k}$, reflecting it in the x-axis if $a < 0$, reflecting it in the y-axis is $k < 0$, vertically translating it c units up or down, and horizontally translating it d units to the right or the left. Therefore, an appropriate flow chart would be as follows:

Perform a vertical stretch/compression by a factor of $|a|$.

↓

Use $\left|\frac{1}{k}\right|$ to determine the horizontal stretch/compression.

↓

Use a and k to determine whether the function is reflected in the y-axis or the x-axis.

↓

Perform a vertical translation of c units up or down.

↓

Perform a horizontal translation of d units to the right or the left.

7.1 Exploring Equivalent Trigonometric Functions, pp. 392–393

1. a) Answers may vary. For example: The graph is that of the trigonometric function $y = \cos\theta$. Since the period of $y = \cos\theta$ is 2π radians, the graph repeats itself every 2π radians. Therefore, three possible equivalent expressions for the graph are $y = \cos(\theta + 2\pi)$, $y = \cos(\theta + 4\pi)$, and $y = \cos(\theta - 2\pi)$.

b) The trigonometric functions $y = \cos\theta$ and $y = \sin\left(\theta + \frac{\pi}{2}\right)$ are equivalent. Since the period of $y = \sin\left(\theta + \frac{\pi}{2}\right)$ is 2π radians, the graph repeats itself every 2π radians. Therefore, three possible equivalent expressions for the graph using the sine function are $y = \sin\left(\theta + \frac{\pi}{2}\right)$, $y = \sin\left(\theta - \frac{3\pi}{2}\right)$, and $y = \sin\left(\theta + \frac{5\pi}{2}\right)$.

2. a) The graph of the trigonometric function $y = \csc\theta$ is symmetric with respect to the origin, so $y = \csc\theta$ is an odd function. Therefore, $\csc(-\theta) = -\csc\theta$. The graph of the trigonometric function $y = \sec\theta$ is symmetric with respect to the y-axis, so $y = \sec\theta$ is an even function. Therefore, $\sec(-\theta) = \sec\theta$. The graph of the trigonometric function $y = \cot\theta$ is symmetric with respect to the origin, so $y = \cot\theta$ is an odd function. Therefore, $\cot(-\theta) = -\cot\theta$.

b) If the graph of the trigonometric function $y = \csc\theta$ is reflected in the y-axis, the equation of the resulting graph is $y = \csc(-\theta)$. Also, if the graph of $y = \csc\theta$ is reflected in the x-axis, the equation of the resulting graph is $y = -\csc\theta$. Since the graph of $y = \csc\theta$ is symmetric with respect to the origin, a reflection in the y-axis is the same as a reflection in the x-axis. Therefore, the equation $\csc(-\theta) = -\csc\theta$ must be true. If the graph of the trigonometric function $y = \sec\theta$ is reflected in the y-axis, the equation of the resulting graph is $y = \sec(-\theta)$. Since the graph of $y = \sec\theta$ is symmetric with respect to the y-axis, a reflection of the function in the y-axis results in the function itself. Therefore, the equation $\sec(-\theta) = \sec\theta$ must be true. If the graph of the trigonometric function $y = \cot\theta$ is reflected in the y-axis, the equation of the resulting graph is $y = \cot(-\theta)$. Also, if the graph of $y = \cot\theta$ is reflected in the x-axis, the equation of the resulting graph is $y = -\cot\theta$. Since the graph of $y = \cot\theta$ is symmetric with respect to the origin, a reflection in the y-axis is the same as a reflection in the x-axis. Therefore, the equation $\cot(-\theta) = -\cot\theta$ must be true.

3. a) Since $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$,

$$\sin\frac{\pi}{6} = \cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \cos\left(\frac{3\pi}{6} - \frac{\pi}{6}\right)$$
$$= \cos\frac{2\pi}{6} = \cos\frac{\pi}{3}$$

b) Since $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$,

$$\cos\frac{5\pi}{12} = \sin\left(\frac{\pi}{2} - \frac{5\pi}{12}\right)$$
$$= \sin\left(\frac{6\pi}{12} - \frac{5\pi}{12}\right) = \sin\frac{\pi}{12}$$

c) Since $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$,

$$\tan\frac{3\pi}{8} = \cot\left(\frac{\pi}{2} - \frac{3\pi}{8}\right)$$
$$= \cot\left(\frac{4\pi}{8} - \frac{3\pi}{8}\right) = \cot\frac{\pi}{8}$$

d) Since $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$,

$$\cos\frac{5\pi}{16} = \sin\left(\frac{\pi}{2} - \frac{5\pi}{16}\right)$$
$$= \sin\left(\frac{8\pi}{16} - \frac{5\pi}{16}\right) = \sin\frac{3\pi}{16}$$

e) Since $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$,

$$\sin\frac{\pi}{8} = \cos\left(\frac{\pi}{2} - \frac{\pi}{8}\right)$$
$$= \cos\left(\frac{4\pi}{8} - \frac{\pi}{8}\right)$$
$$= \cos\frac{3\pi}{8}$$

f) Since $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$,

$$\tan\frac{\pi}{6} = \cot\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \cot\left(\frac{3\pi}{6} - \frac{\pi}{6}\right)$$
$$= \cot\frac{2\pi}{6} = \cot\frac{\pi}{3}$$

4. a) Since $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$,

$$\frac{1}{\sin\theta} = \frac{1}{\cos\left(\frac{\pi}{2} - \theta\right)}$$

Therefore, $\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$.

Since $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$, $\frac{1}{\cos\theta} = \frac{1}{\sin\left(\frac{\pi}{2} - \theta\right)}$.

Therefore, $\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$.

Since $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$, $\frac{1}{\tan\theta} = \frac{1}{\cot\left(\frac{\pi}{2} - \theta\right)}$.

Therefore, $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$.

b) The trigonometric function $y = \sec\left(\frac{\pi}{2} - \theta\right)$ can also be written as $y = \sec\left(-\left(\theta - \frac{\pi}{2}\right)\right)$. If the graph of the trigonometric function $y = \sec\left(-\left(\theta - \frac{\pi}{2}\right)\right)$ is reflected in the y-axis, the equation of the resulting graph is $y = \sec\left(\theta - \frac{\pi}{2}\right)$. Since the graph of $y = \sec\left(-\left(\theta - \frac{\pi}{2}\right)\right)$ is symmetric with respect to the y-axis, a reflection of the function in the y-axis results in the function itself. For this reason, $\sec\left(\frac{\pi}{2} - \theta\right) = \sec\left(\theta - \frac{\pi}{2}\right)$, so $\csc\theta = \sec\left(\theta - \frac{\pi}{2}\right)$. Therefore, $\frac{1}{\csc\theta} = \frac{1}{\sec\left(\theta - \frac{\pi}{2}\right)}$, or $\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$. This is a known identity, so the cofunction identity $\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$ must be true. The trigonometric function $y = \csc\left(\frac{\pi}{2} - \theta\right)$ can also be written as $y = \csc\left(-\left(\theta - \frac{\pi}{2}\right)\right)$. If the graph of the trigonometric function $y = \csc\left(-\left(\theta - \frac{\pi}{2}\right)\right)$ is reflected in the y-axis, the equation of the resulting resulting graph is $y = \csc\left(\theta - \frac{\pi}{2}\right)$. Since the graph of $y = \csc\left(-\left(\theta - \frac{\pi}{2}\right)\right)$ is symmetric with respect to the origin, a reflection in the y-axis is the same as a reflection in the x-axis. For this reason,

$$-\csc\left(\frac{\pi}{2} - \theta\right) = \csc\left(\theta - \frac{\pi}{2}\right), \text{ or}$$
$$\csc\left(\frac{\pi}{2} - \theta\right) = \csc\left(\theta - \frac{\pi}{2} + \pi\right)$$
$$= \csc\left(\theta + \frac{\pi}{2}\right), \text{ so}$$
$$\sec\theta = \csc\left(\theta + \frac{\pi}{2}\right)$$

Therefore, $\frac{1}{\sec\theta} = \frac{1}{\csc\left(\theta + \frac{\pi}{2}\right)}$, or

$$\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$$

This is a known identity, so the cofunction identity $\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$ must be true.

The trigonometric function $y = \tan\left(\frac{\pi}{2} - \theta\right)$ can also be written as $y = \tan\left(-\left(\theta - \frac{\pi}{2}\right)\right)$. If the graph of the trigonometric function $y = \tan\left(-\left(\theta - \frac{\pi}{2}\right)\right)$ is reflected in the y-axis, the equation of the resulting graph is $y = \tan\left(-\left(\theta - \frac{\pi}{2}\right)\right)$.

Since the graph of $y = \tan\left(-\left(\theta - \frac{\pi}{2}\right)\right)$ is symmetric with respect to the origin, a reflection in the y-axis is the same as a reflection in the x-axis. For this reason, $-\tan\left(\frac{\pi}{2} - \theta\right) = \tan\left(\theta - \frac{\pi}{2}\right)$, or $\tan\left(\frac{\pi}{2} - \theta\right) = \tan\left(\theta - \frac{\pi}{2} + \pi\right) = \tan\left(\theta + \frac{\pi}{2}\right)$, so $\cot\theta = \tan\left(\theta + \frac{\pi}{2}\right)$. Therefore, $\frac{1}{\cot\theta} = \frac{1}{\tan\left(\theta + \frac{\pi}{2}\right)}$, or $\tan\theta = \cot\left(\theta + \frac{\pi}{2}\right)$. This is a known identity, so the cofunction identity $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$ must be true.

5. a) Since $\sin\theta = \sin(\pi - \theta)$,

$$\sin\frac{7\pi}{8} = \sin\left(\pi - \frac{7\pi}{8}\right) = \sin\frac{\pi}{8}$$

b) Since $\cos\theta = -\cos(\pi + \theta)$,

$$\cos\frac{13\pi}{12} = -\cos\left(\pi + \frac{13\pi}{12}\right)$$
$$= -\cos\left(\frac{12\pi}{12} + \frac{13\pi}{12}\right) = -\cos\frac{25\pi}{12}$$
$$= -\cos\left(\frac{25\pi}{12} - 2\pi\right)$$
$$= -\cos\left(\frac{25\pi}{12} - \frac{24\pi}{12}\right) = -\cos\frac{\pi}{12}$$

c) Since $\tan\theta = \tan(\pi + \theta)$,

$$\tan\frac{5\pi}{4} = \tan\left(\frac{4\pi}{4} + \frac{5\pi}{4}\right) = \tan\frac{9\pi}{4}$$
$$= \tan\left(\frac{9\pi}{4} - 2\pi\right)$$
$$= \tan\left(\frac{9\pi}{4} - \frac{8\pi}{4}\right) = \tan\frac{\pi}{4}$$

d) Since $\cos\theta = \cos(2\pi - \theta)$,

$$\cos\frac{11\pi}{6} = \cos\left(2\pi - \frac{11\pi}{6}\right)$$
$$= \cos\left(\frac{12\pi}{6} - \frac{11\pi}{6}\right) = \cos\frac{\pi}{6}$$

e) Since $\sin\theta = -\sin(2\pi - \theta)$,

$$\sin\frac{13\pi}{8} = -\sin\left(2\pi - \frac{13\pi}{8}\right)$$
$$= -\sin\left(\frac{16\pi}{8} - \frac{13\pi}{8}\right) = -\sin\frac{3\pi}{8}$$

f) Since $\tan\theta = -\tan(2\pi - \theta)$,

$$\tan\frac{5\pi}{3} = -\tan\left(2\pi - \frac{5\pi}{3}\right)$$
$$= -\tan\left(\frac{6\pi}{3} - \frac{5\pi}{3}\right) = -\tan\frac{\pi}{3}$$

6. a) Assume the circle is a unit circle. Let the coordinates of Q be (x, y). Since P and Q are reflections of each other in the line $y = x$, the coordinates of P are (y, x). Draw a line from P to the positive x-axis. The hypotenuse of the new right triangle makes an angle of $\left(\frac{\pi}{2} - \theta\right)$ with the positive x-axis. Since the x-coordinate of P is y, $\cos\left(\frac{\pi}{2} - \theta\right) = y$. Also, since the y-coordinate of Q is y, $\sin\theta = y$. Therefore, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.

b) Assume the circle is a unit circle. Let the coordinates of the vertex on the circle of the right triangle in the first quadrant be (x, y). Then $\sin\theta = y$, so $-\sin\theta = -y$. The point on the circle that results from rotating the vertex by $\frac{\pi}{2}$ counterclockwise about the origin has coordinates $(-y, x)$, so $\cos\left(\frac{\pi}{2} + \theta\right) = -y$. Therefore, $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$.

7. a) true; the period of cosine is 2π

b) false; Answers may vary. For example: Let $\theta = \frac{\pi}{2}$. Then the left side is $\sin\frac{\pi}{2}$, or 1. The right side is $-\sin\frac{\pi}{2}$ or -1.

c) false; Answers may vary. For example: Let $\theta = \pi$. Then the left side is $\cos\pi$, or -1. The right side is $-\cos 5\pi$, or 1.

d) false; Answers may vary. For example: Let $\theta = \frac{\pi}{4}$. Then the left side is $\tan\frac{3\pi}{4}$, or $-\frac{\sqrt{2}}{2}$. The right side is $\tan\frac{\pi}{4}$, or $\frac{\sqrt{2}}{2}$.

e) false; Answers may vary. For example: Let $\theta = \pi$. Then the left side is $\cot\frac{3\pi}{4}$, or –1. The right side is $\tan\frac{\pi}{4}$, or 1.

f) false; Answers may vary. For example: Let $\theta = \frac{\pi}{2}$. Then the left side is $\sin \frac{5\pi}{2}$, or 1. The right side is $\sin\left(-\frac{\pi}{2}\right)$, or –1.

7.2 Compound Angle Formulas, pp. 400–401

1. a) Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$, $\sin a \cos 2a + \cos a \sin 2a = \sin(a + 2a) = \sin 3a$.

b) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\cos 4x \cos 3x - \sin 4x \sin 3x = \cos(4x + 3x) = \cos 7x$$

2. a) Since $\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$,

$$\frac{\tan 170° - \tan 110°}{1 + \tan 170° \tan 110°} = \tan(170° - 110°) = \tan 60° = \sqrt{3}$$

b) Since $\cos(a - b) = \cos a \cos b + \sin a \sin b$,

$$\cos \frac{5\pi}{12} \cos \frac{\pi}{12} + \sin \frac{5\pi}{12} \sin \frac{\pi}{12} = \cos\left(\frac{5\pi}{12} - \frac{\pi}{12}\right) = \cos \frac{4\pi}{12} = \cos \frac{\pi}{3} = \frac{1}{2}$$

3. a) Two angles from the special triangles are 30° and 45°, so 75° = 30° + 45°.

b) Two angles from the special triangles are 30° and 45°, so –15° = 30° – 45°.

c) Two angles from the special triangles are $\frac{\pi}{6}$ and $\frac{\pi}{3}$, so $-\frac{\pi}{6} = \frac{\pi}{6} - \frac{2\pi}{6} = \frac{\pi}{6} - \frac{\pi}{3}$.

d) Two angles from the special triangles are $\frac{\pi}{6}$ and $\frac{\pi}{4}$, so $\frac{\pi}{12} = \frac{3\pi}{12} - \frac{2\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$.

e) Two angles from the special triangles are 45° and 60°, so 105° = 45° + 60°.

f) Two angles from the special triangles are $\frac{\pi}{3}$ and $\frac{\pi}{2}$, so $\frac{5\pi}{6} = \frac{2\pi}{6} + \frac{3\pi}{6} = \frac{\pi}{3} + \frac{\pi}{2}$.

4. a) Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$,

$$\begin{aligned}\sin 75° &= \sin(30° + 45°)\\ &= \sin 30° \cos 45° + \cos 30° \sin 45°\\ &= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{2} + \sqrt{6}}{4}\end{aligned}$$

b) Since $\cos(a - b) = \cos a \cos b + \sin a \sin b$,

$$\begin{aligned}\cos 15° &= \cos(45° - 30°)\\ &= \cos 45° \cos 30° + \sin 45° \sin 30°\\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{2} + \sqrt{6}}{4}\end{aligned}$$

c) Since $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\begin{aligned}\tan \frac{5\pi}{12} &= \tan\left(\frac{2\pi}{12} + \frac{3\pi}{12}\right) = \tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)\\ &= \frac{\tan \frac{\pi}{6} + \tan \frac{\pi}{4}}{1 - \tan \frac{\pi}{6} \tan \frac{\pi}{4}} = \frac{\frac{\sqrt{3}}{3} + 1}{1 - \left(\frac{\sqrt{3}}{3}\right)(1)}\\ &= \frac{\frac{\sqrt{3}}{3} + \frac{3}{3}}{\frac{3}{3} - \frac{\sqrt{3}}{3}} = \frac{\frac{\sqrt{3} + 3}{3}}{\frac{3 - \sqrt{3}}{3}}\\ &= \frac{\sqrt{3} + 3}{3} \times \frac{3}{3 - \sqrt{3}} = \frac{\sqrt{3} + 3}{3 - \sqrt{3}}\\ &= \frac{(\sqrt{3} + 3)(3 + \sqrt{3})}{(3 - \sqrt{3})(3 + \sqrt{3})}\\ &= \frac{3\sqrt{3} + 9 + 3 + 3\sqrt{3}}{9 - 3\sqrt{3} + 3\sqrt{3} - 3}\\ &= \frac{12 + 6\sqrt{3}}{6} = 2 + \sqrt{3}\end{aligned}$$

d) Since $\sin(a - b) = \sin a \cos b - \cos a \sin b$,

$$\begin{aligned}\sin\left(-\frac{\pi}{12}\right) &= \sin\left(\frac{3\pi}{12} - \frac{4\pi}{12}\right) = \sin\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\\ &= \sin \frac{\pi}{4} \cos \frac{\pi}{3} - \cos \frac{\pi}{4} \sin \frac{\pi}{3}\\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{2} - \sqrt{6}}{4}\end{aligned}$$

e) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\begin{aligned}\cos 105° &= \cos(45° + 60°)\\ &= \cos 45° \cos 60° - \sin 45° \sin 60°\\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\end{aligned}$$

$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$$
$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

f) Since $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\tan \frac{23\pi}{12} = \tan\left(\frac{9\pi}{12} + \frac{14\pi}{12}\right) = \tan\left(\frac{3\pi}{4} + \frac{7\pi}{6}\right)$$
$$= \frac{\tan\frac{3\pi}{4} + \tan\frac{7\pi}{6}}{1 - \tan\frac{3\pi}{4}\tan\frac{7\pi}{6}} = \frac{-1 + \frac{\sqrt{3}}{3}}{1 - (-1)\left(\frac{\sqrt{3}}{3}\right)}$$
$$= \frac{-\frac{3}{3} + \frac{\sqrt{3}}{3}}{\frac{3}{3} - \left(-\frac{\sqrt{3}}{3}\right)} = \frac{\frac{\sqrt{3} - 3}{3}}{\frac{3 + \sqrt{3}}{3}}$$
$$= \frac{\sqrt{3} - 3}{3} \times \frac{3}{3 + \sqrt{3}} = \frac{\sqrt{3} - 3}{3 + \sqrt{3}}$$
$$= \frac{(\sqrt{3} - 3)(3 - \sqrt{3})}{(3 + \sqrt{3})(3 - \sqrt{3})}$$
$$= \frac{3\sqrt{3} - 9 - 3 + 3\sqrt{3}}{9 + 3\sqrt{3} - 3\sqrt{3} - 3}$$
$$= \frac{-12 + 6\sqrt{3}}{6}$$
$$= -2 + \sqrt{3}$$

5. a) Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$,

$$\sin\left(\pi + \frac{\pi}{6}\right) = \sin \pi \cos \frac{\pi}{6} + \cos \pi \sin \frac{\pi}{6}$$
$$= (0)\left(\frac{\sqrt{3}}{2}\right) + (-1)\left(\frac{1}{2}\right)$$
$$= 0 + \left(-\frac{1}{2}\right) = -\frac{1}{2}$$

b) Since $\cos(a - b) = \cos a \cos b + \sin a \sin b$,

$$\cos\left(\pi - \frac{\pi}{4}\right) = \cos \pi \cos \frac{\pi}{4} + \sin \pi \sin \frac{\pi}{4}$$
$$= (-1)\left(\frac{\sqrt{2}}{2}\right) + (0)\left(\frac{\sqrt{2}}{2}\right)$$
$$= -\frac{\sqrt{2}}{2} + 0$$
$$= -\frac{\sqrt{2}}{2}$$

c) Since $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\tan\left(\frac{\pi}{4} + \pi\right) = \frac{\tan\frac{\pi}{4} + \tan \pi}{1 - \tan\frac{\pi}{4}\tan \pi}$$
$$= \frac{1 + 0}{1 - (1)(0)}$$
$$= \frac{1 + 0}{1 - 0}$$
$$= \frac{1}{1} = 1$$

d) Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$,

$$\sin\left(-\frac{\pi}{2} + \frac{\pi}{3}\right) = \sin\left(-\frac{\pi}{2}\right)\cos\frac{\pi}{3}$$
$$+ \cos\left(-\frac{\pi}{2}\right)\sin\frac{\pi}{3}$$
$$= (-1)\left(\frac{1}{2}\right) + (0)\left(\frac{\sqrt{3}}{2}\right)$$
$$= -\frac{1}{2} + 0$$
$$= -\frac{1}{2}$$

e) Since $\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$,

$$\tan\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{6}}$$
$$= \frac{\sqrt{3} - \frac{\sqrt{3}}{3}}{1 + (\sqrt{3})\left(\frac{\sqrt{3}}{3}\right)}$$
$$= \frac{\frac{3\sqrt{3}}{3} - \frac{\sqrt{3}}{3}}{1 + \frac{3}{3}}$$
$$= \frac{\frac{3\sqrt{3} - \sqrt{3}}{3}}{1 + 1}$$
$$= \frac{\frac{2\sqrt{3}}{3}}{2}$$
$$= \frac{2\sqrt{3}}{3} \times \frac{1}{2}$$
$$= \frac{2\sqrt{3}}{6}$$
$$= \frac{\sqrt{3}}{3}$$

f) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\cos\left(\frac{\pi}{2} + \frac{\pi}{3}\right) = \cos\frac{\pi}{2}\cos\frac{\pi}{3} - \sin\frac{\pi}{2}\sin\frac{\pi}{3}$$
$$= (0)\left(\frac{1}{2}\right) - (1)\left(\frac{\sqrt{3}}{2}\right)$$

$= 0 - \frac{\sqrt{3}}{2}$

$= -\frac{\sqrt{3}}{2}$

6. a) Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$,

$$\begin{aligned}\sin(\pi + x) &= \sin \pi \cos x + \cos \pi \sin x\\ &= (0)(\cos x) + (-1)(\sin x)\\ &= 0 + (-\sin x)\\ &= 0 - \sin x\\ &= -\sin x\end{aligned}$$

b) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\begin{aligned}\cos\left(x + \frac{3\pi}{2}\right) &= \cos x \cos \frac{3\pi}{2} - \sin x \sin \frac{3\pi}{2}\\ &= (\cos x)(0) - (\sin x)(-1)\\ &= 0 - (-\sin x) = \sin x\end{aligned}$$

c) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\begin{aligned}\cos\left(x + \frac{\pi}{2}\right) &= \cos x \cos \frac{\pi}{2} - \sin x \sin \frac{\pi}{2}\\ &= (\cos x)(0) - (\sin x)(1)\\ &= 0 - \sin x\\ &= -\sin x\end{aligned}$$

d) Since $\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\begin{aligned}\tan(x + \pi) &= \frac{\tan x + \tan \pi}{1 - \tan x \tan \pi}\\ &= \frac{\tan x + 0}{1 - (\tan x)(0)}\\ &= \frac{\tan x + 0}{1 - 0}\\ &= \frac{\tan x}{1}\\ &= \tan x\end{aligned}$$

e) Since $\sin(a - b) = \sin a \cos b - \cos a \sin b$,

$$\begin{aligned}\sin(x - \pi) &= \sin x \cos \pi - \cos x \sin \pi\\ &= (\sin x)(-1) - (\cos x)(0)\\ &= -\sin x - 0\\ &= -\sin x\end{aligned}$$

f) Since $\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$,

$$\begin{aligned}\tan(2\pi - x) &= \frac{\tan 2\pi - \tan x}{1 + \tan 2\pi \tan x}\\ &= \frac{0 - \tan x}{1 + (0)(\tan x)}\\ &= \frac{-\tan x}{1 + 0}\\ &= \frac{-\tan x}{1}\\ &= -\tan x\end{aligned}$$

7. a) Since $\sin(\pi + x) = \sin(x + \pi)$, $\sin(\pi + x)$ is equivalent to $\sin x$ translated π units to the left, which is equivalent to $-\sin x$.

b) $\cos\left(x + \frac{3\pi}{2}\right)$ is equivalent to $\cos x$ translated $\frac{3\pi}{2}$ units to the left, which is equivalent to $\sin x$.

c) $\cos\left(x + \frac{\pi}{2}\right)$ is equivalent to $\cos x$ translated $\frac{\pi}{2}$ units to the left, which is equivalent to $-\sin x$.

d) $\tan(x + \pi)$ is equivalent to $\tan x$ translated π units to the left, which is equivalent to $\tan x$.

e) $\sin(x - \pi)$ is equivalent to $\sin x$ translated π units to the right, which is equivalent to $-\sin x$.

f) Since $\tan(2\pi - x) = \tan(-x + 2\pi)$, and since the period of the tangent function is 2π, $\tan(2\pi - x)$ is equivalent to $\tan(-x)$, which is equivalent to $\tan x$ reflected in the y-axis, which is equivalent to $-\tan x$.

8. a) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\begin{aligned}\cos 75° &= \cos(30° + 45°)\\ &= \cos 30° \cos 45° - \sin 30° \sin 45°\\ &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

b) Since $\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$,

$$\begin{aligned}\tan(-15°) &= \tan(30° - 45°)\\ &= \frac{\tan 30° - \tan 45°}{1 + \tan 30° \tan 45°}\\ &= \frac{\frac{\sqrt{3}}{3} - 1}{1 + \left(\frac{\sqrt{3}}{3}\right)(1)} = \frac{\frac{\sqrt{3}}{3} - \frac{3}{3}}{\frac{3}{3} + \frac{\sqrt{3}}{3}}\\ &= \frac{\frac{\sqrt{3} - 3}{3}}{\frac{3 + \sqrt{3}}{3}} = \frac{\sqrt{3} - 3}{3} \times \frac{3}{3 + \sqrt{3}}\\ &= \frac{\sqrt{3} - 3}{3 + \sqrt{3}} = \frac{(\sqrt{3} - 3)(3 - \sqrt{3})}{(3 + \sqrt{3})(3 - \sqrt{3})}\\ &= \frac{3\sqrt{3} - 9 - 3 + 3\sqrt{3}}{9 + 3\sqrt{3} - 3\sqrt{3} - 3}\\ &= \frac{-12 + 6\sqrt{3}}{6}\\ &= -2 + \sqrt{3}\end{aligned}$$

c) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\cos\frac{11\pi}{12} = \cos\left(\frac{2\pi}{12} + \frac{9\pi}{12}\right)$$
$$= \cos\left(\frac{\pi}{6} + \frac{3\pi}{4}\right)$$
$$= \cos\frac{\pi}{6}\cos\frac{3\pi}{4} - \sin\frac{\pi}{6}\sin\frac{3\pi}{4}$$
$$= \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$
$$= \frac{-\sqrt{6} - \sqrt{2}}{4}$$

d) Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$,

$$\sin\frac{13\pi}{12} = \sin\left(\frac{3\pi}{12} + \frac{10\pi}{12}\right)$$
$$= \sin\left(\frac{\pi}{4} + \frac{5\pi}{6}\right)$$
$$= \sin\frac{\pi}{4}\cos\frac{5\pi}{6} + \cos\frac{\pi}{4}\sin\frac{5\pi}{6}$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$= -\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$
$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

e) Since $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\tan\frac{7\pi}{12} = \tan\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right)$$
$$= \tan\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$$
$$= \frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{3}}{1 - \tan\frac{\pi}{4}\tan\frac{\pi}{3}}$$
$$= \frac{1 + \sqrt{3}}{1 - (1)(\sqrt{3})}$$
$$= \frac{1 + \sqrt{3}}{1 - \sqrt{3}}$$
$$= \frac{(1 + \sqrt{3})(1 + \sqrt{3})}{(1 - \sqrt{3})(1 + \sqrt{3})}$$
$$= \frac{1 + \sqrt{3} + \sqrt{3} + 3}{1 - \sqrt{3} + \sqrt{3} - 3}$$
$$= \frac{4 + 2\sqrt{3}}{-2} = -2 - \sqrt{3}$$

f) Since $\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$,

$$\tan\left(-\frac{5\pi}{12}\right) = \tan\left(\frac{4\pi}{12} - \frac{9\pi}{12}\right) = \tan\left(\frac{\pi}{3} - \frac{3\pi}{4}\right)$$
$$= \frac{\tan\frac{\pi}{3} - \tan\frac{3\pi}{4}}{1 + \tan\frac{\pi}{3}\tan\frac{3\pi}{4}}$$
$$= \frac{\sqrt{3} - (-1)}{1 + (\sqrt{3})(-1)}$$
$$= \frac{\sqrt{3} + 1}{1 - \sqrt{3}}$$
$$= \frac{1 + \sqrt{3}}{1 - \sqrt{3}}$$
$$= \frac{(1 + \sqrt{3})(1 + \sqrt{3})}{(1 - \sqrt{3})(1 + \sqrt{3})}$$
$$= \frac{1 + \sqrt{3} + \sqrt{3} + 3}{1 - \sqrt{3} + \sqrt{3} - 3}$$
$$= \frac{4 + 2\sqrt{3}}{-2} = -2 - \sqrt{3}$$

9. a) Since $\sin x = \frac{4}{5}$, the leg opposite the angle x in a right triangle has a length of 4, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 4^2 = 5^2$$
$$x^2 + 16 = 25$$
$$x^2 + 16 - 16 = 25 - 16$$
$$x^2 = 9$$
$$x = 3, \text{ in quadrant I}$$

Since $\cos x = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos x = \frac{3}{5}$. In addition, since $\sin y = -\frac{12}{13}$, the leg opposite the angle y in a right triangle has a length of 12, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$
$$x^2 + 144 - 144 = 169 - 144$$

$$x^2 = 25$$
$$x = 5, \text{ in quadrant IV}$$

Since $\cos y = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos y = \frac{5}{13}$. Therefore, since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$= \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)$$
$$= \frac{15}{65} - \left(-\frac{48}{65}\right)$$
$$= \frac{15}{65} + \frac{48}{65}$$
$$= \frac{63}{65}$$

b) Since $\sin x = \frac{4}{5}$, the leg opposite the angle x in a right triangle has a length of 4, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 4^2 = 5^2$$
$$x^2 + 16 = 25$$
$$x^2 + 16 - 16 = 25 - 16$$
$$x^2 = 9$$
$$x = 3, \text{ in quadrant I}$$

Since $\cos x = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos x = \frac{3}{5}$. In addition, since $\sin y = -\frac{12}{13}$, the leg opposite the angle y in a right triangle has a length of 12, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$
$$x^2 + 144 - 144 = 169 - 144$$
$$x^2 = 25$$
$$x = 5, \text{ in quadrant IV}$$

Since $\cos y = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos y = \frac{5}{13}$. Therefore, since $\sin(a + b) = \sin a \cos b + \cos a \sin b$,

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$= \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right)$$
$$= \frac{20}{65} + \left(-\frac{36}{65}\right)$$
$$= \frac{20}{65} - \frac{36}{65}$$
$$= -\frac{16}{65}$$

c) Since $\sin x = \frac{4}{5}$, the leg opposite the angle x in a right triangle has a length of 4, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 4^2 = 5^2$$
$$x^2 + 16 = 25$$
$$x^2 + 16 - 16 = 25 - 16$$
$$x^2 = 9$$
$$x = 3, \text{ in quadrant I}$$

Since $\cos x = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos x = \frac{3}{5}$. In addition, since $\sin y = -\frac{12}{13}$, the leg opposite the angle y in a right triangle has a length of 12, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$
$$x^2 + 144 - 144 = 169 - 144$$
$$x^2 = 25$$
$$x = 5, \text{ in quadrant IV}$$

Since $\cos y = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos y = \frac{5}{13}$. Therefore, since $\cos(a - b) = \cos a \cos b + \sin a \sin b$,

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
$$= \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) = \frac{15}{65} + \left(-\frac{48}{65}\right)$$
$$= \frac{15}{65} - \frac{48}{65} = -\frac{33}{65}$$

d) Since $\sin x = \frac{4}{5}$, the leg opposite the angle x in a right triangle has a length of 4, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 4^2 = 5^2$$
$$x^2 + 16 = 25$$
$$x^2 + 16 - 16 = 25 - 16$$
$$x^2 = 9$$
$$x = 3, \text{ in quadrant I}$$

Since $\cos x = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos x = \frac{3}{5}$. In addition, since $\sin y = -\frac{12}{13}$, the leg opposite the angle y in a right triangle has a length of 12, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$

$$x^2 + 144 - 144 = 169 - 144$$
$$x^2 = 25$$
$$x = 5, \text{ in quadrant IV}$$

Since $\cos y = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos y = \frac{5}{13}$. Therefore, since $\sin(a - b) = \sin a \cos b - \cos a \sin b$,

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
$$= \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) = \frac{20}{65} - \left(-\frac{36}{65}\right)$$
$$= \frac{20}{65} + \frac{36}{65} = \frac{56}{65}$$

e) Since $\sin x = \frac{4}{5}$, the leg opposite the angle x in a right triangle has a length of 4, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 4^2 = 5^2$$
$$x^2 + 16 = 25$$
$$x^2 + 16 - 16 = 25 - 16$$
$$x^2 = 9$$
$$x = 3, \text{ in quadrant I}$$

Since $\tan x = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan x = \frac{4}{3}$. In addition, since $\sin y = -\frac{12}{13}$, the leg opposite the angle y in a right triangle has a length of 12, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$
$$x^2 + 144 - 144 = 169 - 144$$
$$x^2 = 25$$
$$x = 5, \text{ in quadrant IV}$$

Since $\tan y = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan y = -\frac{12}{5}$. (The reason the sign is negative is because angle y is in the fourth quadrant.) Therefore, since

$$\tan(\text{a} + \text{b}) = \frac{\tan a + \tan b}{1 - \tan a \tan b},$$
$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$= \frac{\frac{4}{3} + \left(-\frac{12}{5}\right)}{1 - \left(\frac{4}{3}\right)\left(-\frac{12}{5}\right)} = \frac{\frac{20}{15} + \left(-\frac{36}{15}\right)}{1 - \left(-\frac{48}{15}\right)}$$
$$= \frac{\frac{20}{15} - \frac{36}{15}}{1 + \frac{48}{15}} = \frac{-\frac{16}{15}}{\frac{63}{15}}$$
$$= -\frac{16}{15} \times \frac{15}{63} = -\frac{16}{63}$$

f) Since $\sin x = \frac{4}{5}$, the leg opposite the angle x in a right triangle has a length of 4, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 4^2 = 5^2$$
$$x^2 + 16 = 25$$
$$x^2 + 16 - 16 = 25 - 16$$
$$x^2 = 9$$
$$x = 3, \text{ in quadrant I}$$

Since $\tan x = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan x = \frac{4}{3}$. In addition, since $\sin y = -\frac{12}{13}$, the leg opposite the angle y in a right triangle has a length of 12, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$
$$x^2 + 144 - 144 = 169 - 144$$
$$x^2 = 25$$
$$x = 5, \text{ in quadrant IV}$$

Since $\tan y = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan y = -\frac{12}{5}$. (The reason the sign is negative is because angle y is in the fourth quadrant.) Therefore, since

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b},$$
$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$
$$= \frac{\frac{4}{3} - \left(-\frac{12}{5}\right)}{1 + \left(\frac{4}{3}\right)\left(-\frac{12}{5}\right)} = \frac{\frac{20}{15} + \frac{36}{15}}{1 + \left(-\frac{48}{15}\right)} = \frac{\frac{56}{15}}{-\frac{33}{15}}$$
$$= \frac{56}{15} \times \left(-\frac{15}{33}\right) = -\frac{56}{33}$$

10. Since $\sin \alpha = \frac{7}{25}$, the leg opposite the angle α in a right triangle has a length of 7, while the hypotenuse of the right triangle has a length of 25. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 7^2 = 25^2$$
$$x^2 + 49 = 625$$
$$x^2 + 49 - 49 = 625 - 49$$
$$x^2 = 576$$
$$x = 24, \text{ in quadrant I}$$

Since $\cos \alpha = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos \alpha = \frac{24}{25}$. Also, since $\tan \alpha = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan \alpha = \frac{7}{24}$. In addition, since

$\cos\beta = \frac{5}{13}$, the leg adjacent to the angle β in a right triangle has a length of 5, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$5^2 + y^2 = 13^2$$
$$25 + y^2 = 169$$
$$25 + y^2 - 25 = 169 - 25$$
$$y^2 = 144$$
$$x = 12, \text{ in quadrant I}$$

Since $\sin\beta = \frac{\text{opposite leg}}{\text{hypotenuse}}$, $\sin\beta = \frac{12}{13}$. Also, since $\tan\beta = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan\beta = \frac{12}{5}$. Therefore, since $\sin(a + b) = \sin a \cos b + \cos a \sin b$,

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$= \left(\frac{7}{25}\right)\left(\frac{5}{13}\right) + \left(\frac{24}{25}\right)\left(\frac{12}{13}\right)$$
$$= \frac{35}{325} + \frac{288}{325} = \frac{323}{325}$$

Also, since $\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$= \frac{\frac{7}{24} + \frac{12}{5}}{1 - \left(\frac{7}{24}\right)\left(\frac{12}{5}\right)}$$
$$= \frac{\frac{35}{120} + \frac{288}{120}}{1 - \frac{84}{120}} = \frac{\frac{323}{120}}{\frac{36}{120}}$$
$$= \frac{323}{120} \times \frac{120}{36} = \frac{323}{36}$$

11. a) Since $\cos(a - b) = \cos a \cos b + \sin a \sin b$,

$$\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$$
$$= (0)(\cos x) + (1)(\sin x)$$
$$= 0 + \sin x = \sin x$$

b) Since $\sin(a - b) = \sin a \cos b - \cos a \sin b$,

$$\sin\left(\frac{\pi}{2} - x\right) = \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x$$
$$= (1)(\cos x) - (0)(\sin x)$$
$$= \cos x - 0 = \cos x$$

12. a) Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$, and since $\sin(a - b) = \sin a \cos b - \cos a \sin b$, the expression $\sin(\pi + x) + \sin(\pi - x)$ can be simplified as follows:

$$\sin(\pi + x) + \sin(\pi - x)$$
$$= \sin\pi\cos x + \cos\pi\sin x + \sin\pi\cos x - \cos\pi\sin x$$
$$= 2\sin\pi\cos x = (2)(0)(\sin x) = 0$$

b) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$, and since $\sin(a + b) = \sin a \cos b + \cos a \sin b$, the expression $\cos\left(x + \frac{\pi}{3}\right) - \sin\left(x + \frac{\pi}{6}\right)$ can be simplified as follows:

$$\cos x \cos\frac{\pi}{3} - \sin x \sin\frac{\pi}{3} - \left(\sin x \cos\frac{\pi}{6} + \cos x \sin\frac{\pi}{6}\right)$$
$$= \cos x \cos\frac{\pi}{3} - \sin x \sin\frac{\pi}{3} - \sin x \cos\frac{\pi}{6} - \cos x \sin\frac{\pi}{6}$$
$$= \left(\frac{1}{2}\right)(\cos x) - \left(\frac{\sqrt{3}}{2}\right)(\sin x) - \left(\frac{\sqrt{3}}{2}\right)(\sin x) - \left(\frac{1}{2}\right)(\cos x)$$
$$= -\sqrt{3}\sin x$$

13. Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$, since $\sin(a - b) = \sin a \cos b - \cos a \sin b$, since $\cos(a + b) = \cos a \cos b - \sin a \sin b$, and since $\cos(a - b) = \cos a \cos b + \sin a \sin b$, the expression $\frac{\sin(f + g) + \sin(f - g)}{\cos(f + g) + \cos(f - g)}$ can be simplified as follows:

$$\frac{\sin(f + g) + \sin(f - g)}{\cos(f + g) + \cos(f - g)}$$
$$= \frac{\sin f \cos g + \cos f \sin g + \sin f \cos g - \cos f \sin g}{\cos f \cos g - \sin f \sin g + \cos f \cos g + \sin f \sin g}$$
$$= \frac{2\sin f \cos g}{2\cos f \cos g} = \frac{\sin f}{\cos f}$$
$$= \tan f, \cos f \neq 0, \cos g \neq 0$$

14. If $\sin a$ and $\sin b$ are written as $\frac{y}{r}$, where y is the side opposite angles a and b in a right triangle, and r is the radius of the right triangle, the side x adjacent to angles a and b can be found with the formula $x^2 + y^2 = r^2$. Once x is determined, $\cos a$ and $\cos b$ can be written as $\frac{x}{r}$, and since the terminal arms of angles a and b lie in the first quadrant, $\cos a$ and $\cos b$ are positive. With $\sin a$, $\sin b$, $\cos a$, and $\cos b$ known, $\cos(a + b)$ can be found with the formula $\cos(a + b) = \cos a \cos b - \sin a \sin b$.

Therefore, an appropriate flow chart would be as follows:

Write sin a in terms of $\frac{y}{r}$.
↓
Solve for x using the Pythagorean theorem, $x^2 + y^2 = r^2$.
↓
Since $a \in \left[0, \frac{\pi}{2}\right]$, choose the positive value of x and determine cos a.
↓
Write sin b in terms of $\frac{y}{r}$.
↓
Solve for x using the Pythagorean theorem, $x^2 + y^2 = r^2$.
↓
Since $b \in \left[0, \frac{\pi}{2}\right]$, choose the positive value of x and determine cos b.
↓
Use the formula $\cos(a + b) = \cos a \cos b - \sin a \sin b$ to evaluate $\cos(a + b)$.

15. The compound angle formulas used in this lesson are as follows:

$\sin(x + y) = \sin x \cos y + \cos x \sin y$
$\sin(x - y) = \sin x \cos y - \cos x \sin y$
$\cos(x + y) = \cos x \cos y - \sin x \sin y$
$\cos(x - y) = \cos x \cos y + \sin x \sin y$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

The two sine formulas are the same, except for the operators. The operator for $\sin(x + y)$ is $+$, while the operator for $\sin(x - y)$ is $-$. Remembering that the same operator is used on both the left and right sides in both equations will help you remember the formulas. Similarly, the two cosine formulas are the same, except for the operators. The operator for $\cos(x + y)$ is $-$, while the operator for $\cos(x - y)$ is $+$. Remembering that the operator on the left side is the opposite of the operator on the right side in both equations will help you remember the formulas. The two tangent formulas are the same, except for the operators in the numerator and the denominator on the right side. The operators for $\tan(x + y)$ are $+$ in the numerator and $-$ in the denominator, while the operators for $\tan(x - y)$ are $-$ in the numerator and $+$ in the denominator. Remembering that the operators in the numerator and the denominator are opposite in both equations, and that the operator in the numerator is the same as the operator on the left side, will help you remember the formulas.

16. Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$, and since $\cos(a - b) = \cos a \cos b + \sin a \sin b$, the formula

$\sin C + \sin D = 2 \sin\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right)$ can be developed as follows:

$$2 \sin\left(\frac{C + D}{2}\right) \cos\left(\frac{C - D}{2}\right)$$

$$= (2)\left(\left(\sin\frac{C}{2}\right)\left(\cos\frac{D}{2}\right) + \left(\cos\frac{C}{2}\right)\left(\sin\frac{D}{2}\right)\right) \times \left(\left(\cos\frac{C}{2}\right)\left(\cos\frac{D}{2}\right) + \left(\sin\frac{C}{2}\right)\left(\sin\frac{D}{2}\right)\right)$$

$$= (2)\left(\left(\sin\frac{C}{2}\right)\left(\cos\frac{C}{2}\right)\left(\cos^2\frac{D}{2}\right) + \left(\sin\frac{D}{2}\right) \times \left(\cos\frac{D}{2}\right)\left(\cos^2\frac{C}{2}\right) + \left(\sin\frac{D}{2}\right)\left(\cos\frac{D}{2}\right) \times \left(\sin^2\frac{C}{2}\right) + \left(\sin\frac{C}{2}\right)\left(\cos\frac{C}{2}\right)\left(\sin^2\frac{D}{2}\right)\right)$$

$$= 2\left(\sin\frac{C}{2}\right)\left(\cos\frac{C}{2}\right)\left(\cos^2\frac{D}{2} + \sin^2\frac{D}{2}\right) + 2\left(\sin\frac{D}{2}\right)\left(\cos\frac{D}{2}\right)\left(\cos^2\frac{C}{2} + \sin^2\frac{C}{2}\right)$$

$$= 2\left(\sin\frac{C}{2}\right)\left(\cos\frac{C}{2}\right) + 2\left(\sin\frac{D}{2}\right)\left(\cos\frac{D}{2}\right)$$

$$= \sin\left(2\left(\frac{C}{2}\right)\right) + \sin\left(2\left(\frac{D}{2}\right)\right)$$

$$= \sin C + \sin D$$

17. Since $\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$, $\cot(x + y)$ in terms of $\cot x$ and $\cot y$ can be determined as follows:

$\cot(x + y)$

$$= \frac{1}{\tan(x + y)} = \frac{1}{\frac{\tan x + \tan y}{1 - \tan x \tan y}} = \frac{1 - \tan x \tan y}{\tan x + \tan y}$$

$$= \frac{1 - \frac{1}{\cot x \cot y}}{\frac{1}{\cot x} + \frac{1}{\cot y}} = \frac{\frac{\cot x \cot y}{\cot x \cot y} - \frac{1}{\cot x \cot y}}{\frac{\cot y}{\cot x \cot y} + \frac{\cot x}{\cot x \cot y}}$$

$$= \frac{\frac{\cot x \cot y - 1}{\cot x \cot y}}{\frac{\cot x + \cot y}{\cot x \cot y}} = \frac{\cot x \cot y - 1}{\cot x \cot y} \times \frac{\cot x \cot y}{\cot x + \cot y}$$

$$= \frac{\cot x \cot y - 1}{\cot x + \cot y}$$

18. Let $C = x + y$ and let $D = x - y$.

$\cos C + \cos D = \cos(x + y) + \cos(x - y)$
$= \cos x \cos y - \sin x \sin y$
$+ \cos x \cos y + \sin x \sin y = 2 \cos x \cos y$

$$\frac{C + D}{2} = \frac{x + y + x - y}{2} = x$$

$$\frac{C - D}{2} = \frac{x + y - x + y}{2} = y$$

So, $\cos C + \cos D = 2 \cos\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right)$.

19. Let $C = x + y$ and let $D = x - y$.

$\cos C - \cos D = \cos(x + y) - \cos(x - y)$
$= \cos x \cos y - \sin x \sin y$
$- (\cos x \cos y - \sin x \sin y) = -2 \sin x \sin y$

$$\frac{C + D}{2} = \frac{x + y + x - y}{2} = x$$

$$\frac{C - D}{2} = \frac{x + y - x + y}{2} = y$$

So, $\cos C - \cos D = -2 \sin\left(\frac{C + D}{2}\right)\sin\left(\frac{C - D}{2}\right)$.

7.3 Double Angle Formulas, pp. 407–408

1. a) Since $\sin 2\theta = 2 \sin\theta \cos\theta$,
$2 \sin 5x \cos 5x = \sin 2(5x) = \sin 10x$.

b) Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,
$\cos^2\theta - \sin^2\theta = \cos 2\theta$.

c) Since $\cos 2\theta = 1 - 2\sin^2\theta$,
$1 - 2\sin^2 3x = \cos 2(3x) = \cos 6x$.

d) Since $\tan 2\theta = \frac{2 \tan\theta}{1 - \tan^2\theta}$,

$$\frac{2 \tan 4x}{1 - \tan^2 4x} = \tan 2(4x) = \tan 8x.$$

e) Since $\sin 2\theta = 2 \sin\theta \cos\theta$,
$4 \sin\theta \cos\theta = (2)(2 \sin\theta \cos\theta) = 2 \sin 2\theta$.

f) Since $\cos 2\theta = 2\cos^2\theta - 1$,

$$2\cos^2\frac{\theta}{2} - 1 = \cos 2\left(\frac{\theta}{2}\right) = \cos\theta.$$

2. a) Since $\sin 2\theta = 2 \sin\theta \cos\theta$,
$2 \sin 45° \cos 45° = \sin 2(45°) = \sin 90° = 1$

b) Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,

$$\cos^2 30° - \sin^2 30° = \cos 2(30°) = \cos 60° = \frac{1}{2}$$

c) Since $\sin 2\theta = 2 \sin\theta \cos\theta$,

$$2 \sin\frac{\pi}{12} \cos\frac{\pi}{12} = \sin 2\left(\frac{\pi}{12}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$$

d) Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,

$$\cos^2\frac{\pi}{12} - \sin^2\frac{\pi}{12} = \cos 2\left(\frac{\pi}{12}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

e) Since $\cos 2\theta = 1 - 2\sin^2\theta$,

$$1 - 2\sin^2\frac{3\pi}{8} = \cos 2\left(\frac{3\pi}{8}\right) = \cos\frac{6\pi}{8}$$

$$= \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$$

f) Since $\sin 2\theta = 2 \sin\theta \cos\theta$,

$$2 \tan 60° \cos^2 60° = (2)\left(\frac{\sin 60°}{\cos 60°}\right)(\cos^2 60°)$$

$$= 2 \sin 60° \cos 60° = \sin 2(60°)$$

$$= \sin 120° = \frac{\sqrt{3}}{2}$$

3. a) Since $\sin 2\theta = 2 \sin\theta \cos\theta$,
$\sin 4\theta = 2 \sin 2\theta \cos 2\theta$

b) Since $\cos 2\theta = 2\cos^2\theta - 1$,
$\cos 3x = 2\cos^2(1.5x) - 1$

c) Since $\tan 2\theta = \frac{2 \tan\theta}{1 - \tan^2\theta}$,

$$\tan x = \frac{2 \tan(0.5x)}{1 - \tan^2(0.5x)}$$

d) Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,
$\cos 6\theta = \cos^2 3\theta - \sin^2 3\theta$

e) Since $\sin 2\theta = 2 \sin\theta \cos\theta$,
$\sin x = 2 \sin(0.5x) \cos(0.5x)$

f) Since $\tan 2\theta = \frac{2 \tan\theta}{1 - \tan^2\theta}$,

$$\tan 5\theta = \frac{2 \tan(2.5\theta)}{1 - \tan^2(2.5\theta)}$$

4. Since $\cos\theta = \frac{3}{5}$, the leg adjacent to the angle θ in a right triangle has a length of 3, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$3^2 + y^2 = 5^2$$
$$9 + y^2 = 25$$
$$9 + y^2 - 9 = 25 - 9$$

$y^2 = 16$
$y = 4$, in quadrant I

Since $\sin\theta = \frac{\text{opposite leg}}{\text{hypotenuse}}$, $\sin\theta = \frac{4}{5}$.

Therefore, since $\sin 2\theta = 2\sin\theta\cos\theta$,

$$\sin 2\theta = (2)\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{24}{25}.$$

Also, since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,

$$\cos 2\theta = \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}.$$

Finally, since $\tan\theta = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan\theta = \frac{4}{3}$.

Since $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$,

$$\tan 2\theta = \frac{(2)(\frac{4}{3})}{1-(\frac{4}{3})^2} = \frac{\frac{8}{3}}{1-\frac{16}{9}} = \frac{\frac{8}{3}}{\frac{9}{9}-\frac{16}{9}} = \frac{\frac{8}{3}}{-\frac{7}{9}} = \frac{8}{3} \times -\frac{9}{7} = -\frac{24}{7}$$

5. Since $\tan\theta = -\frac{7}{24}$, the leg opposite the angle θ in a right triangle has a length of 7, while the leg adjacent to the angle θ has a length of 24. For this reason, the hypotenuse of the right triangle can be calculated as follows:

$7^2 + 24^2 = c^2$
$49 + 576 = c^2$
$625 = c^2$
$c = 25$, in quadrant II

Since $\sin\theta = \frac{\text{opposite leg}}{\text{hypotenuse}}$, $\sin\theta = \frac{7}{25}$, and since $\cos\theta = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos\theta = -\frac{24}{25}$. (The reason the sign is negative is because angle θ is in the second quadrant.) Therefore, since $\sin 2\theta = 2\sin\theta\cos\theta$, $\sin 2\theta = (2)\left(\frac{7}{25}\right)\left(-\frac{24}{25}\right) = -\frac{336}{625}$. Also, since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,

$$\cos 2\theta = \left(-\frac{24}{25}\right)^2 - \left(\frac{7}{25}\right)^2 = \frac{576}{625} - \frac{49}{625} = \frac{527}{625}$$

Finally, since $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$,

$$\tan 2\theta = \frac{(2)(-\frac{7}{24})}{1-(-\frac{7}{24})^2} = \frac{-\frac{7}{12}}{1-\frac{49}{576}} = \frac{-\frac{7}{12}}{\frac{576}{576}-\frac{49}{576}} = \frac{-\frac{7}{12}}{\frac{527}{576}} = -\frac{7}{12} \times \frac{576}{527} = -\frac{336}{527}$$

6. Since $\sin\theta = -\frac{12}{13}$, the leg opposite the angle θ in a right triangle has a length of 12, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$x^2 + 12^2 = 13^2$
$x^2 + 144 = 169$
$x^2 + 144 - 144 = 169 - 144$
$x^2 = 25$
$x = 5$, in quadrant IV

Since $\cos\theta = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos\theta = \frac{5}{13}$. Therefore, since $\sin 2\theta = 2\sin\theta\cos\theta$, $\sin 2\theta = (2)\left(-\frac{12}{13}\right)\left(\frac{5}{13}\right) = -\frac{120}{169}$. Also, since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,

$\cos 2\theta = \left(\frac{5}{13}\right)^2 - \left(-\frac{12}{13}\right)^2 = \frac{25}{169} - \frac{144}{169} = -\frac{119}{169}$. Finally, since $\tan\theta = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan\theta = -\frac{12}{5}$. (The reason the sign is negative is because angle θ is in the fourth quadrant.) Since

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$,

$$\tan 2\theta = \frac{(2)(-\frac{12}{5})}{1-(-\frac{12}{5})^2} = \frac{-\frac{24}{5}}{1-\frac{144}{25}} = \frac{-\frac{24}{5}}{\frac{25}{25}-\frac{144}{25}} = \frac{-\frac{24}{5}}{-\frac{119}{25}} = \frac{-\frac{24}{5}}{-\frac{119}{25}} = -\frac{24}{5} \times -\frac{25}{119} = \frac{120}{119}$$

7. Since $\cos\theta = -\frac{4}{5}$, the leg adjacent to the angle θ in a right triangle has a length of 4, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$4^2 + y^2 = 5^2$
$16 + y^2 = 25$
$16 + y^2 - 16 = 25 - 16$
$y^2 = 9$
$y = 3$, in quadrant II

Since $\sin\theta = \frac{\text{opposite leg}}{\text{hypotenuse}}$, $\sin\theta = \frac{3}{5}$.

Therefore, since $\sin 2\theta = 2\sin\theta\cos\theta$,

$\sin 2\theta = (2)\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) = -\frac{24}{25}$.

Also, since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,

$$\cos 2\theta = \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2$$
$$= \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$$

Finally, since $\tan\theta = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan\theta = -\frac{3}{4}$. (The reason the sign is negative is because angle θ is in the second quadrant.) Since $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$,

$$\tan 2\theta = \frac{(2)(-\frac{3}{4})}{1 - (-\frac{3}{4})^2} = \frac{-\frac{3}{2}}{1 - \frac{9}{16}} = \frac{-\frac{3}{2}}{\frac{16}{16} - \frac{9}{16}}$$
$$= \frac{-\frac{3}{2}}{\frac{7}{16}} = -\frac{3}{2} \times \frac{16}{7} = -\frac{24}{7}$$

8. To determine the value of a, first rearrange the equation as follows:

$2\tan x - \tan 2x + 2a = 1 - \tan 2x \tan^2 x$
$2\tan x - \tan 2x + 2a + \tan 2x$
$= 1 - \tan 2x \tan^2 x + \tan 2x$
$2\tan x + 2a = 1 - \tan 2x \tan^2 x + \tan 2x$
$2\tan x + 2a - 1 = 1 - \tan 2x \tan^2 x + \tan 2x - 1$
$2\tan x + 2a - 1 = \tan 2x - \tan 2x \tan^2 x$
$2\tan x + 2a - 1 = (\tan 2x)(1 - \tan^2 x)$
$$\frac{2\tan x + 2a - 1}{1 - \tan^2 x} = \frac{(\tan 2x)(1 - \tan^2 x)}{1 - \tan^2 x}$$
$$\tan 2x = \frac{2\tan x + 2a - 1}{1 - \tan^2 x}$$

Since $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$, the value of $2a - 1$ must equal 0. Therefore, a can be solved for as follows:

$2a - 1 = 0$
$2a - 1 + 1 = 0 + 1$
$2a = 1$
$$\frac{2a}{2} = \frac{1}{2}$$
$$a = \frac{1}{2}$$

9. Jim can find the sine of $\frac{\pi}{8}$ by using the formula $\cos 2x = 1 - 2\sin^2 x$ and isolating $\sin x$ on one side of the equation as follows:

$\cos 2x = 1 - 2\sin^2 x$
$\cos 2x + 2\sin^2 x = 1 - 2\sin^2 x + 2\sin^2 x$
$\cos 2x + 2\sin^2 x = 1$
$\cos 2x + 2\sin^2 x - \cos 2x = 1 - \cos 2x$
$2\sin^2 x = 1 - \cos 2x$
$$\frac{2\sin^2 x}{2} = \frac{1 - \cos 2x}{2}$$
$$\sin^2 x = \frac{1 - \cos 2x}{2}$$
$$\sin x = \pm\sqrt{\frac{1 - \cos 2x}{2}}$$

The cosine of $\frac{\pi}{4}$ is $\frac{\sqrt{2}}{2}$, so

$$\sin\frac{\pi}{8} = \pm\sqrt{\frac{1 - \cos((2)(\frac{\pi}{8}))}{2}}$$
$$= \pm\sqrt{\frac{1 - \cos\frac{\pi}{4}}{2}} = \pm\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}}$$
$$= \pm\sqrt{\frac{\frac{2}{2} - \frac{\sqrt{2}}{2}}{2}} = \pm\sqrt{\frac{\frac{2 - \sqrt{2}}{2}}{2}}$$
$$= \pm\sqrt{\frac{2 - \sqrt{2}}{2} \times \frac{1}{2}}$$
$$= \pm\sqrt{\frac{2 - \sqrt{2}}{4}} = \pm\frac{\sqrt{2 - \sqrt{2}}}{2}$$

Since $\frac{\pi}{8}$ is in the first quadrant, the sign of $\sin\frac{\pi}{8}$ is positive. Therefore, $\sin\frac{\pi}{8} = \frac{\sqrt{2 - \sqrt{2}}}{2}$.

10. Marion can find the cosine of $\frac{\pi}{12}$ by using the formula $\cos 2x = 2\cos^2 x - 1$ and isolating $\cos x$ on one side of the equation as follows:

$\cos 2x = 2\cos^2 x - 1$
$\cos 2x + 1 = 2\cos^2 x - 1 + 1$
$\cos 2x + 1 = 2\cos^2 x$
$$\frac{\cos 2x + 1}{2} = \frac{2\cos^2 x}{2}$$
$$\cos^2 x = \frac{1 + \cos 2x}{2}$$
$$\cos x = \pm\sqrt{\frac{1 + \cos 2x}{2}}$$

The cosine of $\frac{\pi}{6}$ is $\frac{\sqrt{3}}{2}$, so

$$\cos\frac{\pi}{12} = \pm\sqrt{\frac{1 + \cos((2)(\frac{\pi}{12}))}{2}}$$
$$= \pm\sqrt{\frac{1 + \cos\frac{\pi}{6}}{2}} = \pm\sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}}$$
$$= \pm\sqrt{\frac{\frac{2}{2} + \frac{\sqrt{3}}{2}}{2}} = \pm\sqrt{\frac{\frac{2 + \sqrt{3}}{2}}{2}}$$
$$= \pm\sqrt{\frac{2 + \sqrt{3}}{2} \times \frac{1}{2}}$$
$$= \pm\sqrt{\frac{2 + \sqrt{3}}{4}} = \pm\frac{\sqrt{2 + \sqrt{3}}}{2}$$

Since $\frac{\pi}{12}$ is in the first quadrant, the sign of $\cos \frac{\pi}{12}$ is positive. Therefore, $\cos \frac{\pi}{12} = \frac{\sqrt{2 + \sqrt{3}}}{2}$.

11. a) Since $\sin 2\theta = 2 \sin \theta \cos \theta$,

$\sin 4x = 2 \sin 2x \cos 2x$

$= (2)(2 \sin x \cos x)(\cos 2x)$. At this point, either the formula $\cos 2\theta = 2\cos^2 \theta - 1$, $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$, or $\cos 2\theta = 1 - 2\sin^2 \theta$ can be used to simplify for $\cos 2x$. If the formula $\cos 2\theta = 2\cos^2 \theta - 1$ is used, the formula for $\sin 4x$ can be developed as follows:

$\sin 4x = (2)(2 \sin x \cos x)(\cos 2x)$

$= (2)(2 \sin x \cos x)(2\cos^2 x - 1)$

$= (4 \sin x \cos x)(2\cos^2 x - 1)$

$= 8\cos^3 x \sin x - 4 \sin x \cos x$

If the formula $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ is used, the formula for $\sin 4x$ can be developed as follows:

$\sin 4x = (2)(2 \sin x \cos x)(\cos 2x)$

$= (2)(2 \sin x \cos x)(\cos^2 x - \sin^2 x)$

$= (4 \sin x \cos x)(\cos^2 x - \sin^2 x)$

$= 4\cos^3 x \sin x - 4\sin^3 x \cos x$

If the formula $\cos 2\theta = 1 - 2\sin^2 \theta$ is used, the formula for $\sin 4x$ can be developed as follows:

$\sin 4x = (2)(2 \sin x \cos x)(\cos 2x)$

$= (2)(2 \sin x \cos x)(1 - 2\sin^2 x)$

$= (4 \sin x \cos x)(1 - 2\sin^2 x)$

$= 4 \sin x \cos x - 8\sin^3 x \cos x$

b) The value of $\sin \frac{2\pi}{3}$ is $\frac{\sqrt{3}}{2}$. Using the formula $\sin 4x = 4 \sin x \cos x - 8\sin^3 x \cos x$, $\sin \frac{2\pi}{3} = \sin \frac{8\pi}{3}$ can be verified as follows:

$\sin 4x = 4 \sin x \cos x - 8\sin^3 x \cos x$

$\sin 4\left(\frac{2\pi}{3}\right) = 4 \sin \frac{2\pi}{3} \cos \frac{2\pi}{3} - 8\sin^3 \frac{2\pi}{3} \cos \frac{2\pi}{3}$

$\sin \frac{8\pi}{3} = (4)\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) - (8)\left(\frac{\sqrt{3}}{2}\right)^3\left(-\frac{1}{2}\right)$

$\sin \frac{8\pi}{3} = -\frac{4\sqrt{3}}{4} - (-4)\left(\frac{3\sqrt{3}}{8}\right)$

$\sin \frac{8\pi}{3} = -\frac{4\sqrt{3}}{4} - \left(-\frac{3\sqrt{3}}{2}\right)$

$\sin \frac{8\pi}{3} = -\frac{4\sqrt{3}}{4} - \left(-\frac{6\sqrt{3}}{4}\right)$

$\sin \frac{8\pi}{3} = -\frac{4\sqrt{3}}{4} + \frac{6\sqrt{3}}{4}$

$\sin \frac{8\pi}{3} = \frac{2\sqrt{3}}{4}$

$\sin \frac{8\pi}{3} = \frac{\sqrt{3}}{2}$

12. a) Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$, $\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta \cos \theta + \cos 2\theta \sin \theta$. Since $\sin 2\theta = 2 \sin \theta \cos \theta$,

$\sin 3\theta = (2 \sin \theta \cos \theta)(\cos \theta) + \cos 2\theta \sin \theta$

$= 2\cos^2 \theta \sin \theta + \cos 2\theta \sin \theta$. At this point, either the formula $\cos 2\theta = 2\cos^2 \theta - 1$, $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$, or $\cos 2\theta = 1 - 2\sin^2 \theta$ can be used to simplify for $\cos 2\theta$. If the formula $\cos 2\theta = 2\cos^2 \theta - 1$ is used, the formula for $\sin 3\theta$ can be developed as follows:

$\sin 3\theta = 2\cos^2 \theta \sin \theta + \cos 2\theta \sin \theta$

$\sin 3\theta = 2\cos^2 \theta \sin \theta + (2\cos^2 \theta - 1)(\sin \theta)$

$\sin 3\theta = 2\cos^2 \theta \sin \theta + 2\cos^2 \theta \sin \theta - \sin \theta$

$\sin 3\theta = 4\cos^2 \theta \sin \theta - \sin \theta$

If the formula $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ is used, the formula for $\sin 3\theta$ can be developed as follows:

$\sin 3\theta = 2\cos^2 \theta \sin \theta + \cos 2\theta \sin \theta$

$\sin 3\theta = 2\cos^2 \theta \sin \theta + (\cos^2 \theta - \sin^2 \theta)(\sin \theta)$

$\sin 3\theta = 2\cos^2 \theta \sin \theta + \cos^2 \theta \sin \theta - \sin^3 \theta$

$\sin 3\theta = 3\cos^2 \theta \sin \theta - \sin^3 \theta$

If the formula $\cos 2\theta = 1 - 2\sin^2 \theta$ is used, the formula for $\sin 3\theta$ can be developed as follows:

$\sin 3\theta = 2\cos^2 \theta \sin \theta + \cos 2\theta \sin \theta$

$\sin 3\theta = 2\cos^2 \theta \sin \theta + (1 - 2\sin^2 \theta)(\sin \theta)$

$\sin 3\theta = 2\cos^2 \theta \sin \theta + \sin \theta - 2\sin^3 \theta$

b) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$, $\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos \theta - \sin \theta \sin 2\theta$. Since $\sin 2\theta = 2 \sin \theta \cos \theta$,

$\cos 3\theta = \cos 2\theta \cos \theta - (\sin \theta)(2 \sin \theta \cos \theta)$

$= \cos 2\theta \cos \theta - 2\sin^2 \theta \cos \theta$. At this point, either the formula $\cos 2\theta = 2\cos^2 \theta - 1$, $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$, or $\cos 2\theta = 1 - 2\sin^2 \theta$ can be used to simplify for $\cos 2\theta$. If the formula $\cos 2\theta = 2\cos^2 \theta - 1$ is used, the formula for $\cos 3\theta$ can be developed as follows:

$\cos 3\theta = \cos 2\theta \cos \theta - 2\sin^2 \theta \cos \theta$

$\cos 3\theta = (2\cos^2 \theta - 1)(\cos \theta) - 2\sin^2 \theta \cos \theta$

$\cos 3\theta = 2\cos^3 \theta - \cos \theta - 2\sin^2 \theta \cos \theta$

If the formula $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ is used, the formula for $\cos 3\theta$ can be developed as follows:

$\cos 3\theta = \cos 2\theta \cos \theta - 2\sin^2 \theta \cos \theta$

$\cos 3\theta = (\cos^2 \theta - \sin^2 \theta)(\cos \theta) - 2\sin^2 \theta \cos \theta$

$\cos 3\theta = \cos^3 \theta - \sin^2 \theta \cos \theta - 2\sin^2 \theta \cos \theta$

$\cos 3\theta = \cos^3 \theta - 3\sin^2 \theta \cos \theta$

If the formula $\cos 2\theta = 1 - 2\sin^2\theta$ is used, the formula for $\cos 3\theta$ can be developed as follows:

$\cos 3\theta = \cos 2\theta \cos\theta - 2\sin^2\theta\cos\theta$

$\cos 3\theta = (1 - 2\sin^2\theta)(\cos\theta) - 2\sin^2\theta\cos\theta$

$\cos 3\theta = \cos\theta - 2\sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta$

$\cos 3\theta = \cos\theta - 4\sin^2\theta\cos\theta$

c) Since $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\tan 3\theta = \tan(2\theta + \theta) = \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\tan\theta}$$

Since $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$,

$$\tan 3\theta = \frac{\dfrac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)(\tan\theta)}$$

$$= \frac{\dfrac{2\tan\theta}{1-\tan^2\theta} + \dfrac{\tan\theta - \tan^3\theta}{1-\tan^2\theta}}{1 - \dfrac{2\tan^2\theta}{1-\tan^2\theta}}$$

$$= \frac{\dfrac{2\tan\theta}{1-\tan^2\theta} + \dfrac{\tan\theta - \tan^3\theta}{1-\tan^2\theta}}{\dfrac{1-\tan^2\theta}{1-\tan^2\theta} - \dfrac{2\tan^2\theta}{1-\tan^2\theta}}$$

$$= \frac{\dfrac{2\tan\theta + \tan\theta - \tan^3\theta}{1-\tan^2\theta}}{\dfrac{1-\tan^2\theta - 2\tan^2\theta}{1-\tan^2\theta}}$$

$$= \frac{\dfrac{3\tan\theta - \tan^3\theta}{1-\tan^2\theta}}{\dfrac{1-3\tan^2\theta}{1-\tan^2\theta}}$$

$$= \frac{3\tan\theta - \tan^3\theta}{1-\tan^2\theta} \times \frac{1-\tan^2\theta}{1-3\tan^2\theta}$$

$$= \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

13. a) Since $\sin^2 x = \frac{8}{9}$, and since the angle x is in the second quadrant, $\sin x = \sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{\sqrt{9}} = \frac{2\sqrt{2}}{3}$.

Since $\sin x = \frac{2\sqrt{2}}{3}$, the leg opposite the angle x in a right triangle has a length of $2\sqrt{2}$, while the hypotenuse of the right triangle has a length of 3. For this reason, the other leg of the right triangle can be calculated as follows:

$x^2 + (2\sqrt{2})^2 = 3^2$

$x^2 + 8 = 9$

$x^2 + 8 - 8 = 9 - 8$

$x^2 = 1$

$x = -1$, in quadrant II

Since $\cos x = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, and since the angle x is in the second quadrant, $\cos x = -\frac{1}{3}$.

Since $\sin 2\theta = 2\sin\theta\cos\theta$,

$$\sin 2x = 2\sin x\cos x = (2)\left(\frac{2\sqrt{2}}{3}\right)\left(-\frac{1}{3}\right) = -\frac{4\sqrt{2}}{9}$$

b) Since $\sin^2 x = \frac{8}{9}$, and since the angle x is in the second quadrant, $\sin x = \sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{\sqrt{9}} = \frac{2\sqrt{2}}{3}$.

Since $\sin x = \frac{2\sqrt{2}}{3}$, the leg opposite the angle x in a right triangle has a length of $2\sqrt{2}$, while the hypotenuse of the right triangle has a length of 3. For this reason, the other leg of the right triangle can be calculated as follows:

$x^2 + (2\sqrt{2})^2 = 3^2$

$x^2 + 8 = 9$

$x^2 + 8 - 8 = 9 - 8$

$x^2 = 1$

$x = -1$, in quadrant II

Since $\cos x = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, and since the angle x is in the second quadrant, $\cos x = -\frac{1}{3}$. Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= \left(-\frac{1}{3}\right)^2 - \left(\frac{2\sqrt{2}}{3}\right)^2$$

$$= \frac{1}{9} - \frac{8}{9} = -\frac{7}{9}$$

(The formulas $\cos 2\theta = 2\cos^2\theta - 1$ and $\cos 2\theta = 1 - 2\sin^2\theta$ could also have been used.)

c) Since $\sin^2 x = \frac{8}{9}$, and since the angle x is in the second quadrant, $\sin x = \sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{\sqrt{9}} = \frac{2\sqrt{2}}{3}$.

Since $\sin x = \frac{2\sqrt{2}}{3}$, the leg opposite the angle x in a right triangle has a length of $2\sqrt{2}$, while the hypotenuse of the right triangle has a length of 3. For this reason, the other leg of the right triangle can be calculated as follows:

$x^2 + (2\sqrt{2})^2 = 3^2$

$x^2 + 8 = 9$

$x^2 + 8 - 8 = 9 - 8$

$x^2 = 1$

$x = -1$, in quadrant II

Since $\cos x = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, and since the angle x is in the second quadrant, $\cos x = -\frac{1}{3}$. Since $\cos 2\theta = 2\cos^2\theta - 1$, $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$, and since $\cos x = -\frac{1}{3}$, $-\frac{1}{3} = 2\cos^2\frac{\theta}{2} - 1$. The value of $\cos\frac{\theta}{2}$ can now be determined as follows:

$$-\frac{1}{3} = 2\cos^2\frac{\theta}{2} - 1$$
$$-\frac{1}{3} + 1 = 2\cos^2\frac{\theta}{2} - 1 + 1$$
$$\frac{2}{3} = 2\cos^2\frac{\theta}{2}$$
$$\frac{\frac{2}{3}}{2} = \frac{2\cos^2\frac{\theta}{2}}{2}$$
$$\cos^2\frac{\theta}{2} = \frac{2}{3} \times \frac{1}{2}$$
$$\cos^2\frac{\theta}{2} = \frac{2}{6} = \frac{1}{3}$$
$$\cos\frac{\theta}{2} = \sqrt{\frac{1}{3}} = \frac{\sqrt{1}}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

d) Since $\sin^2 x = \frac{8}{9}$, and since the angle x is in the second quadrant, $\sin x = \sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{\sqrt{9}} = \frac{2\sqrt{2}}{3}$. Since $\sin x = \frac{2\sqrt{2}}{3}$, the leg opposite the angle x in a right triangle has a length of $2\sqrt{2}$, while the hypotenuse of the right triangle has a length of 3. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + (2\sqrt{2})^2 = 3^2$$
$$x^2 + 8 = 9$$
$$x^2 + 8 - 8 = 9 - 8$$
$$x^2 = 1$$
$$x = -1, \text{ in quadrant II}$$

Since $\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, and since the angle x is in the second quadrant, $\cos x = -\frac{1}{3}$. Since $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$,

$$\sin 3x = 3\cos^2 x\sin x - \sin^3 x$$
$$= (3)\left(-\frac{1}{3}\right)^2\left(\frac{2\sqrt{2}}{3}\right) - \left(\frac{2\sqrt{2}}{3}\right)^3$$
$$= (3)\left(\frac{1}{9}\right)\left(\frac{2\sqrt{2}}{3}\right) - \frac{16\sqrt{2}}{27}$$
$$= \frac{6\sqrt{2}}{27} - \frac{16\sqrt{2}}{27}$$
$$= -\frac{10\sqrt{2}}{27}$$

14. If $\sin a$ is written as $\frac{y}{r}$, where y is the side opposite angle a in a right triangle, and r is the radius of the right triangle, the side x adjacent to angle a can be found with the formula $x^2 + y^2 = r^2$. Once x is determined, $\cos a$ can be written as $\frac{x}{r}$, and since the terminal arm of angle a lies in the second quadrant, $\cos a$ is negative. With $\sin a$ and $\cos a$ known, $\sin 2a$ can be found with the formula $\sin 2\theta = 2\sin\theta\cos\theta$, Therefore, an appropriate flow chart would be as follows:

Write $\sin a$ in terms of $\frac{y}{r}$.

↓

Solve for x using the Pythagorean theorem, $x^2 + y^2 = r^2$.

↓

Choose the negative value of x since $a \in \left[\frac{\pi}{2}, \pi\right]$, and determine $\cos a$.

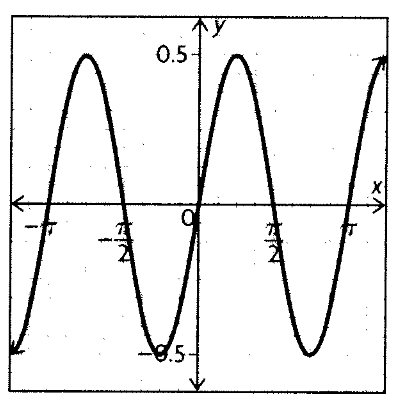
↓

Write $\cos a$ in terms of $\frac{x}{r}$.

↓

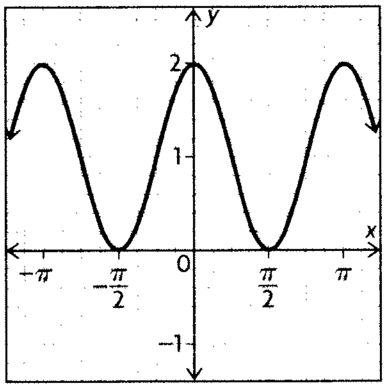
Use the formula $\sin 2a = 2\sin a\cos a$ to evaluate $\sin 2a$.

15. a) Since $\sin 2\theta = 2\sin\theta\cos\theta$, $\sin 2x = 2\sin x\cos x$. For this reason, $\frac{\sin 2x}{2} = \frac{2\sin x\cos x}{2} = \sin x\cos x$. Therefore, the graph of $f(x) = \sin x\cos x$ is the same as that of $f(x) = \frac{\sin 2x}{2}$. The graph of $f(x) = \frac{\sin 2x}{2}$ can be obtained by vertically compressing $f(x) = \sin x$ by a factor of $\frac{1}{2}$ and horizontally compressing it by a factor of $\frac{1}{2}$. The graph is shown below:



b) Since $\cos 2\theta = 2\cos^2\theta - 1$, $\cos 2x = 2\cos^2 x - 1$. For this reason, $\cos 2x + 1 = 2\cos^2 x - 1 + 1 = 2\cos^2 x$.

Therefore, the graph of $f(x) = 2\cos^2 x$ is the same as that of $f(x) = \cos 2x + 1$. The graph of $f(x) = \cos 2x + 1$ can be obtained by horizontally compressing $f(x) = \cos x$ by a factor of $\frac{1}{2}$ and vertically translating it 1 unit up. The graph is shown below:

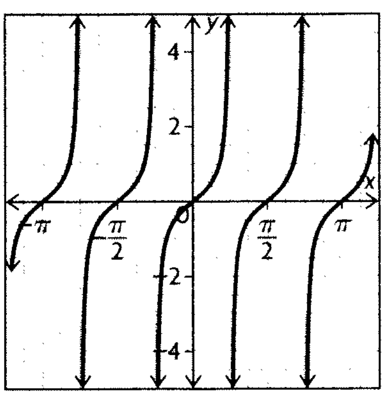


c) Since $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$, $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$.

For this reason, $\dfrac{\tan 2x}{2} = \dfrac{\frac{2\tan x}{1 - \tan^2 x}}{2}$

$= \dfrac{2\tan x}{1 - \tan^2 x} \times \dfrac{1}{2} = \dfrac{\tan x}{1 - \tan^2 x}$. Therefore, the graph of $f(x) = \dfrac{\tan x}{1 - \tan^2 x}$ is the same as that of $f(x) = \dfrac{\tan 2x}{2}$.

The graph of $f(x) = \dfrac{\tan 2x}{2}$ can be obtained by vertically compressing $f(x) = \tan x$ by a factor of $\frac{1}{2}$ and horizontally compressing it by a factor of $\frac{1}{2}$. The graph is shown below:



16. a) To eliminate A from the equations $x = \tan 2A$ and $y = \tan A$ to find an equation that relates x to y, first take the $\tan^{-1}$ of both sides of both equations and solve for A as follows:

$$x = \tan 2A$$
$$\tan^{-1} x = \tan^{-1}(\tan 2A)$$
$$\tan^{-1} x = 2A$$
$$A = \frac{\tan^{-1} x}{2}$$
$$y = \tan A$$
$$\tan^{-1} y = \tan^{-1}(\tan A)$$
$$A = \tan^{-1} y$$

Since $A = \dfrac{\tan^{-1} x}{2}$ and $A = \tan^{-1} y$,

$$\frac{\tan^{-1} x}{2} = \tan^{-1} y$$

b) To eliminate A from the equations $x = \cos 2A$ and $y = \cos A$ to find an equation that relates x to y, first take the $\cos^{-1}$ of both sides of both equations and solve for A as follows:

$$x = \cos 2A$$
$$\cos^{-1} x = \cos^{-1}(\cos 2A)$$
$$\cos^{-1} x = 2A$$
$$A = \frac{\cos^{-1} x}{2}$$
$$y = \cos A$$
$$\cos^{-1} y = \cos^{-1}(\cos A)$$
$$A = \cos^{-1} y$$

Since $A = \dfrac{\cos^{-1} x}{2}$ and $A = \cos^{-1} y$,

$$\frac{\cos^{-1} x}{2} = \cos^{-1} y$$

c) To eliminate A from the equations $x = \cos 2A$ and $y = \csc A$ to find an equation that relates x to y, first take the $\cos^{-1}$ of both sides of the first equation and the $\csc^{-1}$ of both sides of the second equation and solve for A as follows:

$$x = \cos 2A$$
$$\cos^{-1} x = \cos^{-1}(\cos 2A)$$
$$\cos^{-1} x = 2A$$
$$A = \frac{\cos^{-1} x}{2}$$
$$A = \csc A$$
$$\csc^{-1} y = \csc^{-1}(\csc A)$$
$$A = \csc^{-1} y$$

Since $A = \dfrac{\cos^{-1} x}{2}$ and $A = \cos^{-1} y$,

$$\frac{\cos^{-1} x}{2} = \csc^{-1} y, \text{ or } \frac{\cos^{-1} x}{2} = \sin^{-1}\left(\frac{1}{y}\right)$$

d) To eliminate A from the equations $x = \sin 2A$ and $y = \sec 4A$ to find an equation that relates x to y, first take the $\sin^{-1}$ of both sides of the first equation and the $\sec^{-1}$ of both sides of the second equation and solve for A as follows:

$$x = \sin 2A$$
$$\sin^{-1} x = \sin^{-1}(\sin 2A)$$
$$\sin^{-1} x = 2A$$
$$A = \frac{\sin^{-1} x}{2}$$
$$y = \sec 4A$$
$$\sec^{-1} y = \sec^{-1}(\sec 4A)$$
$$\sec^{-1} y = 4A$$
$$A = \frac{\sec^{-1} y}{4}$$

Since $A = \frac{\sin^{-1} x}{2}$ and $A = \frac{\sec^{-1} y}{4}$,

$$\frac{\sin^{-1} x}{2} = \frac{\sec^{-1} y}{4}, \text{ or } \frac{\sin^{-1} x}{2} = \frac{\cos^{-1}\left(\frac{1}{y}\right)}{4}$$

17. a) Since $\cos 2\theta = 1 - 2\sin^2\theta$, the equation $\cos 2x = \sin x$ can be rewritten $1 - 2\sin^2 x = \sin x$. This equation can be solved as follows:

$$1 - 2\sin^2 x = \sin x$$
$$1 - 2\sin^2 x + 2\sin^2 x - 1 = \sin x + 2\sin^2 x - 1$$
$$2\sin^2 x + \sin x - 1 = 0$$
$$(2\sin x - 1)(\sin x + 1) = 0$$
$$2\sin x - 1 = 0$$
$$2\sin x - 1 + 1 = 0 + 1$$
$$2\sin x = 1$$
$$\frac{2\sin x}{2} = \frac{1}{2}$$
$$\sin x = \frac{1}{2}$$
$$x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$
$$\text{or } \sin x + 1 = 0$$
$$\sin x + 1 - 1 = 0 - 1$$
$$\sin x = -1$$
$$x = \frac{3\pi}{2}$$

Therefore, $x = \frac{\pi}{6}, \frac{5\pi}{6}$, or $\frac{3\pi}{2}$.

b) Since $\sin 2\theta = 2\sin\theta\cos\theta$, and since $\cos 2\theta = 2\cos^2\theta - 1$, the equation $\sin 2x - 1 = \cos 2x$ can be rewritten $2\sin x\cos x - 1 = 2\cos^2 x - 1$. This equation can be solved as follows:

$$2\sin x\cos x - 1 = 2\cos^2 x - 1$$
$$2\sin x\cos x - 1 + 1 = 2\cos^2 x - 1 + 1$$
$$2\sin x\cos x = 2\cos^2 x$$
$$2\sin x\cos x - 2\sin x\cos x$$
$$= 2\cos^2 x - 2\sin x\cos x$$
$$2\cos^2 x - 2\sin x\cos x = 0$$
$$(2\cos x)(\cos x - \sin x) = 0$$
$$2\cos x = 0$$
$$\frac{2\cos x}{2} = \frac{0}{2}$$
$$\cos x = 0$$
$$x = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$
$$\text{or } \cos x - \sin x = 0$$
$$\cos x - \sin x + \sin x = 0 + \sin x$$
$$\cos x = \sin x$$
$$x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

Therefore, $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}$, or $\frac{3\pi}{2}$.

18. First write $\sin\theta$ and $\cos\theta$ in terms of $\tan\theta$.

$$\sin\theta = \frac{\sin\theta}{\cos\theta} \times \cos\theta$$
$$= \frac{\tan\theta}{\sec\theta}$$
$$= \frac{\tan\theta}{\sqrt{1 + \tan^2\theta}}$$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$, so

$$\cos\theta = \frac{\sin\theta}{\tan\theta}$$
$$= \frac{\frac{\tan\theta}{\sqrt{1 + \tan^2\theta}}}{\tan\theta}$$
$$= \frac{1}{\sqrt{1 + \tan^2\theta}}$$

a) $\sin 2\theta = 2\sin\theta\cos\theta$

$$= 2\left(\frac{\tan\theta}{\sqrt{1 + \tan^2\theta}}\right)\left(\frac{1}{\sqrt{1 + \tan^2\theta}}\right)$$
$$= \frac{2\tan\theta}{1 + \tan^2\theta}$$

b) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$= \left(\frac{1}{\sqrt{1 + \tan^2\theta}}\right)^2 - \left(\frac{\tan\theta}{\sqrt{1 + \tan^2\theta}}\right)^2$$
$$= \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$$

c) Use the results from parts **a)** and **b)**

$$\frac{\sin 2\theta}{1 + \cos 2\theta} = \frac{\frac{2\tan\theta}{1 + \tan^2\theta}}{1 + \frac{1 - \tan^2\theta}{1 + \tan^2\theta}}$$

$$= \frac{\dfrac{2\tan\theta}{1+\tan^2\theta}}{\dfrac{1+\tan^2\theta+1-\tan^2\theta}{1+\tan^2\theta}}$$

$$= \frac{\dfrac{2\tan\theta}{1+\tan^2\theta}}{\dfrac{2}{1+\tan^2\theta}}$$

$$= \frac{2\tan\theta}{1+\tan^2\theta} \times \frac{1+\tan^2\theta}{2}$$

$$= \tan\theta$$

d) Use the results from parts **a)** and **b)**

$$\frac{1-\cos 2\theta}{\sin 2\theta} = \frac{1-\dfrac{1-\tan^2\theta}{1+\tan^2\theta}}{\dfrac{2\tan\theta}{1+\tan^2\theta}}$$

$$= \frac{\dfrac{1+\tan^2\theta-1+\tan^2\theta}{1+\tan^2\theta}}{\dfrac{2\tan\theta}{1+\tan^2\theta}}$$

$$= \frac{\dfrac{2\tan^2\theta}{1+\tan^2\theta}}{\dfrac{2\tan\theta}{1+\tan^2\theta}}$$

$$= \frac{2\tan^2\theta}{1+\tan^2\theta} \times \frac{1+\tan^2\theta}{2\tan\theta}$$

$$= \tan\theta$$

Mid-Chapter Review, p. 411

1. a) Since $\cos(2\pi - \theta) = \cos\theta$, $\cos\theta = \cos(2\pi - \theta)$. Therefore,

$$\cos\frac{\pi}{16} = \cos\left(2\pi - \frac{\pi}{16}\right)$$

$$= \cos\left(\frac{32\pi}{16} - \frac{\pi}{16}\right)$$

$$= \cos\frac{31\pi}{16}$$

b) Since $\sin(\pi - \theta) = \sin\theta$, $\sin\theta = \sin(\pi - \theta)$.

Therefore, $\sin\frac{7\pi}{9} = \sin\left(\pi - \frac{7\pi}{9}\right)$

$$= \sin\left(\frac{9\pi}{9} - \frac{7\pi}{9}\right) = \sin\frac{2\pi}{9}$$

c) Since $\tan(\pi + \theta) = \tan\theta$, $\tan\theta = \tan(\pi + \theta)$.

Therefore, $\tan\frac{9\pi}{10} = \tan\left(\pi + \frac{9\pi}{10}\right)$

$$= \tan\left(\frac{10\pi}{10} + \frac{9\pi}{10}\right) = \tan\frac{19\pi}{10}$$

d) Since $-\cos\theta = \cos(\pi + \theta)$,

$$-\cos\frac{2\pi}{5} = \cos\left(\pi + \frac{2\pi}{5}\right)$$

$$= \cos\left(\frac{5\pi}{5} + \frac{2\pi}{5}\right) = \cos\frac{7\pi}{5}$$

e) Since $-\sin\theta = \sin(\pi + \theta)$,

$$-\sin\frac{9\pi}{7} = \sin\left(\pi + \frac{9\pi}{7}\right)$$

$$= \sin\left(\frac{7\pi}{7} + \frac{9\pi}{7}\right) = \sin\frac{16\pi}{7}$$

Since $\sin\theta = \sin(\theta - 2\pi)$,

$$\sin\frac{16\pi}{7} = \sin\left(\frac{16\pi}{7} - 2\pi\right)$$

$$= \sin\left(\frac{16\pi}{7} - \frac{14\pi}{7}\right)$$

$$= \sin\frac{2\pi}{7}$$

Therefore, $-\sin\frac{9\pi}{7} = \sin\frac{2\pi}{7}$.

f) Since $\tan(\pi + \theta) = \tan\theta$, $\tan\theta = \tan(\pi + \theta)$.

Therefore, $\tan\frac{3\pi}{4} = \tan\left(\pi + \frac{3\pi}{4}\right)$

$$= \tan\left(\frac{4\pi}{4} + \frac{3\pi}{4}\right) = \tan\frac{7\pi}{4}$$

2. Since $\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$, the equation

$y = -6\cos\left(x + \frac{\pi}{2}\right) + 4$ can be rewritten

$y = -6\sin\left(x + \frac{\pi}{2} + \frac{\pi}{2}\right) + 4$

$= -6\sin(x + \pi) + 4$. Since a horizontal translation of π to the left or right is equivalent to a reflection in the x-axis, the equation $y = -6\sin(x + \pi) + 4$ can be rewritten $y = 6\sin x + 4$. Therefore, the equation $y = -6\cos\left(x + \frac{\pi}{2}\right) + 4$ can be rewritten $y = 6\sin x + 4$.

3. a) Since $\cos(a + b) = \cos a\cos b - \sin a\sin b$,

$$\cos\left(x + \frac{5\pi}{3}\right) = \cos x\cos\frac{5\pi}{3} - \sin x\sin\frac{5\pi}{3}$$

$$= (\cos x)\left(\frac{1}{2}\right) - (\sin x)\left(-\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x$$

b) Since $\sin(a+b)\sin a\cos b+\cos a\sin b$,

$$\sin\left(x+\frac{5\pi}{6}\right)=(\sin x)\left(\cos\frac{5\pi}{6}\right)+(\cos x)\left(\sin\frac{5\pi}{6}\right)$$
$$=(\sin x)\left(-\frac{\sqrt{3}}{2}\right)+(\cos x)\left(\frac{1}{2}\right)$$
$$=\frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x$$

c) Since $\tan(a+b)=\dfrac{\tan a+\tan b}{1-\tan a\tan b}$,

$$\tan\left(x+\frac{5\pi}{4}\right)=\frac{\tan x+\tan\frac{5\pi}{4}}{1-\tan x\tan\frac{5\pi}{4}}$$
$$=\frac{\tan x+1}{1-(\tan x)(1)}$$
$$=\frac{1+\tan x}{1-\tan x}$$

d) Since $\cos(a+b)=\cos a\cos b-\sin a\sin b$,

$$\cos\left(x+\frac{4\pi}{3}\right)=\cos x\cos\frac{4\pi}{3}-\sin x\sin\frac{4\pi}{3}$$
$$=(\cos x)\left(-\frac{1}{2}\right)-(\sin x)\left(-\frac{\sqrt{3}}{2}\right)$$
$$=\frac{\sqrt{3}}{2}\sin x-\frac{1}{2}\cos x$$

4. a) Since $\sin(a-b)=\sin a\cos b-\cos a\cos b$,

$$\sin\left(x-\frac{11\pi}{6}\right)=\sin x\cos\frac{11\pi}{6}-\cos x\sin\frac{11\pi}{6}$$
$$=(\sin x)\left(\frac{\sqrt{3}}{2}\right)-(\cos x)\left(-\frac{1}{2}\right)$$
$$=\frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x$$

b) Since $\tan(a-b)=\dfrac{\tan a-\tan b}{1+\tan a\tan b}$,

$$\tan\left(x-\frac{\pi}{3}\right)=\frac{\tan x-\tan\frac{\pi}{3}}{1+\tan x\tan\frac{\pi}{3}}$$
$$=\frac{\tan x-\sqrt{3}}{1+(\tan x)(\sqrt{3})}=\frac{\tan x-\sqrt{3}}{1+\sqrt{3}\tan x}$$

c) Since $\cos(a-b)=\cos a\cos b+\sin a\sin b$,

$$\cos\left(x-\frac{7\pi}{4}\right)=(\cos x)\left(\cos\frac{7\pi}{4}\right)+(\sin x)\left(\sin\frac{7\pi}{4}\right)$$
$$=(\cos x)\left(\frac{\sqrt{2}}{2}\right)+(\sin x)\left(-\frac{\sqrt{2}}{2}\right)$$
$$=\frac{\sqrt{2}}{2}\cos x-\frac{\sqrt{2}}{2}\sin x$$

d) Since $\sin(a-b)-\sin a\cos b-\cos a\sin b$,

$$\sin\left(x-\frac{2\pi}{3}\right)=(\sin x)\left(\cos\frac{2\pi}{3}\right)-(\cos x)\left(\sin\frac{2\pi}{3}\right)$$
$$=(\sin x)\left(-\frac{1}{2}\right)-(\cos x)\left(\frac{\sqrt{3}}{2}\right)$$
$$=-\frac{1}{2}\sin x-\frac{\sqrt{3}}{2}\cos x$$

5. a) Since $\tan(a-b)=\dfrac{\tan a-\tan b}{1+\tan a\tan b}$,

$$\frac{\tan\frac{8\pi}{9}-\tan\frac{5\pi}{9}}{1+\tan\frac{8\pi}{9}\tan\frac{5\pi}{9}}=\tan\left(\frac{8\pi}{9}-\frac{5\pi}{9}\right)$$
$$=\tan\frac{3\pi}{9}=\tan\frac{\pi}{3}=\sqrt{3}$$

b) Since $\sin(a-b)=\sin a\cos b-\cos a\sin b$,

$$\sin\frac{299\pi}{298}\cos\frac{\pi}{298}-\cos\frac{299\pi}{298}\sin\frac{\pi}{298}$$
$$=\sin\left(\frac{299\pi}{298}-\frac{\pi}{298}\right)=\sin\frac{298\pi}{298}$$
$$=\sin\pi=0$$

c) Since $\sin(a-b)=\sin a\cos b-\cos a\sin b$,

$$\sin 50°\cos 20°-\cos 50°\sin 20°$$
$$=\sin(50°-20°)=\sin 30°=\frac{1}{2}$$

d) Since $\sin(a+b)=\sin a\cos b+\cos a\sin b$,

$$\sin\frac{3\pi}{8}\cos\frac{\pi}{8}+\cos\frac{3\pi}{8}\sin\frac{\pi}{8}=\sin\left(\frac{3\pi}{8}+\frac{\pi}{8}\right)$$
$$=\sin\frac{4\pi}{8}=\sin\frac{\pi}{2}=1$$

6. a) Since $\tan(a+b)=\dfrac{\tan a+\tan b}{1-\tan a\tan b}$,

$$\frac{2\tan x}{1-\tan^2 x}=\frac{\tan x+\tan x}{1-(\tan x)(\tan x)}$$
$$=\tan(x+x)=\tan 2x$$

b) Since $\sin(a+b)=\sin a\cos b+\cos a\sin b$,

$$\sin\frac{x}{5}\cos\frac{4x}{5}+\cos\frac{x}{5}\sin\frac{4x}{5}=\sin\left(\frac{x}{5}+\frac{4x}{5}\right)$$
$$=\sin\frac{5x}{5}=\sin x$$

c) $$\cos\left(\frac{\pi}{2}-x\right)=\cos\frac{\pi}{2}\cos x+\sin\frac{\pi}{2}\sin x$$
$$=(0)\cos x+(1)\sin x=\sin x$$

d) $$\sin\left(\frac{\pi}{2}+x\right)=\sin\frac{\pi}{2}\cos x+\cos\frac{\pi}{2}\sin x$$
$$=(1)\cos x+(0)\sin x=\cos x$$

e) $\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}+x\right)=2\cos\left(\frac{\pi}{4}+x\right)$

$=2\left[\cos\frac{\pi}{4}\cos x-\sin\frac{\pi}{4}\sin x\right]$

$=2\left[\left(\frac{\sqrt{2}}{2}\right)\cos x-\left(\frac{\sqrt{2}}{2}\right)\sin x\right]$

$=\sqrt{2}(\cos x-\sin x)$

f) $\tan\left(x-\frac{\pi}{4}\right)=\dfrac{\tan x-\tan\left(\frac{\pi}{4}\right)}{1+\tan x\tan\left(\frac{\pi}{4}\right)}=\dfrac{\tan x-1}{1+\tan x}$

7. $a=\sqrt{3}$ and $b=-3$, so

$R=\sqrt{(\sqrt{3})^2+(-3)^2}=\sqrt{12}=2\sqrt{3}$

$\cos\alpha=\dfrac{\sqrt{3}}{2\sqrt{3}}=\dfrac{1}{2}$

$\sin\alpha=\dfrac{-3}{2\sqrt{3}}=\dfrac{-\sqrt{3}}{2}$

Since $\cos\alpha$ is positive and $\sin\alpha$ is negative, α is in the fourth quadrant. $\alpha=-\frac{\pi}{3}$

So, $\sqrt{3}\cos x-3\sin x=2\sqrt{3}\cos\left(x+\frac{\pi}{3}\right)$.

8. a) Since $\cos 2\theta=2\cos^2\theta-1$,

$2\cos^2\frac{2\pi}{3}-1=\cos\left((2)\left(\frac{2\pi}{3}\right)\right)$

$=\cos\frac{4\pi}{3}=-\frac{1}{2}$

b) Since $\sin 2\theta=2\sin\theta\cos\theta$,

$2\sin\frac{11\pi}{12}\cos\frac{11\pi}{12}=\sin\left((2)\left(\frac{11\pi}{12}\right)\right)$

$=\sin\frac{22\pi}{12}=\sin\frac{11\pi}{6}=-\frac{1}{2}$

c) Since $\cos 2\theta=\cos^2\theta-\sin^2\theta$,

$\cos^2\frac{7\pi}{8}-\sin^2\frac{7\pi}{8}=\cos\left((2)\left(\frac{7\pi}{8}\right)\right)$

$=\cos\frac{14\pi}{8}=\cos\frac{7\pi}{8}=\frac{\sqrt{2}}{2}$

d) Since $\cos 2\theta=1-2\sin^2\theta$,

$1-2\sin^2\frac{\pi}{2}=\cos\left((2)\left(\frac{\pi}{2}\right)\right)=\cos\pi=-1$

9. a) Since $\cos^2 x=\frac{10}{11}$, $\cos x=\pm\sqrt{\frac{10}{11}}$ $=\pm\frac{\sqrt{10}}{\sqrt{11}}=\pm\frac{\sqrt{110}}{11}$, and since angle x is in the third quadrant, $\cos x$ is negative. For this reason, $\cos x=-\frac{\sqrt{110}}{11}$. Since $\cos x=-\frac{\sqrt{110}}{11}$, the leg adjacent to the angle x in a right triangle has a length of $\sqrt{110}$, while the hypotenuse of the right triangle has a length of 11. For this reason, the other leg of the right triangle can be calculated as follows:

$(\sqrt{110})^2+y^2=11^2$

$110+y^2=121$

$110+y^2-110=121-110$

$y^2=11$

$y=-\sqrt{11}$, in quadrant II

Since $\sin x=\dfrac{\text{opposite leg}}{\text{hypotenuse}}$, and since angle x is in the third quadrant, $\sin x=-\frac{\sqrt{11}}{11}$.

b) Since $\cos^2 x=\frac{10}{11}$, $\cos x=\pm\sqrt{\frac{10}{11}}=\pm\frac{\sqrt{10}}{\sqrt{11}}$ $=\pm\frac{\sqrt{110}}{11}$, and since angle x is in the third quadrant, $\cos x$ is negative. For this reason, $\cos x=-\frac{\sqrt{110}}{11}$.

c) Since $\cos^2 x=\frac{10}{11}$, $\cos x=\pm\sqrt{\frac{10}{11}}=\pm\frac{\sqrt{10}}{\sqrt{11}}$ $=\pm\frac{\sqrt{110}}{11}$, and since angle x is in the third quadrant, $\cos x$ is negative. For this reason, $\cos x=-\frac{\sqrt{110}}{11}$. Since $\cos x=-\frac{\sqrt{110}}{11}$, the leg adjacent to the angle x in a right triangle has a length of $\sqrt{110}$, while the hypotenuse of the right triangle has a length of 11. For this reason, the other leg of the right triangle can be calculated as follows:

$(\sqrt{110})^2+y^2=11^2$

$110+y^2=121$

$110+y^2-110=121-110$

$y^2=11$

$y=-\sqrt{11}$, in quadrant II

Since $\sin x=\dfrac{\text{opposite leg}}{\text{hypotenuse}}$, and since angle x is in the third quadrant, $\sin x=-\frac{\sqrt{11}}{11}$. Since $\sin 2\theta=2\sin\theta\cos\theta$, $\sin 2x=2\sin x\cos x$

$=(2)\left(-\frac{\sqrt{11}}{11}\right)\left(-\frac{\sqrt{110}}{11}\right)=\frac{2\sqrt{10}}{11}$.

d) Since $\cos^2 x=\frac{10}{11}$, $\cos x=\pm\sqrt{\frac{10}{11}}=\pm\frac{\sqrt{10}}{\sqrt{11}}$ $=\pm\frac{\sqrt{110}}{11}$, and since angle x is in the third quadrant, $\cos x$ is negative. For this reason, $\cos x=-\frac{\sqrt{110}}{11}$. Since $\cos x=-\frac{\sqrt{110}}{11}$, the leg adjacent to the angle x in a right triangle has a length of $\sqrt{110}$, while the hypotenuse of the right triangle has a length of 11. For this reason, the other leg of the right triangle can be calculated as follows:

$(\sqrt{110})^2+y^2=11^2$

$110+y^2=121$

$110+y^2-110=121-110$

$$y^2 = 11$$
$$y = -\sqrt{11}, \text{ in quadrant II}$$

Since $\sin x = \frac{\text{opposite leg}}{\text{hypotenuse}}$, and since angle x is in the third quadrant, $\sin x = -\frac{\sqrt{11}}{11}$. Since $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$, $\cos 2x = \cos^2 x - \sin^2 x$ $= \frac{10}{11} - \left(-\frac{\sqrt{11}}{11}\right)^2 = \frac{10}{11} - \frac{1}{11} = \frac{9}{11}$. (The formulas $\cos 2\theta = 2\cos^2 \theta - 1$ and $\cos 2\theta = 1 - 2\sin^2 \theta$ could also have been used.)

10. Since $\sin x = \frac{3}{5}$, the leg opposite the angle x in a right triangle has a length of 3, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 3^2 = 5^2$$
$$x^2 + 9 = 25$$
$$x^2 + 9 - 9 = 25 - 9$$
$$x^2 = 16$$
$$x = 4, \text{ in quadrant I}$$

Since $\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos x = \frac{4}{5}$. Therefore, since $\sin 2\theta = 2 \sin \theta \cos \theta$,

$$\sin 2x = 2 \sin x \cos x = (2)\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}.$$

Also, since $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$, $\cos 2x = \cos^2 x - \sin^2 x = \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2$ $= \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$. (The formulas $\cos 2\theta = 2\cos^2 \theta - 1$ and $\cos 2\theta = 1 - 2\sin^2 \theta$ could also have been used.)

11. Since $\sin x = \frac{5}{13}$, the leg opposite the angle x in a right triangle has a length of 5, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 5^2 = 13^2$$
$$x^2 + 25 = 169$$
$$x^2 + 25 - 25 = 169 - 25$$
$$x^2 = 144$$
$$x = 12, \text{ in quadrant I}$$

Since $\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos x = \frac{12}{13}$. Therefore, since $\sin 2\theta = 2 \sin \theta \cos \theta$,

$$\sin 2x = 2 \sin x \cos x = (2)\left(\frac{5}{13}\right)\left(\frac{12}{13}\right) = \frac{120}{169}.$$

12. Since $\cos x = -\frac{4}{5}$, the leg adjacent to the angle x in a right triangle has a length of 4, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$4^2 + y^2 = 5^2$$
$$16 + y^2 = 25$$
$$16 + y^2 - 16 = 25 - 16$$
$$y^2 = 9$$
$$y = 3, \text{ in quadrant III}$$

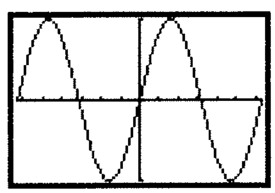
Since $\tan x = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan x = \frac{3}{4}$. (The reason the sign is positive is because angle x is in the third quadrant.) Since $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$,

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{(2)\left(\frac{3}{4}\right)}{1 - \left(\frac{3}{4}\right)^2} = \frac{\frac{3}{2}}{1 - \frac{9}{16}}$$
$$= \frac{\frac{3}{2}}{\frac{16}{16} - \frac{9}{16}} = \frac{\frac{3}{2}}{\frac{7}{16}} = \frac{3}{2} \times \frac{16}{7} = \frac{24}{7}$$

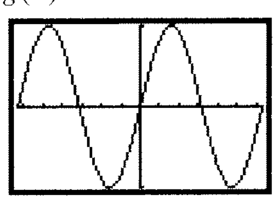
7.4 Proving Trigonometric Identities, pp. 417–418

1. Although $\sin x = \cos x$ is true for $x = \frac{\pi}{4}$, it is not true for all values of x, and therefore, it is not an identity. A counterexample is a value of x for which $\sin x = \cos x$ is not true. Many counterexamples exist, so answers may vary. One counterexample is $x = \frac{\pi}{6}$, since $\sin \frac{\pi}{6} = \frac{1}{2}$, and since $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

2. a) The graphs of $f(x) = \sin x$ and $g(x) = \tan x \cos x$ are as follows: $f(x) = \sin x$:



$g(x) = \tan x \cos x$:



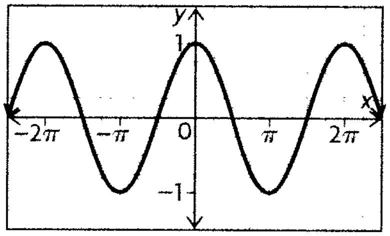
b) Since the graphs of $f(x) = \sin x$ and $g(x) = \tan x \cos x$ are the same, $\sin x = \tan x \cos x$.

c) To prove that the identity $\sin x = \tan x \cos x$ is true, $\tan x \cos x$ can be simplified as follows:

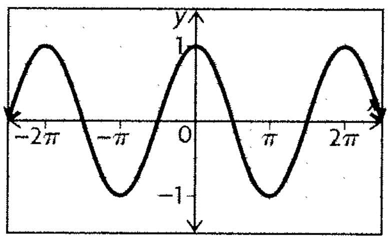
$$\tan x \cos x = \left(\frac{\sin x}{\cos x}\right)(\cos x) = \frac{\sin x \cos x}{\cos x} = \sin x$$

d) The identity is not true when $\cos x = 0$ because when $\cos x = 0$, $\tan x$, or $\frac{\sin x}{\cos x}$, is undefined. This is because 0 cannot be in the denominator of a fraction.

3. a) The graph of $y = \sin x \cot x$ is as follows:

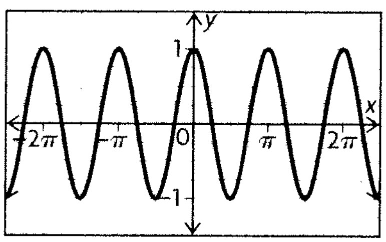


The graph of $y = \cos x$ is as follows:

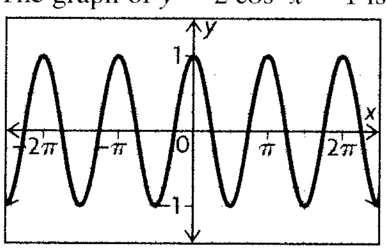


Since the graphs are the same, the answer is C.

b) The graph of $y = 1 - 2\sin^2 x$ is as follows:

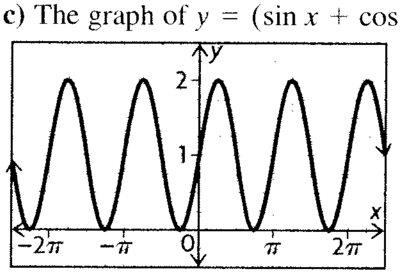


The graph of $y = 2\cos^2 x - 1$ is as follows:

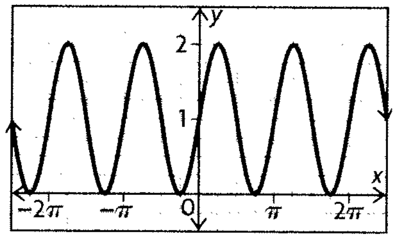


Since the graphs are the same, the answer is D.

c) The graph of $y = (\sin x + \cos x)^2$ is as follows:

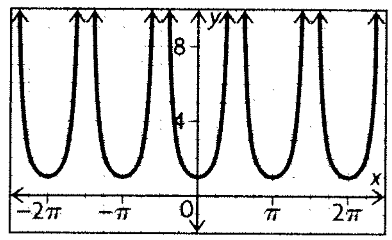


The graph of $y = 1 + 2\sin x \cos x$ is as follows:

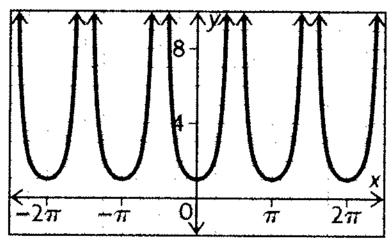


Since the graphs are the same, the answer is B.

d) The graph of $y = \sec^2 x$ is as follows:



The graph of $y = \sin^2 x + \cos^2 x + \tan^2 x$ is as follows:



Since the graphs are the same, the answer is A.

4. a) The identity $\sin x \cot x = \cos x$ can be proven as follows:

$$\sin x \cot x = (\sin x)\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x \cos x}{\sin x} = \cos x$$

b) The identity $1 - 2\sin^2 x = 2\cos^2 x - 1$ can be proven as follows:

$$\begin{aligned} 1 - 2\sin^2 x &= 2\cos^2 x - 1 \\ 1 - 2\sin^2 x - 2\cos^2 x + 1 &= 0 \\ 2 - 2\sin^2 x - 2\cos^2 x &= 0 \\ 2 - 2(\sin^2 x + \cos^2 x) &= 0 \\ 2 - 2(1) &= 0 \\ 2 - 2 &= 0 \\ 0 &= 0 \end{aligned}$$

c) The identity $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$ can be proven as follows:

$$\begin{aligned} (\sin x + \cos x)^2 &= \sin^2 x + 2\sin x \cos x + \cos^2 x \\ &= (\sin^2 x + \cos^2 x) + 2\sin x \cos x \\ &= 1 + 2\sin x \cos x \end{aligned}$$

d) The identity $\sec^2 x = \sin^2 x + \cos^2 x + \tan^2 x$ can be proven as follows:

$$\sin^2 x + \cos^2 x + \tan^2 x$$
$$= (\sin^2 x + \cos^2 x) + \tan^2 x$$
$$= 1 + \tan^2 x$$
$$= \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}$$
$$= \frac{\sin^2 x + \cos^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x}$$
$$= \sec^2 x$$

5. a) The equation $\cos x = \frac{1}{\cos x}$ is not true for all values of x, and therefore, it is not an identity. A counterexample is a value of x for which $\cos x = \frac{1}{\cos x}$ is not true. Many counterexamples exist, so answers may vary. One counterexample is $x = \frac{\pi}{6}$, since $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, and since $\frac{1}{\cos \frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

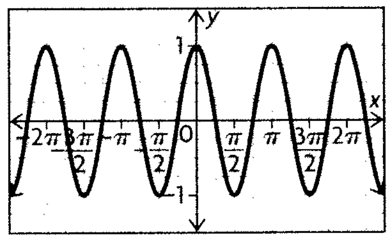
b) The equation $1 - \tan^2 x = \sec^2 x$ is not true for all values of x, and therefore, it is not an identity. A counterexample is a value of x for which $1 - \tan^2 x = \sec^2 x$ is not true. Many counterexamples exist, so answers may vary. One counterexample is $x = \frac{\pi}{4}$, since $1 - \tan^2\left(\frac{\pi}{4}\right) = 1 - (1)^2 = 1 - 1 = 0$, and since $\sec^2\left(\frac{\pi}{4}\right) = (\sqrt{2})^2 = 2$.

c) The equation $\sin(x + y) = \cos x \cos y + \sin x \sin y$ is not true for all values of x and y, and therefore, it is not an identity. A counterexample is values for x and y for which $\sin(x + y) = \cos x \cos y + \sin x \sin y$ is not true. Many counterexamples exist, so answers may vary. One counterexample is $x = \frac{\pi}{2}$ and $y = \pi$, since $\sin\left(\frac{\pi}{2} + \pi\right) = \sin\left(\frac{3\pi}{2}\right) = -1$, and since $\cos\left(\frac{\pi}{2}\right)\cos \pi + \sin\left(\frac{\pi}{2}\right)\sin \pi = (0)(-1) + (1)(0) = 0 + 0 = 0$.

d) The equation $\cos 2x = 1 + 2\sin^2 x$ is not true for all values of x, and therefore, it is not an identity. A counterexample is a value of x for which $\cos 2x = 1 + 2\sin^2 x$ is not true. Many counterexamples exist, so answers may vary. One counterexample is $x = \frac{\pi}{3}$, since $\cos\left(2\left(\frac{\pi}{3}\right)\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$, and since

$$1 + 2\sin^2\left(\frac{\pi}{3}\right) = 1 + (2)\left(\frac{\sqrt{3}}{2}\right)^2$$
$$= 1 + (2)\left(\frac{3}{4}\right) = \frac{4}{4} + \frac{6}{4} = \frac{10}{4} = \frac{5}{2}$$

6. Answers may vary. For example, the graph of the function $y = \frac{1 - \tan^2 x}{1 + \tan^2 x}$ is as follows:



The graph is the same as that of the function $y = \cos 2x$, so an appropriate conjecture is that $\cos 2x$ is another expression that is equivalent to $\frac{1 - \tan^2 x}{1 + \tan^2 x}$.

7. The identity $\frac{1 - \tan^2 x}{1 + \tan^2 x} = \cos 2x$ can be proven as follows:

$$\frac{1 - \tan^2 x}{1 + \tan^2 x} = \frac{\frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}}{\sec^2 x} = \frac{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}}{\sec^2 x}$$
$$= \frac{\cos^2 x - \sin^2 x}{\cos^2 x} \times \frac{1}{\sec^2 x}$$
$$= \frac{\cos^2 x - \sin^2 x}{\cos^2 x} \times \cos^2 x = \cos^2 x - \sin^2 x$$
$$= \cos 2x$$

8. The identity $\frac{1 + \tan x}{1 + \cot x} = \frac{1 - \tan x}{\cot x - 1}$ can be proven as follows:

$$\text{LS} = \frac{1 + \tan x}{1 + \cot x} = \frac{1 + \tan x}{1 + \frac{1}{\tan x}} = \frac{1 + \tan x}{\frac{\tan x + 1}{\tan x}} = \tan x$$

$$\text{RS} = \frac{1 - \tan x}{\cot x - 1} = \frac{1 - \tan x}{\frac{1}{\tan x} - 1} = \frac{1 - \tan x}{\frac{1 - \tan x}{\tan x}} = \tan x$$

Since the right side and the left side are equal, $\frac{1 + \tan x}{1 + \cot x} = \frac{1 - \tan x}{\cot x - 1}$

9. a) The identity $\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin\theta\cos\theta} = 1 - \tan\theta$ can be proven as follows:

$$\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin\theta\cos\theta}$$
$$= \frac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{(\cos\theta)(\cos\theta + \sin\theta)}$$
$$= \frac{\cos\theta - \sin\theta}{\cos\theta} = \frac{\cos\theta}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = 1 - \tan\theta$$

b) The identity $\tan^2 x - \sin^2 x = \sin^2 x\tan^2 x$ can be proven as follows:

$$\text{LS} = \tan^2 x - \sin^2 x$$
$$= \frac{\sin^2 x}{\cos^2 x} - \sin^2 x$$
$$= \sin^2 x\left(\frac{1}{\cos^2 x} - 1\right)$$
$$= \sin^2 x(\sec^2 x - 1)$$
$$= \sin^2 x\tan^2 x$$
$$= \text{RS}$$

So $\tan^2 x - \sin^2 x = \sin^2 x\tan^2 x$.
Since $\csc^2 x = 1 + \cot^2 x$ is a known identity, $\tan^2 x - \sin^2 x$ must equal $\sin^2 x\tan^2 x$.

c) The identity

$\tan^2 x - \cos^2 x = \frac{1}{\cos^2 x} - 1 - \cos^2 x$ can be proven as follows:

$$\tan^2 x - \cos^2 x = \frac{1}{\cos^2 x} - 1 - \cos^2 x;$$
$$\tan^2 x - \cos^2 x + \cos^2 x = \frac{1}{\cos^2 x} - 1 - \cos^2 x + \cos^2 x;$$
$$\tan^2 x = \frac{1}{\cos^2 x} - 1;$$
$$\tan^2 x = \frac{1}{\cos^2 x} - \frac{\cos^2 x}{\cos^2 x};$$
$$\tan^2 x = \frac{1 - \cos^2 x}{\cos^2 x};$$
$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x};$$
$$\tan^2 x = \tan^2 x$$

d) The identity $\frac{1}{1 + \cos\theta} + \frac{1}{1 - \cos\theta} = \frac{2}{\sin^2\theta}$ can be proven as follows:

$$\frac{1}{1 + \cos\theta} + \frac{1}{1 - \cos\theta} = \frac{1 - \cos\theta}{(1 + \cos\theta)(1 - \cos\theta)} + \frac{1 + \cos\theta}{(1 - \cos\theta)(1 + \cos\theta)}$$
$$= \frac{1 - \cos\theta}{1 - \cos^2\theta} + \frac{1 + \cos\theta}{1 - \cos^2\theta}$$
$$= \frac{1 - \cos\theta + 1 + \cos\theta}{1 - \cos^2\theta} = \frac{2}{1 - \cos^2\theta} = \frac{2}{\sin^2\theta}$$

10. a) The identity $\cos x\tan^3 x = \sin x\tan^2 x$ can be proven as follows:

$$\cos x\tan^3 x = \sin x\tan^2 x$$
$$\frac{\cos x\tan^3 x}{\tan^2 x} = \frac{\sin x\tan^2 x}{\tan^2 x}$$
$$\cos x\tan x = \sin x$$
$$(\cos x)\left(\frac{\sin x}{\cos x}\right) = \sin x$$
$$\sin x = \sin x$$

b) The identity $\sin^2\theta + \cos^4\theta = \cos^2\theta + \sin^4\theta$ can be proven as follows:

$$\sin^2\theta + \cos^4\theta = \cos^2\theta + \sin^4\theta$$
$$\sin^2\theta + \cos^4\theta - \sin^4\theta = \cos^2\theta + \sin^4\theta - \sin^4\theta$$
$$\sin^2\theta + \cos^4\theta - \sin^4\theta = \cos^2\theta$$
$$\sin^2\theta + \cos^4\theta - \sin^4\theta - \sin^2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$$
$$(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) = \cos^2\theta - \sin^2\theta$$
$$\cos^2\theta + \sin^2\theta = 1$$
$$1 = 1$$

c) The identity

$(\sin x + \cos x)\left(\frac{\tan^2 x + 1}{\tan x}\right) = \frac{1}{\cos x} + \frac{1}{\sin x}$

can be proven as follows:

$$(\sin x + \cos x)\left(\frac{\tan^2 x + 1}{\tan x}\right) = \frac{1}{\cos x} + \frac{1}{\sin x}$$
$$(\sin x + \cos x)\left(\frac{\sec^2 x}{\tan x}\right) = \frac{\sin x}{\cos x\sin x} + \frac{\cos x}{\sin x\cos x}$$
$$(\sin x + \cos x)\left(\frac{1}{\cos^2 x}\right)\left(\frac{1}{\tan x}\right) = \frac{\sin x + \cos x}{\cos x\sin x}$$
$$(\sin x + \cos x)\left(\frac{1}{\cos^2 x}\right)\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x + \cos x}{\cos x\sin x}$$
$$(\sin x + \cos x)\left(\frac{1}{\cos x\sin x}\right) = \frac{\sin x + \cos x}{\cos x\sin x}$$
$$\frac{\sin x + \cos x}{\cos x\sin x} = \frac{\sin x + \cos x}{\cos x\sin x}$$

d) The identity $\tan^2\beta + \cos^2\beta + \sin^2\beta = \frac{1}{\cos^2\beta}$ can be proven as follows:

$$\tan^2\beta + \cos^2\beta + \sin^2\beta = \frac{1}{\cos^2\beta}$$
$$\tan^2\beta + 1 = \frac{1}{\cos^2\beta}$$

$\tan^2 \beta + 1 = \sec^2 \beta$

Since $\tan^2 \beta + 1 = \sec^2 \beta$ is a known identity, $\tan^2 \beta + \cos^2 \beta + \sin^2 \beta$ must equal $\frac{1}{\cos^2 \beta}$.

e) The identity

$\sin\left(\frac{\pi}{4} + x\right) + \sin\left(\frac{\pi}{4} - x\right) = \sqrt{2}\cos x$

can be proven as follows:

$$\sin\left(\frac{\pi}{4} + x\right) + \sin\left(\frac{\pi}{4} - x\right) = \sqrt{2}\cos x;$$

$$\sin\frac{\pi}{4}\cos x + \cos\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x = \sqrt{2}\cos x;$$

$$2\sin\frac{\pi}{4}\cos x = \sqrt{2}\cos x;$$

$$(2)\left(\frac{\sqrt{2}}{2}\right)(\cos x) = \sqrt{2}\cos x;$$

$$\sqrt{2}\cos x = \sqrt{2}\cos x$$

f) The identity $\sin\left(\frac{\pi}{2} - x\right)\cot\left(\frac{\pi}{2} + x\right) = -\sin x$ can be proven as follows:

$$\sin\left(\frac{\pi}{2} - x\right)\cot\left(\frac{\pi}{2} + x\right) = -\sin x;$$

$$\sin\left(\frac{\pi}{2} - x\right)\left(\frac{\cos\left(\frac{\pi}{2} + x\right)}{\sin\left(\frac{\pi}{2} + x\right)}\right) = -\sin x;$$

$$\left(\sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x\right) \times \left(\frac{\cos\frac{\pi}{2}\cos x - \sin\frac{\pi}{2}\sin x}{\sin\frac{\pi}{2}\cos x + \cos\frac{\pi}{2}\sin x}\right) = -\sin x;$$

$$((1)(\cos x) - (0)(\sin x))\left(\frac{(0)(\cos x) - (1)(\sin x)}{(1)(\cos x) + (0)(\sin x)}\right) = -\sin x;$$

$$(\cos x - 0)\left(\frac{0 - \sin x}{\cos x + 0}\right) = -\sin x;$$

$$(\cos x)\left(-\frac{\sin x}{\cos x}\right) = -\sin x;$$

$$-\sin x = -\sin x$$

11. a) The identity $\frac{\cos 2x + 1}{\sin 2x} = \cot x$ can be proven as follows:

$$\frac{\cos 2x + 1}{\sin 2x} = \cot x$$

$$\frac{2\cos^2 x - 1 + 1}{2\sin x\cos x} = \cot x$$

$$\frac{2\cos^2 x}{2\sin x\cos x} = \cot x$$

$$\frac{\cos x}{\sin x} = \cot x$$

$$\cot x = \cot x$$

b) The identity $\frac{\sin 2x}{1 - \cos 2x} = \cot x$ can be proven as follows:

$$\frac{\sin 2x}{1 - \cos 2x} = \cot x$$

$$\frac{2\sin x\cos x}{1 - (1 - 2\sin^2 x)} = \cot x$$

$$\frac{2\sin x\cos x}{1 - 1 + 2\sin^2 x} = \cot x$$

$$\frac{2\sin x\cos x}{2\sin^2 x} = \cot x$$

$$\frac{\cos x}{\sin x} = \cot x$$

$$\cot x = \cot x$$

c) The identity $(\sin x + \cos x)^2 = 1 + \sin 2x$ can be proven as follows:

$(\sin x + \cos x)^2 = 1 + \sin 2x;$

$\sin^2 x + \sin x\cos x + \sin x\cos x + \cos^2 x = 1 + 2\sin x\cos x;$

$\sin^2 x + 2\sin x\cos x + \cos^2 x = 1 + 2\sin x\cos x;$

$(\cos^2 x + \sin^2 x) + 2\sin x\cos x = 1 + 2\sin x\cos x;$

$1 + 2\sin x\cos x = 1 + 2\sin x\cos x$

d) The identity $\cos^4 \theta - \sin^4 \theta = \cos 2\theta$ can be proven as follows:

$$\cos^4 \theta - \sin^4 \theta = \cos 2\theta$$

$$(\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta) = \cos^2 \theta - \sin^2 \theta$$

$$(1)(\cos^2 \theta - \sin^2 \theta) = \cos^2 \theta - \sin^2 \theta$$

$$\cos^2 \theta - \sin^2 \theta = \cos^2 \theta - \sin^2 \theta$$

e) The identity $\cot \theta - \tan \theta = 2\cot 2\theta$ can be proven as follows:

$$\cot \theta - \tan \theta = 2\cot 2\theta$$

$$\frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} = 2\frac{\cos 2\theta}{\sin 2\theta}$$

$$\frac{\cos^2 \theta}{\sin \theta\cos \theta} - \frac{\sin^2 \theta}{\cos \theta\sin \theta} = (2)\left(\frac{\cos 2\theta}{2\cos \theta\sin \theta}\right)$$

$$\frac{\cos^2 \theta - \sin^2 \theta}{\sin \theta\cos \theta} = \frac{\cos 2\theta}{\cos \theta\sin \theta}$$

$$\frac{\cos 2\theta}{\cos \theta\sin \theta} = \frac{\cos 2\theta}{\cos \theta\sin \theta}$$

f) The identity $\cot\theta + \tan\theta = 2\csc 2\theta$ can be proven as follows:

$$\cot\theta + \tan\theta = 2\csc 2\theta$$
$$\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = 2\frac{1}{\sin 2\theta}$$
$$\frac{\cos^2\theta}{\sin\theta\cos\theta} + \frac{\sin^2\theta}{\cos\theta\sin\theta} = (2)\left(\frac{1}{2\cos\theta\sin\theta}\right)$$
$$\frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} = \frac{1}{\cos\theta\sin\theta}$$
$$\frac{1}{\cos\theta\sin\theta} = \frac{1}{\cos\theta\sin\theta}$$

g) The identity $\frac{1+\tan x}{1-\tan x} = \tan\left(x + \frac{\pi}{4}\right)$ can be proven as follows:

$$\frac{1+\tan x}{1-\tan x} = \tan\left(x + \frac{\pi}{4}\right)$$
$$\frac{1+\tan x}{1-\tan x} = \frac{\tan x + \tan\frac{\pi}{4}}{1 - \tan x\tan\frac{\pi}{4}}$$
$$\frac{1+\tan x}{1-\tan x} = \frac{\tan x + 1}{1 - (\tan x)(1)}$$
$$\frac{1+\tan x}{1-\tan x} = \frac{1+\tan x}{1-\tan x}$$

h) The identity $\csc 2x + \cot 2x = \cot x$ can be proven as follows:

$$\csc 2x + \cot 2x = \cot x;$$
$$\frac{1}{\sin 2x} + \frac{1}{\tan 2x} = \cot x;$$
$$\frac{1}{2\sin x\cos x} + \frac{1}{\frac{2\tan x}{1-\tan^2 x}} = \cot x;$$
$$\frac{1}{2\sin x\cos x} + \frac{1-\tan^2 x}{2\tan x} = \cot x;$$
$$\frac{1}{2\sin x\cos x} + \frac{1-\tan^2 x}{2\frac{\sin x}{\cos x}} = \cot x;$$
$$\frac{1}{2\sin x\cos x} + \frac{(\cos x)(1-\tan^2 x)}{2\sin x} = \frac{\cos x}{\sin x};$$
$$\frac{1}{2\sin x\cos x} + \frac{(\cos x)(1-\tan^2 x)(\cos x)}{2\sin x\cos x}$$
$$= \frac{(\cos x)(2\cos x)}{(\sin x)(2\cos x)};$$
$$\frac{1}{2\sin x\cos x} + \frac{(\cos^2 x)(1-\tan^2 x)}{2\sin x\cos x}$$
$$= \frac{2\cos^2 x}{2\sin x\cos x};$$
$$\frac{1}{2\sin x\cos x} + \frac{\cos^2 x - (\tan^2 x)(\cos^2 x)}{2\sin x\cos x}$$
$$= \frac{2\cos^2 x}{2\sin x\cos x};$$
$$\frac{1}{2\sin x\cos x} + \frac{\cos^2 x - \sin^2 x}{2\sin x\cos x} = \frac{2\cos^2 x}{2\sin x\cos x};$$
$$\frac{1+\cos^2 x - \sin^2 x}{2\sin x\cos x} = \frac{2\cos^2 x}{2\sin x\cos x};$$
$$\frac{1+\cos^2 x - \sin^2 x}{2\sin x\cos x} - \frac{2\cos^2 x}{2\sin x\cos x}$$
$$= \frac{2\cos^2 x}{2\sin x\cos x} - \frac{2\cos^2 x}{2\sin x\cos x};$$
$$\frac{1+\cos^2 x - \sin^2 x - 2\cos^2 x}{2\sin x\cos x} = 0;$$
$$\frac{1-\sin^2 x - \cos^2 x}{2\sin x\cos x} = 0;$$
$$\frac{1-(\sin^2 x + \cos^2 x)}{2\sin x\cos x} = 0;$$
$$\frac{1-1}{2\sin x\cos x} = 0;$$
$$\frac{0}{2\sin x\cos x} = 0;$$
$$0 = 0$$

i) The identity $\frac{2\tan x}{1+\tan^2 x} = \sin 2x$ can be proven as follows:

$$\frac{2\tan x}{1+\tan^2 x} = \sin 2x$$
$$\frac{2\tan x}{\sec^2 x} = \sin 2x$$
$$\frac{2\tan x}{\frac{1}{\cos^2 x}} = \sin 2x$$
$$(2\tan x)(\cos^2 x) = \sin 2x$$
$$\left(\frac{2\sin x}{\cos x}\right)(\cos^2 x) = \sin 2x$$
$$\sin 2x = 2\sin x\cos x$$

Since $\sin 2x = 2\sin x\cos x$ is a known identity, $\frac{2\tan x}{1-\tan^2 x}$ must equal $\sin 2x$.

j) The identity $\sec 2t = \frac{\csc t}{\csc t - 2\sin t}$ can be proven as follows:

$$\sec 2t = \frac{\csc t}{\csc t - 2\sin t}$$
$$\frac{1}{\cos 2t} = \frac{\frac{1}{\sin t}}{\frac{1}{\sin t} - 2\sin t}$$
$$\frac{1}{\cos 2t} = \frac{\frac{1}{\sin t}}{\frac{1}{\sin t} - \frac{2\sin^2 t}{\sin t}}$$

$$\frac{1}{\cos 2t} = \frac{\frac{1}{\sin t}}{\frac{1 - 2\sin^2 t}{\sin t}}$$

$$\frac{1}{\cos 2t} = \frac{1}{\sin t} \times \frac{\sin t}{1 - 2\sin^2 t}$$

$$\frac{1}{\cos 2t} = \frac{1}{1 - 2\sin^2 t}$$

$$\frac{1}{\cos 2t} = \frac{1}{\cos 2t}$$

k) The identity $\csc 2\theta = \frac{1}{2}\sec\theta\csc\theta$ can be proven as follows:

$$\csc 2\theta = \frac{1}{2}\sec\theta\csc\theta$$

$$\frac{1}{\sin 2\theta} = \left(\frac{1}{2}\right)\left(\frac{1}{\cos\theta}\right)\left(\frac{1}{\sin\theta}\right)$$

$$\frac{1}{\sin 2\theta} = \frac{1}{2\cos\theta\sin\theta}$$

$$\frac{1}{2\sin\theta\cos\theta} = \frac{1}{2\sin\theta\cos\theta}$$

l) The identity $\sec t = \frac{\sin 2t}{\sin t} - \frac{\cos 2t}{\cos t}$ can be proven as follows:

$$\frac{1}{\cos t} = \frac{2\sin t\cos t}{\sin t} - \frac{2\cos^2 t - 1}{\cos t}$$

$$\frac{\sin t}{\cos t\sin t} = \frac{2\sin t\cos^2 t}{\sin t\cos t} - \frac{(\sin t)(2\cos^2 t - 1)}{\cos t\sin t}$$

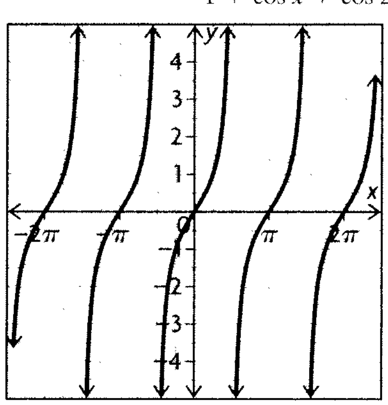
$$\frac{\sin t}{\cos t\sin t} = \frac{2\sin t\cos^2 t}{\sin t\cos t} - \frac{2\cos^2 t\sin t - \sin t}{\cos t\sin t}$$

$$\frac{\sin t}{\cos t\sin t} = \frac{2\sin t\cos^2 t - (2\cos^2 t\sin t - \sin t)}{\sin t\cos t}$$

$$\frac{\sin t}{\cos t\sin t} = \frac{2\sin t\cos^2 t - 2\sin t\cos^2 t + \sin t}{\sin t\cos t}$$

$$\frac{\sin t}{\cos t\sin t} = \frac{\sin t}{\cos t\sin t}$$

12. Answers may vary. For example, the graph of the function $y = \frac{\sin x + \sin 2x}{1 + \cos x + \cos 2x}$ is as follows:



The graph is the same as that of the function $y = \tan x$, so an expression equivalent to $\frac{\sin x + \sin 2x}{1 + \cos x + \cos 2x}$ is $\tan x$.

13. The identity $\frac{\sin x + \sin 2x}{1 + \cos x + \cos 2x} = \tan x$ can be proven as follows:

$$\frac{\sin x + \sin 2x}{1 + \cos x + \cos 2x} = \tan x$$

$$\frac{\sin x + 2\sin x\cos x}{1 + \cos x + \cos 2x} = \tan x$$

$$\frac{(\sin x)(1 + 2\cos x)}{1 + \cos x + \cos 2x} = \tan x$$

$$\frac{(\sin x)(1 + 2\cos x)}{1 + \cos x + 2\cos^2 x - 1} = \tan x$$

$$\frac{(\sin x)(1 + 2\cos x)}{\cos x + 2\cos^2 x} = \tan x$$

$$\frac{(\sin x)(1 + 2\cos x)}{(\cos x)(1 + 2\cos x)} = \tan x$$

$$\frac{\sin x}{\cos x} = \tan x$$

$$\tan x = \tan x$$

14. A trigonometric identity is a statement of the equivalence of two trigonometric expressions. To prove it, both sides of the equation must be shown to be equivalent through graphing or simplifying/rewriting. Therefore, the chart can be completed as follows:

Definition	**Methods of Proof**
A statement of the equivalence of two trigonometric expressions	Both sides of the equation must be shown to be equivalent through graphing or simplifying/rewriting.
Trigonometric Identities	
Examples	**Non-Examples**
$\cos 2x + \sin^2 x = \cos^2 x$ $\cos 2x + 1 = 2\cos^2 x$	$\cos 2x - 2\sin^2 x = 1$ $\cot^2 x + \csc^2 x = 1$

15. She can determine whether the equation $2\sin x\cos x = \cos 2x$ is an identity by trying to simplify and/or rewrite the left side of the equation so that it is equivalent to the right side of the equation. Alternatively, she can graph the functions $y = 2 \sin x \cos x$ and $y = \cos 2x$ and see if the graphs are the same. If they're the same, it's an identity, but if they're not the same, it's not an identity. By doing this she can determine it's not an identity, but she can make it an identity by changing the equation to $2 \sin x \cos x = \sin 2x$.

16. a) To write the expression $2 \cos^2 x + 4 \sin x \cos x$ in the form $a \sin 2x + b \cos 2x + c$, rewrite the expression as follows:

$2 \cos^2 x + 4 \sin x \cos x$

$= 2 \cos^2 x + (2)(2 \sin x \cos x)$

$= (2 \cos^2 x - 1) + (2)(2 \sin x \cos x) + 1$

$= \cos 2x + 2 \sin 2x + 1$

$= 2 \sin 2x + \cos 2x + 1$

Since the expression can be rewritten as $2 \sin 2x + \cos 2x + 1$, the values of a, b, and c are $a = 2$, $b = 1$, and $c = 1$.

b) To write the expression $-2 \sin x \cos x - 4 \sin^2 x$ in the form $a \sin 2x + b \cos 2x + c$, rewrite the expression as follows:

$-2 \sin x \cos x - 4 \sin^2 x$

$= -2 \sin x \cos x + (2 - 4 \sin^2 x) - 2$

$= -2 \sin x \cos x + (2)(1 - 2 \sin^2 x) - 2$

$= -\sin 2x + 2 \cos 2x - 2$

Since the expression can be rewritten as $-\sin 2x + 2 \cos 2x - 2$, the values of a, b, and c are $a = -1$, $b = 2$, and $c = -2$.

17. To write the expression $8 \cos^4 x$ in the form $a \cos 4x + b \cos 2x + c$, it's first necessary to develop a formula for $\cos 4x$. Since $\cos 2x = 2 \cos^2 x - 1$, $\cos 4x = 2 \cos^2 2x - 1$. The formula $\cos 2x = 2 \cos^2 x - 1$ can now be used again to further develop the formula for $\cos 4x$ as follows:

$\cos 4x = 2 \cos^2 2x - 1$

$\cos 4x = (2)(2 \cos^2 x - 1)^2 - 1$

$\cos 4x = (2)(4 \cos^4 x - 2 \cos^2 x$
$\quad - 2 \cos^2 x + 1) - 1$

$\cos 4x = (2)(4 \cos^4 x - 4 \cos^2 x + 1) - 1$

$\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 2 - 1$

$\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1$

Since $\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1$, the value of a must be 1. Now, to turn the expression $8 \cos^4 x - 8 \cos^2 x + 1$ into the expression $8 \cos^4 x$, the terms $-8 \cos^2 x$ and 1 must be eliminated. First, to eliminate $-8 \cos^2 x$, $4 \cos 2x$, or $(4)(2 \cos^2 x - 1)$, can be added to the expression $8 \cos^4 x - 8 \cos^2 x + 1$ as follows:

$8 \cos^4 x - 8 \cos^2 x + 1 + (4)(2 \cos^2 x - 1)$

$8 \cos^4 x - 8 \cos^2 x + 1 + 8 \cos^2 x - 4$

$8 \cos^4 x - 3$

Since adding $4 \cos 2x$ to the expression eliminated the term $-8 \cos^2 x$, the value of b must be 4. Now, to turn the expression $8 \cos^4 x - 3$ into the expression $8 \cos^4 x$, 3 must be added to it as follows:

$8 \cos^4 x - 3 + 3$

$8 \cos^4 x$

Therefore, the value of c must be 3, and the expression must be $\cos 4x + 4 \cos 2x + 3$.

$a = 1, b = 4, c = 3$

7.5 Solving Linear Trigonometric Equations, pp. 426–428

1. a) From the graph of $y = \sin \theta$, the equation $\sin \theta = 1$ is true when $\theta = \frac{\pi}{2}$.

b) From the graph of $y = \sin \theta$, the equation $\sin \theta = -1$ is true when $\theta = \frac{3\pi}{2}$.

c) From the graph of $y = \sin \theta$, the equation $\sin \theta = 0.5$ is true when $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

d) From the graph of $y = \sin \theta$, the equation $\sin \theta = -0.5$ is true when $\theta = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$.

e) From the graph of $y = \sin \theta$, the equation $\sin \theta = 0$ is true when $\theta = 0, \pi$, or 2π.

f) From the graph of $y = \sin \theta$, the equation $\sin \theta = \frac{\sqrt{3}}{2}$ is true when $\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$.

2. a) From the graph of $y = \cos \theta$, the equation $\cos \theta = 1$ is true when $\theta = 0$ or 2π.

b) From the graph of $y = \cos \theta$, the equation $\cos \theta = -1$ is true when $\theta = \pi$.

c) From the graph of $y = \cos \theta$, the equation $\cos \theta = 0.5$ is true when $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.

d) From the graph of $y = \cos \theta$, the equation $\cos \theta = -0.5$ is true when $\theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

e) From the graph of $y = \cos \theta$, the equation $\cos \theta = 0$ is true when $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

f) From the graph of $y = \cos \theta$, the equation $\cos \theta = \frac{\sqrt{3}}{2}$ is true when $\theta = \frac{\pi}{6}$ or $\frac{11\pi}{6}$.

3. a) Given $\sin x = \frac{\sqrt{3}}{2}$ and $0 \le x \le 2\pi$, 2 solutions must be possible, since $\sin x = \frac{\sqrt{3}}{2}$ in 2 of the 4 quadrants.

b) Given $\sin x = \frac{\sqrt{3}}{2}$ and $0 \le x \le 2\pi$, the solutions for x must occur in the 1st and 2nd quadrants, since the sine function is positive in these quadrants.

c) Given $\sin x = \frac{\sqrt{3}}{2}$ and $0 \le x \le 2\pi$, the related acute angle for the equation must be $x = \frac{\pi}{3}$, since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

d) Given $\sin x = \frac{\sqrt{3}}{2}$ and $0 \le x \le 2\pi$, the solutions to the equation must be $x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$, since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$.

4. a) Given $\cos x = -0.8667$ and $0° \le x \le 360°$, two solutions must be possible, since $\cos x = -0.8667$ in two of the four quadrants.
b) Given $\cos x = -0.8667$ and $0° \le x \le 360°$, the solutions for x must occur in the second and third quadrants, since the cosine function is negative in these quadrants.
c) Given $\cos x = -0.8667$ and $0° \le x \le 360°$, the related acute angle for the equation must be $x = 30°$, since $\cos 30° = -0.866$.
d) Given $\cos x = -0.8667$ and $0° \le x \le 360°$, the solutions to the equation must be $x = 150°$ or $210°$, since $\cos 150° = -0.866$ and $\cos 210° = -0.866$.
5. a) Given $\tan \theta = 2.7553$ and $0 \le \theta \le 2\pi$, two solutions must be possible, since $\tan \theta = 2.7553$ in two of the four quadrants.
b) Given $\tan \theta = 2.7553$ and $0 \le \theta \le 2\pi$, the solutions for θ must occur in the first and third quadrants, since the tangent function is positive in these quadrants.
c) Given $\tan \theta = 2.7553$ and $0 \le \theta \le 2\pi$, the related acute angle for the equation must be $\theta = 1.22$, since $\tan 1.22 = 2.7553$.
d) Given $\tan \theta = 2.7553$ and $0 \le \theta \le 2\pi$, the solutions to the equation must be $\theta = 1.22$ or 4.36, since $\tan 1.22 = 2.7553$ and $\tan 4.36 = 2.7553$.
6. a) Since $\tan \frac{\pi}{4} = 1$ and $\tan \frac{5\pi}{4} = 1$, the solutions to the equation $\tan \theta = 1$ are $\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.
b) Since $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}$, the solutions solutions to the equation $\sin \theta = \frac{1}{\sqrt{2}}$ are $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$.

c) Since $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\cos \frac{11\pi}{6} = \frac{\sqrt{3}}{2}$, the solutions to the equation $\cos \theta = \frac{\sqrt{3}}{2}$ are $\theta = \frac{\pi}{6}$ or $\frac{11\pi}{6}$.
d) Since $\sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$ and $\sin \frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$, the solutions to the equation $\sin \theta = -\frac{\sqrt{3}}{2}$ are $\theta = \frac{4\pi}{3}$ or $\frac{5\pi}{3}$.
e) Since $\cos \frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$ and $\cos \frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$, the solutions to the equation $\cos \theta = -\frac{1}{\sqrt{2}}$ are $\theta = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$.
f) Since $\tan \frac{\pi}{3} = \sqrt{3}$ and $\tan \frac{4\pi}{3} = \sqrt{3}$, the solutions to the equation $\tan \theta = \sqrt{3}$ are $\theta = \frac{\pi}{3}$ or $\frac{4\pi}{3}$.
7. a) The equation $2 \sin \theta = -1$ can be rewritten as follows:

$$2 \sin \theta = -1$$
$$\frac{2 \sin \theta}{2} = -\frac{1}{2}$$
$$\sin \theta = -\frac{1}{2}$$

Given $\sin \theta = -\frac{1}{2}$ and $0° \le \theta \le 360°$, the solutions to the equation must be $\theta = 210°$ or $330°$, since $\sin 210° = -\frac{1}{2}$ and $\sin 330° = -\frac{1}{2}$.
b) The equation $3 \cos \theta = -2$ can be rewritten as follows:

$$3 \cos \theta = -2$$
$$\frac{3 \cos \theta}{3} = -\frac{2}{3}$$
$$\cos \theta = -\frac{2}{3}$$

Given $\cos \theta = -\frac{2}{3}$ and $0° \le \theta \le 360°$, the solutions to the equation must be $\theta = 131.8°$ or $228.2°$.
c) The equation $2 \tan \theta = 3$ can be rewritten as follows:

$$2 \tan \theta = 3$$
$$\frac{2 \tan \theta}{2} = \frac{3}{2}$$
$$\tan \theta = \frac{3}{2}$$

Given $\tan \theta = \frac{3}{2}$ and $0° \le \theta \le 360°$, the solutions to the equation must be $\theta = 56.3°$ or $236.3°$.

d) The equation $-3\sin\theta - 1 = 1$ can be rewritten as follows:

$$-3\sin\theta - 1 = 1$$
$$-3\sin\theta - 1 + 1 = 1 + 1$$
$$-3\sin\theta = 2$$
$$\frac{-3\sin\theta}{-3} = \frac{2}{-3}$$
$$\sin\theta = -\frac{2}{3}$$

Given $\sin\theta = -\frac{2}{3}$ and $0° \le \theta \le 360°$, the solutions to the equation must be $\theta = 221.8°$ or $318.2°$.

e) The equation $-5\cos\theta + 3 = 2$ can be rewritten as follows:

$$-5\cos\theta + 3 = 2$$
$$-5\cos\theta + 3 - 3 = 2 - 3$$
$$-5\cos\theta = -1$$
$$\frac{-5\cos\theta}{-5} = \frac{-1}{-5}$$
$$\cos\theta = \frac{1}{5}$$

Given $\cos\theta = \frac{1}{5}$ and $0° \le \theta \le 360°$, the solutions to the equation must be $\theta = 78.5°$ or $281.5°$.

f) The equation $8 - \tan\theta = 10$ can be rewritten as follows:

$$8 - \tan\theta = 10$$
$$8 - \tan\theta + \tan\theta = 10 + \tan\theta$$
$$8 = 10 + \tan\theta$$
$$8 - 10 = 10 + \tan\theta - 10$$
$$\tan\theta = -2$$

Given $\tan\theta = -2$ and $0° \le \theta \le 360°$, the solutions to the equation must be $\theta = 116.6°$ or $296.6°$.

8. a) The equation $3\sin x = \sin x + 1$ can be rewritten as follows:

$$3\sin x = \sin x + 1$$
$$3\sin x - \sin x = \sin x + 1 - \sin x$$
$$2\sin x = 1$$
$$\frac{2\sin x}{2} = \frac{1}{2}$$
$$\sin x = \frac{1}{2}$$

Given $\sin x = \frac{1}{2}$ and $0 \le x \le 2\pi$, the solutions to the equation must be $x = 0.52$ or 2.62.

b) The equation $5\cos x - \sqrt{3} = 3\cos x$ can be rewritten as follows:

$$5\cos x - \sqrt{3} = 3\cos x$$
$$5\cos x - \sqrt{3} - 3\cos x = 3\cos x - 3\cos x$$
$$2\cos x - \sqrt{3} = 0$$
$$2\cos x - \sqrt{3} + \sqrt{3} = 0 + \sqrt{3}$$
$$2\cos x = \sqrt{3}$$
$$\frac{2\cos x}{2} = \frac{\sqrt{3}}{2}$$
$$\cos x = \frac{\sqrt{3}}{2}$$

Given $\cos x = \frac{\sqrt{3}}{2}$ and $0 \le x \le 2\pi$, the solutions to the equation must be $x = 0.52$ or 5.76.

c) The equation $\cos x - 1 = -\cos x$ can be rewritten as follows:

$$\cos x - 1 = -\cos x$$
$$\cos x - 1 + \cos x = -\cos x + \cos x$$
$$2\cos x - 1 = 0$$
$$2\cos x - 1 + 1 = 0 + 1$$
$$2\cos x = 1$$
$$\frac{2\cos x}{2} = \frac{1}{2}$$
$$\cos x = \frac{1}{2}$$

Given $\cos x = \frac{1}{2}$ and $0 \le x \le 2\pi$, the solutions to the equation must be $x = 1.05$ or 5.24.

d) The equation $5\sin x + 1 = 3\sin x$ can be rewritten as follows:

$$5\sin x + 1 = 3\sin x$$
$$5\sin x + 1 - 3\sin x = 3\sin x - 3\sin x$$
$$2\sin x + 1 = 0$$
$$2\sin x + 1 - 1 = 0 - 1$$
$$2\sin x = -1$$
$$\frac{2\sin x}{2} = -\frac{1}{2}$$
$$\sin x = -\frac{1}{2}$$

Given $\sin x = -\frac{1}{2}$ and $0 \le x \le 2\pi$, the solutions to the equation must be $x = 3.67$ or 5.76.

9. a) The equation $2 - 2\cot x = 0$ can be rewritten as follows:

$$2 - 2\cot x = 0$$
$$2 - 2\cot x + 2\cot x = 0 + 2\cot x$$
$$2 = 2\cot x$$
$$\frac{2}{2} = \frac{2\cot x}{2}$$
$$\cot x = 1$$

Given $\cot x = 1$ and $0 \le x \le 2\pi$, the solutions to the equation must be $x = 0.79$ or 3.93.

b) The equation $\csc x - 2 = 0$ can be rewritten as follows:

$$\csc x - 2 = 0$$
$$\csc x - 2 + 2 = 0 + 2$$
$$\csc x = 2$$

Given $\csc x = 2$ and $0 \le x \le 2\pi$, the solutions to the equation must be $x = 0.52$ or 2.62.

c) The equation $7 \sec x = 7$ can be rewritten as follows:

$$7 \sec x = 7$$
$$\frac{7 \sec x}{7} = \frac{7}{7}$$
$$\sec x = 1$$

Given $\sec x = 1$ and $0 \le x \le 2\pi$, the solutions to the equation must be $x = 0$ or 6.28.

d) The equation $2 \csc x + 17 = 15 + \csc x$ can be rewritten as follows:

$$2 \csc x + 17 = 15 + \csc x$$
$$2 \csc x + 17 - \csc x = 15 + \csc x - \csc x$$
$$\csc x + 17 = 15$$
$$\csc x + 17 - 17 = 15 - 17$$
$$\csc x = -2$$

Given $\csc x = -2$ and $0 \le x \le 2\pi$, the solutions to the equation must be $x = 3.67$ or 5.76.

e) The equation $2 \sec x + 1 = 6$ can be rewritten as follows:

$$2 \sec x + 1 = 6$$
$$2 \sec x + 1 - 1 = 6 - 1$$
$$2 \sec x = 5$$
$$\frac{2 \sec x}{2} = \frac{5}{2}$$
$$\sec x = \frac{5}{2}$$

Given $\sec x = \frac{5}{2}$ and $0 \le x \le 2\pi$, the solutions to the equation must be $x = 1.16$ or 5.12.

f) The equation $8 + 4 \cot x = 10$ can be rewritten as follows:

$$8 + 4 \cot x = 10$$
$$8 + 4 \cot x - 8 = 10 - 8$$
$$4 \cot x = 2$$
$$\frac{4 \cot x}{4} = \frac{2}{4}$$
$$\cot x = \frac{1}{2}$$

Given $\cot x = \frac{1}{2}$ and $0 \le x \le 2\pi$, the solutions to the equation must be $x = 1.11$ or 4.25.

10. a) Given $\sin 2x = \frac{1}{\sqrt{2}}$ and $0 \le x \le 2\pi$, $2x$ must equal $0.79 + 2k\pi$ or $2.36 + 2k\pi$. Therefore, the solutions to the equation must be $x = \frac{0.79}{2} = 0.39$, $\frac{2.36}{2} = 1.18$, 3.53, or 4.32.

b) Given $\sin 4x = \frac{1}{2}$ and $0 \le x \le 2\pi$, $4x$ must equal $0.52 + 2k\pi$ or $2.62 + 2k\pi$. Therefore, the solutions to the equation must be $x = \frac{0.52}{4} = 0.13$, $\frac{2.62}{4} = 0.65$, 1.70, 2.23, 3.27, 3.80, 4.84, and 5.37.

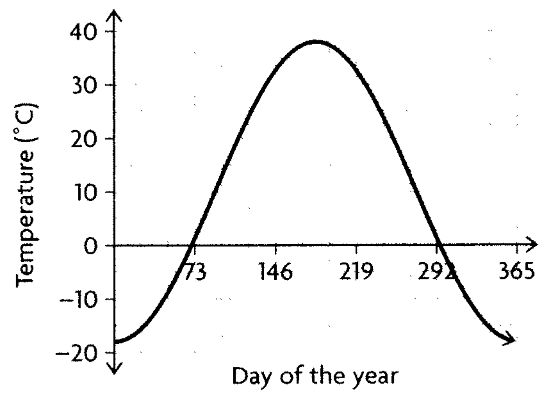
c) Given $\sin 3x = -\frac{\sqrt{3}}{2}$ and $0 \le x \le 2\pi$, $3x$ must equal $4.19 + 2k\pi$ or $5.24 + 2k\pi$. Therefore, the solutions to the equation must be $x = \frac{4.19}{3} = 1.40$, $\frac{5.24}{3} = 1.75$, 3.49, 3.84, 5.59, or 5.93.

d) Given $\cos 4x = -\frac{1}{\sqrt{2}}$ and $0 \le x \le 2\pi$, $4x$ must equal $2.36 + 2k\pi$ or $3.93 + 2k\pi$. Therefore, the solutions to the equation must be $x = \frac{2.36}{4} = 0.59$, $\frac{3.93}{4} = 0.985$, 2.16, 2.55, 3.73, 4.12, 5.30, or 5.697.

e) Given $\cos 2x = -\frac{1}{2}$ and $0 \le x \le 2\pi$, $2x$ must equal $2.09 + 2k\pi$ or $4.19 + 2k\pi$. Therefore, the solutions to the equation must be $x = \frac{2.09}{2} = 1.05$, $\frac{4.19}{2} = 2.09$, 4.19, or 5.24.

f) Given $\cos \frac{x}{2} = \frac{\sqrt{3}}{2}$ and $0 \le x \le 2\pi$, $\frac{x}{2}$ must equal 0.52 or 5.76. Therefore, the solution to the equation must be $x = (2)(0.52) = 1.05$. 5.76 is not a solution to the equation since $(2)(5.76) = 11.52$ is greater than 2π.

11. When graphed, the function modelling the city's daily high temperature is as follows:



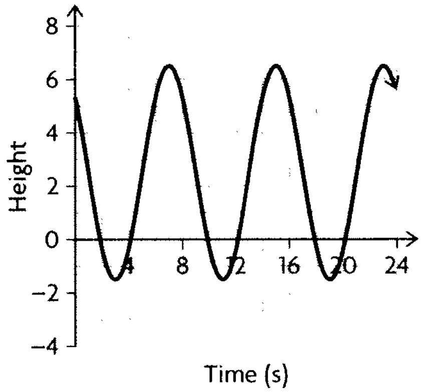
First it's necessary to find the first day when the temperature is approximately 32°C. This can be done as follows:

$$32 = -28 \cos \frac{2\pi}{365}d + 10$$
$$32 - 10 = -28 \cos \frac{2\pi}{365}d + 10 - 10$$
$$22 = -28 \cos \frac{2\pi}{365}d$$
$$\frac{22}{-28} = \frac{-28}{-28} \cos \frac{2\pi}{365}d$$
$$-0.7857 = \cos \frac{2\pi}{365}d$$
$$\cos^{-1}(-0.7857) = \cos^{-1}\left(\cos \frac{2\pi}{365}d\right)$$
$$2.4746 = \frac{2\pi}{365}d$$

$$2.4746 \times \frac{365}{2\pi} = \frac{2\pi}{365}d \times \frac{365}{2\pi}$$
$$d = 143.76$$

Since the first day when the temperature is approximately 32 °C is day 144, and since the period of the function is 365 days, the other day when the temperature is approximately 32 °C is $365 - 144 = 221$. Therefore, the temperature is approximately 32 °C or above from about day 144 to about day 221, and those are the days of the year when the air conditioners are running at the City Hall.

12. When graphed, the function modelling the height of the nail above the surface of the water is as follows:

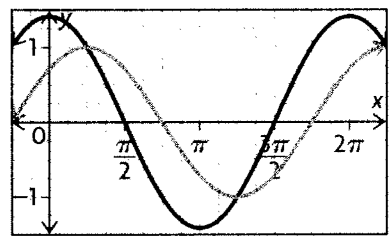


First it's necessary to find the first time when the nail is at the surface of the water. This can be done as follows:

$$0 = -4 \sin \frac{\pi}{4}(t - 1) + 2.5$$
$$0 - 2.5 = -4 \sin \frac{\pi}{4}(t - 1) + 2.5 - 2.5$$
$$-2.5 = -4 \sin \frac{\pi}{4}(t - 1)$$
$$\frac{-2.5}{-4} = \frac{-4}{-4} \sin \frac{\pi}{4}(t - 1)$$
$$0.625 = \sin \frac{\pi}{4}(t - 1)$$
$$\sin^{-1}(0.625) = \sin^{-1}\left(\sin \frac{\pi}{4}(t - 1)\right)$$
$$0.6751 = \frac{\pi}{4}(t - 1)$$
$$0.6751 \times \frac{4}{\pi} = \frac{\pi}{4}(t - 1) \times \frac{4}{\pi}$$
$$t - 1 = 0.8596$$
$$t - 1 + 1 = 0.8596 + 1$$
$$t = 1.86$$

Since the first time when the nail is at the surface of the water is 1.86 s, and since the period of the function is 8 s, the next time the nail is at the surface of the water is $6 - 1.86 = 4.14$ s. Therefore, the nail is below the water when $1.86 \text{ s} < t < 4.14$ s. Since the cycle repeats itself two more times in the first 24 s that the wheel is rotating, the nail is also below the water when $9.86 \text{ s} < t < 12.14$ and when $17.86 \text{ s} < t < 20.14$ s.

13. To solve $\sin\left(x + \frac{\pi}{4}\right) = \sqrt{2} \cos x$ for $0 \le x \le 2\pi$, graph the functions $y = \sin\left(x + \frac{\pi}{4}\right)$ and $y = \sqrt{2} \cos x$ on the same coordinate grid as follows:

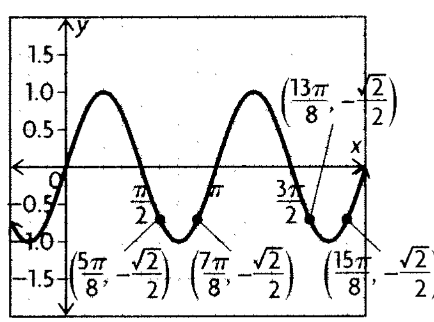


Since the graphs intersect when $x = \frac{\pi}{4}$ and when $x = \frac{5\pi}{4}$, the solution to the equation is $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

14. Given $\sin 2\theta = -\frac{1}{\sqrt{2}}$ and $0 \le x \le 2\pi$, 2θ must equal $\frac{5\pi}{4} + 2\pi k$ or $\frac{7\pi}{4} + 2\pi k$. Therefore, the solutions to the equation must be

$$\theta = \frac{\frac{5\pi}{4} + 2\pi k}{2} = \frac{5\pi}{8} + \pi k \text{ or}$$
$$\frac{\frac{7\pi}{4} + 2\pi k}{2} = \frac{7\pi}{8} + \pi k$$

So the solution are $\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}$, and $\frac{15\pi}{8}$. When the solutions are plotted on the graph of the function $y = \sin 2\theta$ for $0 \le \theta \le 2\pi$, the graph appears as follows:



15. The value of $f(x) = \sin x$ is the same at x and $\pi - x$. In other words, it is the same at x and half the period minus x. Since the period of $f(x) = 25 \sin \frac{\pi}{50}(x + 20) - 55$ is 100, if the function were not horizontally translated, its value at x would be the same as at $50 - x$. The function is horizontally translated 20 units to the left, however, so it goes through half its period from $x = -20$ to $x = 30$. At $x = 3$, the function is 23 units away from the left end of the range, so it will have the same value at $x = 30 - 23$ or $x = 7$, which is 23 units away from the right end of the range.

16. To solve a trigonometric equation **algebraically**, first isolate the trigonometric function on one side of the equation. For example, the trigonometric equation $5 \cos x - 3 = 2$ would become $5 \cos x = 5$, which would then become $\cos x = 1$. Next, apply the inverse of the trigonometric function to both sides of the equation. For example, the trigonometric equation $\cos x = 1$ would become $x = \cos^{-1} 1$. Finally, simplify the equation. For example, $x = \cos^{-1} 1$ would become $x = 0 + 2n\pi$, where $n \in \mathbf{I}$.

To solve a trigonometric equation **graphically**, first isolate the trigonometric function on one side of the equation. For example, the trigonometric equation $5 \cos x - 3 = 2$ would become $5 \cos x = 5$, which would then become $\cos x = 1$. Next, graph both sides of the equation. For example, the functions $f(x) = \cos x$ and $f(x) = 1$ would both be graphed. Finally, find the points where the two graphs intersect. For example, $f(x) = \cos x$ and $f(x) = 1$ would intersect at $x = 0 + 2n\pi$, where $n \in \mathbf{I}$.

Similarity: Both trigonometric functions are first isolated on one side of the equation.

Differences: The inverse of a trigonometric function is not applied in the graphical method, and the points of intersection are not obtained in the algebraic method.

17. To solve the trigonometric equation $2 \sin x \cos x + \sin x = 0$, first factor $\sin x$ from the left side of the equation as follows:

$2 \sin x \cos x + \sin x = 0$

$(\sin x)(2 \cos x + 1) = 0$

Since $(\sin x)(2 \cos x + 1) = 0$, either $\sin x = 0$ or $2 \cos x + 1 = 0$ (or both). If $\sin x = 0$, one solution to the equation is $x = 0 + n\pi$, where $n \in \mathbf{I}$. If $2 \cos x + 1 = 0$, x can be found by first rewriting the equation as follows:

$$2 \cos x + 1 = 0$$
$$2 \cos x + 1 - 1 = 0 - 1$$
$$2 \cos x = -1$$
$$\frac{2 \cos x}{2} = -\frac{1}{2}$$
$$\cos x = -\frac{1}{2}$$

The solutions to the equation $\cos x = -\frac{1}{2}$ occur at $x = \frac{2\pi}{3} + 2n\pi$ or $\frac{4\pi}{3} + 2n\pi$, where $n \in \mathbf{I}$, since the period of the cosine function is 2π. Therefore, the solutions to the equation $2 \sin x \cos x + \sin x = 0$ are $x = 0 + n\pi, \frac{2\pi}{3} + 2n\pi$, or $\frac{4\pi}{3} + 2n\pi$, where $n \in \mathbf{I}$.

18. a) Since $\sin 2\theta = 2 \sin \theta \cos \theta$, the equation $\sin 2x - 2 \cos^2 x = 0$ can be rewritten as follows:

$$\sin 2x - 2 \cos^2 x = 0$$
$$2 \sin x \cos x - 2 \cos^2 x = 0$$
$$(2 \cos x)(\sin x - \cos x) = 0$$

Since $(2 \cos x)(\sin x - \cos x) = 0$, either $2 \cos x = 0$ or $\sin x - \cos x = 0$ (or both). If $2 \cos x = 0$, x can be found by first rewriting the equation as follows:

$$2 \cos x = 0$$
$$\frac{2 \cos x}{2} = \frac{0}{2}$$
$$\cos x = 0$$

The solutions to the equation $\cos x = 0$ occur at $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. If $\sin x - \cos x = 0$, x can be found by first rewriting the equation as follows:

$$\sin x - \cos x = 0$$
$$\sin x - \cos x + \cos x = 0 + \cos x$$
$$\sin x = \cos x$$

The solutions to the equation $\sin x = \cos x$ occur at $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. Therefore, the solutions to the equation $\sin 2x - 2 \cos^2 x = 0$ are $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}$, or $\frac{3\pi}{2}$.

b) Since $\cos 2\theta = 1 - 2 \sin^2 \theta$, the equation $3 \sin x + \cos 2x = 2$ can be rewritten as follows:

$$3 \sin x + \cos 2x = 2$$
$$3 \sin x + 1 - 2 \sin^2 x = 2$$
$$3 \sin x + 1 - 2 \sin^2 x - 2 = 2 - 2$$
$$3 \sin x - 1 - 2 \sin^2 x = 0$$
$$-2 \sin^2 x + 3 \sin x - 1 = 0$$
$$(-1)(-2 \sin^2 x + 3 \sin x - 1) = (-1)(0)$$
$$2 \sin^2 x - 3 \sin x + 1 = 0$$
$$(2 \sin x - 1)(\sin x - 1) = 0$$

Since $(2 \sin x - 1)(\sin x - 1) = 0$, either $2 \sin x - 1 = 0$ or $\sin x - 1 = 0$ (or both).

If $\sin x - 1 = 0$, x can be found by first rewriting the equation as follows:

$$2 \sin x - 1 = 0$$
$$2 \sin x - 1 + 1 = 0 + 1$$
$$2 \sin x = 1$$
$$\frac{2 \sin x}{2} = \frac{1}{2}$$
$$\sin x = \frac{1}{2}$$

The solutions to the equation $\sin x = \frac{1}{2}$ occur at $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. If $\sin x - 1 = 0$, x can be found by first rewriting the equation as follows:

$$\sin x - 1 = 0$$
$$\sin x - 1 + 1 = 0 + 1$$
$$\sin x = 1$$

The solution to the equation $\sin x = 1$ occurs at $x = \frac{\pi}{2}$. Therefore, the solutions to the equation $3 \sin x + \cos 2x = 2$ are $x = \frac{\pi}{6}, \frac{\pi}{2}$, or $\frac{5\pi}{6}$.

7.6 Solving Quadratic Trigonometric Equations, pp. 435–437

1. a) The expression $\sin^2 \theta - \sin \theta$ can be factored as follows:

$\sin^2 \theta - \sin \theta$

$(\sin \theta)(\sin \theta - 1)$

b) The expression $\cos^2 \theta - 2 \cos \theta + 1$ can be factored as follows:

$\cos^2 \theta - 2 \cos \theta + 1$

$(\cos \theta - 1)(\cos \theta - 1)$

c) The expression $3 \sin^2 \theta - \sin \theta - 2$ can be factored as follows:

$(3 \sin \theta + 2)(\sin \theta - 1)$

d) The expression $4 \cos^2 \theta - 1$ can be factored as follows:

$4 \cos^2 \theta - 1$

$(2 \cos \theta - 1)(2 \cos \theta + 1)$

e) The expression $24 \sin^2 x - 2 \sin x - 2$ can be factored as follows:

$24 \sin^2 x - 2 \sin x - 2$

$(6 \sin x - 2)(4 \sin x + 1)$

f) The expression $49 \tan^2 x - 64$ can be factored as follows:

$49 \tan^2 x - 64$

$(7 \tan x + 8)(7 \tan x - 8)$

2. a) The equation $y^2 = \frac{1}{3}$ can be solved as follows:

$$y^2 = \frac{1}{3}$$
$$\sqrt{y^2} = \pm\sqrt{\frac{1}{3}}$$
$$y = \pm\sqrt{\frac{1}{3}}$$
$$y = \pm\frac{\sqrt{3}}{3}$$

Likewise, the equation $\tan^2 x = \frac{1}{3}$ can be solved as follows:

$$\tan^2 x = \frac{1}{3}$$
$$\sqrt{\tan^2 x} = \pm\sqrt{\frac{1}{3}}$$
$$\tan x = \pm\sqrt{\frac{1}{3}}$$
$$\tan x = \pm\frac{\sqrt{3}}{3}$$

The solutions to the equation $\tan x = \pm\frac{\sqrt{3}}{3}$ occur at $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$.

b) The equation $y^2 + y = 0$ can be solved as follows:

$$y^2 + y = 0$$
$$(y)(y + 1) = 0$$

Since $(y)(y + 1) = 0$, either $y = 0$ or $y + 1 = 0$ (or both). If $y + 1 = 0$, y can be solved for as follows:

$$y + 1 = 0$$
$$y + 1 - 1 = 0 - 1$$
$$y = -1$$

Therefore, the solutions to the equation $y^2 + y = 0$ are $y = 0$ or $y = -1$. Likewise, the equation $\sin^2 x + \sin x = 0$ can be solved as follows:

$$\sin^2 x + \sin x = 0$$
$$(\sin x)(\sin x + 1) = 0$$

Since $(\sin x)(\sin x + 1) = 0$, either $\sin x = 0$ or $\sin x + 1 = 0$ (or both). The solutions to the equation $\sin x = 0$ occur at $x = 0, \pi$, or 2π. If $\sin x + 1 = 0$, x can be solved for as follows:

$$\sin x + 1 = 0$$
$$\sin x + 1 - 1 = 0 - 1$$
$$\sin x = -1$$

The solution to the equation $\sin x = -1$ occurs at $x = \frac{3\pi}{2}$, so the solutions to the equation $\sin^2 x + \sin x = 0$ are $x = 0, \pi, \frac{3\pi}{2}$, or 2π.

c) The equation $y - 2yz = 0$ can be solved as follows:

$y - 2yz = 0$
$(y)(1 - 2z) = 0$

Since $(y)(1 - 2z) = 0$, either $y = 0$ or $1 - 2z = 0$ (or both). If $1 - 2z = 0$, z can be solved for as follows:

$1 - 2z = 0$
$1 - 2z + 2z = 0 + 2z$
$1 = 2z$
$\frac{1}{2} = \frac{2z}{2}$
$z = \frac{1}{2}$

Likewise, the equation $\cos x - 2\cos x \sin x = 0$ can be solved as follows:

$\cos x - 2\cos x \sin x = 0$
$(\cos x)(1 - 2\sin x) = 0$

Since $(\cos x)(1 - 2\sin x) = 0$, either $\cos x = 0$ or $1 - 2\sin x = 0$ (or both). The solutions to the equation $\cos x = 0$ occur at $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. If $1 - 2\sin x = 0$, x can be solved for as follows:

$1 - 2\sin x = 0$
$1 - 2\sin x + 2\sin x = 0 + 2\sin x$
$1 = 2\sin x$
$\frac{1}{2} = \frac{2\sin x}{2}$
$\sin x = \frac{1}{2}$

The solutions to the equation $\sin x = \frac{1}{2}$ occur at $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$, so the solutions to the equation $\cos x - 2\cos x \sin x = 0$ are $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, or $\frac{3\pi}{2}$.

d) The equation $yz = y$ can be solved as follows:

$yz = y$
$yz - y = y - y$
$yz - y = 0$
$(y)(z - 1) = 0$

Since $(y)(z - 1) = 0$, either $y = 0$ or $z - 1 = 0$ (or both). If $z - 1 = 0$, z can be solved for as follows:

$z - 1 = 0$
$z - 1 + 1 = 0 + 1$
$z = 1$

Therefore, the solutions to the equation $yz = y$ are $y = 0$ or $z = 1$. Likewise, the equation $\tan x \sec x = \tan x$ can be solved as follows:

$\tan x \sec x = \tan x$
$\tan x \sec x - \tan x = \tan x - \tan x$
$\tan x \sec x - \tan x = 0$
$(\tan x)(\sec x - 1) = 0$

Since $(\tan x)(\sec x - 1) = 0$, either $\tan x = 0$ or $\sec x - 1 = 0$ (or both). The solutions to the equation $\tan x = 0$ occur at $x = 0$, π, or 2π. If $\sec x - 1 = 0$, x can be solved for as follows:

$\sec x - 1 = 0$
$\sec x - 1 + 1 = 0 + 1$
$\sec x = 1$

The solutions to the equation $\sec x = 1$ occur at $x = 0$ or 2π, so the solutions to the equation $(\tan x)(\sec x - 1) = 0$ are $x = 0$, π, or 2π.

3. a) The equation $6y^2 - y - 1 = 0$ can be solved as follows:

$6y^2 - y - 1 = 0$
$(2y - 1)(3y + 1) = 0$

Since $(2y - 1)(3y + 1) = 0$, either $2y - 1 = 0$ or $3y + 1 = 0$ (or both). If $2y - 1 = 0$, y can be solved for as follows:

$2y - 1 = 0$
$2y - 1 + 1 = 0 + 1$
$2y = 1$
$\frac{2y}{2} = \frac{1}{2}$
$y = \frac{1}{2}$

If $3y + 1 = 0$, y can be solved for as follows:

$3y + 1 = 0$
$3y + 1 - 1 = 0 - 1$
$3y = -1$
$\frac{3y}{3} = -\frac{1}{3}$
$y = -\frac{1}{3}$

Therefore, the solutions to the equation $6y^2 - y - 1 = 0$ are $y = \frac{1}{2}$ or $-\frac{1}{3}$.

b) The equation $6\cos^2 x - \cos x - 1 = 0$ for $0 \le x \le 2\pi$ can be solved as follows:

$6\cos^2 x - \cos x - 1 = 0$
$(2\cos x - 1)(3\cos x + 1) = 0$

Since $(2\cos x - 1)(3\cos x + 1) = 0$, either $2\cos x - 1 = 0$ or $3\cos x + 1 = 0$ (or both). If $2\cos x - 1 = 0$, x can be solved for as follows:

$2\cos x - 1 = 0$
$2\cos x - 1 + 1 = 0 + 1$

$$2\cos x = 1$$
$$\frac{2\cos x}{2} = \frac{1}{2}$$
$$\cos x = \frac{1}{2}$$
The solutions to the equation $\cos x = \frac{1}{2}$ occur at $x = 1.05$ or 5.24. If $3\cos x + 1 = 0$, x can be solved for as follows:
$$3\cos x + 1 = 0$$
$$3\cos x + 1 - 1 = 0 - 1$$
$$3\cos x = -1$$
$$\frac{3\cos x}{3} = -\frac{1}{3}$$
$$\cos x = -\frac{1}{3}$$
The solutions to the equation $\cos x = -\frac{1}{3}$ occur at $x = 1.91$ or 4.37. Therefore, the solutions to the equation $6\cos^2 x - \cos x - 1 = 0$ are $x = 1.05$, 1.91, 4.37, or 5.24.

4. a) The equation $\sin^2\theta = 1$ can be solved as follows:
$$\sin^2\theta = 1$$
$$\sqrt{\sin^2\theta} = \pm\sqrt{1}$$
$$\sin\theta = \pm 1$$
The solutions to the equation $\sin\theta = \pm 1$ occur at $\theta = 90°$ or $270°$.

b) The equation $\cos^2\theta = 1$ can be solved as follows:
$$\cos^2\theta = 1$$
$$\sqrt{\cos^2\theta} = \pm\sqrt{1}$$
$$\cos\theta = \pm 1$$
The solutions to the equation $\cos\theta = \pm 1$ occur at $\theta = 0°, 180°$, or $360°$.

c) The equation $\tan^2\theta = 1$ can be solved as follows:
$$\tan^2\theta = 1$$
$$\sqrt{\tan^2\theta} = \pm\sqrt{1}$$
$$\tan\theta = \pm 1$$
The solutions to the equation $\tan\theta = \pm 1$ occur at $\theta = 45°, 135°, 225°$, or $315°$.

d) The equation $4\cos^2\theta = 1$ can be solved as follows:
$$4\cos^2\theta = 1$$
$$\frac{4\cos^2\theta}{4} = \frac{1}{4}$$
$$\cos^2\theta = \frac{1}{4}$$
$$\sqrt{\cos^2\theta} = \pm\sqrt{\frac{1}{4}}$$
$$\cos\theta = \pm\frac{1}{2}$$
The solutions to the equation $\cos\theta = \pm\frac{1}{2}$ occur at $\theta = 60°, 120°, 240°$, or $300°$.

e) The equation $3\tan^2\theta = 1$ can be solved as follows:
$$3\tan^2\theta = 1$$
$$\frac{3\tan^2\theta}{3} = \frac{1}{3}$$
$$\tan^2\theta = \frac{1}{3}$$
$$\sqrt{\tan^2\theta} = \pm\sqrt{\frac{1}{3}}$$
$$\tan\theta = \pm\frac{\sqrt{3}}{3}$$
The solutions to the equation $\tan\theta = \pm\frac{\sqrt{3}}{3}$ occur at $\theta = 30°, 150°, 210°$, or $330°$.

f) The equation $2\sin^2\theta = 1$ can be solved as follows:
$$2\sin^2\theta = 1$$
$$\frac{2\sin^2\theta}{2} = \frac{1}{2}$$
$$\sin^2\theta = \frac{1}{2}$$
$$\sqrt{\sin^2\theta} = \pm\sqrt{\frac{1}{2}}$$
$$\sin\theta = \pm\frac{\sqrt{2}}{2}$$
The solutions to the equation $\sin\theta = \pm\frac{\sqrt{2}}{2}$ occur at $\theta = 45°, 135°, 225°$, or $315°$.

5. a) Since $\sin x\cos x = 0$, either $\sin x = 0$ or $\cos x = 0$ (or both). If $\sin x = 0$, the solutions for x occur at $x = 0°, 180°$, or $360°$. If $\cos x = 0$, the solutions for x occur at $x = 90°$ or $270°$. Therefore, the solutions to the equation occur at $x = 0°, 90°, 180°, 270°$, or $360°$.

b) Since $\sin x(\cos x - 1) = 0$, either $\sin x = 0$ or $\cos x - 1 = 0$ (or both). If $\sin x = 0$, the solutions for x occur at $x = 0°, 180°$, or $360°$. If $\cos x - 1 = 0$, x can be solved for as follows:
$$\cos x - 1 = 0$$
$$\cos x - 1 + 1 = 0 + 1$$
$$\cos x = 1$$
The solutions to the equation $\cos x = 1$ occur at $x = 0°$ or $360°$. Therefore, the solutions to the equation $\sin x(\cos x - 1) = 0$ occur at $x = 0°$, $180°$, or $360°$.

c) Since $(\sin x + 1)\cos x = 0$, either $\sin x + 1 = 0$ or $\cos x = 0$ (or both). If $\sin x + 1 = 0$, x can be solved for as follows:

$$\sin x + 1 = 0$$
$$\sin x + 1 - 1 = 0 - 1$$
$$\sin x = -1$$

The solution to the equation $\sin x = -1$ occurs at $x = 270°$. If $\cos x = 0$, the solutions for x occur at $x = 90°$ or $270°$. Therefore, the solutions to the equation $(\sin x + 1)\cos x = 0$ occur at $x = 90°$ or $270°$.

d) Since $\cos x\,(2\sin x - \sqrt{3}) = 0$, either $\cos x = 0$ or $2\sin x - \sqrt{3} = 0$ (or both). If $\cos x = 0$, the solutions for x occur at $x = 90°$ or $270°$. If $2\sin x - \sqrt{3} = 0$, x can be solved for as follows:

$$2\sin x - \sqrt{3} = 0$$
$$2\sin x - \sqrt{3} + \sqrt{3} = 0 + \sqrt{3}$$
$$2\sin x = \sqrt{3}$$
$$\frac{2\sin x}{2} = \frac{\sqrt{3}}{2}$$
$$\sin x = \frac{\sqrt{3}}{2}$$

The solutions to the equation $\sin x = \frac{\sqrt{3}}{2}$ occur at $x = 60°$ and $120°$. Therefore, the solutions to the equation $\cos x(2\sin x - \sqrt{3}) = 0$ occur at $x = 60°$, $90°$, $120°$, or $270°$.

e) Since $(\sqrt{2}\sin x - 1)(\sqrt{2}\sin x + 1) = 0$, either $\sqrt{2}\sin x - 1 = 0$ or $\sqrt{2}\sin x + 1 = 0$ (or both). If $\sqrt{2}\sin x - 1 = 0$, x can be solved for as follows:

$$\sqrt{2}\sin x - 1 = 0$$
$$\sqrt{2}\sin x - 1 + 1 = 0 + 1$$
$$\sqrt{2}\sin x = 1$$
$$\frac{\sqrt{2}\sin x}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$
$$\sin x = \frac{1}{\sqrt{2}}$$
$$\sin x = \frac{\sqrt{2}}{2}$$

The solutions to the equation $\sin x = \frac{\sqrt{2}}{2}$ occur at $x = 45°$ or $135°$. If $\sqrt{2}\sin x + 1 = 0$, x can be solved for as follows:

$$\sqrt{2}\sin x + 1 = 0$$
$$\sqrt{2}\sin x + 1 - 1 = 0 - 1$$
$$\sqrt{2}\sin x = -1$$
$$\frac{\sqrt{2}\sin x}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$
$$\sin x = -\frac{1}{\sqrt{2}}$$
$$\sin x = -\frac{\sqrt{2}}{2}$$

The solutions to the equation $\sin x = -\frac{\sqrt{2}}{2}$ occur at $x = 225°$ or $315°$. Therefore, the solutions to the equation $(\sqrt{2}\sin x - 1)(\sqrt{2}\sin x + 1) = 0$ occur at $x = 45°$, $135°$, $225°$, or $315°$.

f) Since $(\sin x - 1)(\cos x + 1) = 0$, either $\sin x - 1 = 0$ or $\cos x + 1 = 0$ (or both). If $\sin x - 1 = 0$, x can be solved for as follows:

$$\sin x - 1 = 0$$
$$\sin x - 1 + 1 = 0 + 1$$
$$\sin x = 1$$

The solution to the equation $\sin x = 1$ occurs at $x = 90°$. If $\cos x + 1 = 0$, x can be solved for as follows:

$$\cos x + 1 = 0$$
$$\cos x + 1 - 1 = 0 - 1$$
$$\cos x = -1$$

The solution to the equation $\cos x = -1$ occurs at $x = 180°$. Therefore, the solutions to the equation $(\sin x - 1)(\cos x + 1) = 0$ occur at $x = 90°$ or $180°$.

6. a) Since $(2\sin x - 1)\cos x = 0$, either $2\sin x - 1 = 0$ or $\cos x = 0$. If $2\sin x - 1 = 0$, x can be solved for as follows:

$$2\sin x - 1 = 0$$
$$2\sin x - 1 + 1 = 0 + 1$$
$$2\sin x = 1$$
$$\frac{2\sin x}{2} = \frac{1}{2}$$
$$\sin x = \frac{1}{2}$$

The solutions to the equation $\sin x = \frac{1}{2}$ occur at $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. Also, the solutions to the equation $\cos x = 0$ occur at $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Therefore, the solutions to the equation $(2\sin x - 1)\cos x = 0$ occur at $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, or $\frac{3\pi}{2}$.

b) Since $(\sin x + 1)^2 = 0$, $\sin x + 1 = 0$, so x can be solved for as follows:

$$\sin x + 1 = 0$$
$$\sin x + 1 - 1 = 0 - 1$$
$$\sin x = -1$$

The solution to the equation $\sin x = -1$ occurs at $x = \frac{3\pi}{2}$, so the solution to the equation $(\sin x + 1)^2 = 0$ occurs at $x = \frac{3\pi}{2}$.

c) Since $(2\cos x + \sqrt{3})\sin x = 0$, either $2\cos x + \sqrt{3} = 0$ or $\sin x = 0$ (or both). If $2\cos x + \sqrt{3} = 0$, x can be solved for as follows:

$$2\cos x + \sqrt{3} = 0$$
$$2\cos x + \sqrt{3} - \sqrt{3} = 0 - \sqrt{3}$$
$$2\cos x = -\sqrt{3}$$
$$\frac{2\cos x}{2} = -\frac{\sqrt{3}}{2}$$
$$\cos x = -\frac{\sqrt{3}}{2}$$

The solutions to the equation $\cos x = -\frac{\sqrt{3}}{2}$ occur at $x = \frac{5\pi}{6}$ or $\frac{7\pi}{6}$. Also, the solutions to the equation $\sin x = 0$ occur at $x = 0$, π, or 2π. Therefore, the solutions to the equation $(2\cos x + \sqrt{3})\sin x = 0$ occur at $x = 0, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}$, or 2π.

d) Since $(2\cos x - 1)(2\sin x + \sqrt{3}) = 0$, either $2\cos x - 1 = 0$ or $2\sin x + \sqrt{3} = 0$. If $2\cos x - 1 = 0$, x can be solved for as follows:

$$2\cos x - 1 = 0$$
$$2\cos x - 1 + 1 = 0 + 1$$
$$2\cos x = 1$$
$$\frac{2\cos x}{2} = \frac{1}{2}$$
$$\cos x = \frac{1}{2}$$

The solutions to the equation $\cos x = \frac{1}{2}$ occur at $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$. If $2\sin x + \sqrt{3} = 0$, x can be solved for as follows:

$$2\sin x + \sqrt{3} = 0$$
$$2\sin x + \sqrt{3} - \sqrt{3} = 0 - \sqrt{3}$$
$$2\sin x = -\sqrt{3}$$
$$\frac{2\sin x}{2} = -\frac{\sqrt{3}}{2}$$
$$\sin x = -\frac{\sqrt{3}}{2}$$

The solutions to the equation $\sin x = -\frac{\sqrt{3}}{2}$ occur at $x = \frac{4\pi}{3}$ or $\frac{5\pi}{3}$. Therefore, the solutions to the equation $(2\cos x - 1)(2\sin x + \sqrt{3}) = 0$ occur at $x = \frac{\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$.

e) Since $(\sqrt{2}\cos x - 1)(\sqrt{2}\cos x + 1) = 0$, either $\sqrt{2}\cos x - 1 = 0$ or $\sqrt{2}\cos x + 1 = 0$ (or both). If $\sqrt{2}\cos x - 1 = 0$, x can be solved for as follows:

$$\sqrt{2}\cos x - 1 = 0$$
$$\sqrt{2}\cos x - 1 + 1 = 0 + 1$$
$$\sqrt{2}\cos x = 1$$
$$\frac{\sqrt{2}\cos x}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$
$$\cos x = \frac{1}{\sqrt{2}}$$
$$\cos x = \frac{\sqrt{2}}{2}$$

The solutions to the equation $\cos x = \frac{\sqrt{2}}{2}$ occur at $x = \frac{\pi}{4}$ or $\frac{7\pi}{4}$. If $\sqrt{2}\cos x + 1 = 0$, x can be solved for as follows:

$$\sqrt{2}\cos x + 1 = 0$$
$$\sqrt{2}\cos x + 1 - 1 = 0 - 1$$
$$\sqrt{2}\cos x = -1$$
$$\frac{\sqrt{2}\cos x}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$
$$\cos x = -\frac{1}{\sqrt{2}}$$
$$\cos x = -\frac{\sqrt{2}}{2}$$

The solutions to the equation $\cos x = -\frac{\sqrt{2}}{2}$ occur at $x = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$. Therefore, the solutions to the equation $(\sqrt{2}\cos x - 1)(\sqrt{2}\cos x + 1) = 0$ occur at $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$.

f) Since $(\sin x + 1)(\cos x - 1) = 0$, either $\sin x + 1 = 0$ or $\cos x - 1 = 0$ (or both). If $\sin x + 1 = 0$, x can be solved for as follows:

$$\sin x + 1 = 0$$
$$\sin x + 1 - 1 = 0 - 1$$
$$\sin x = -1$$

The solution to the equation $\sin x = -1$ occurs at $x = \frac{3\pi}{2}$. If $\cos x - 1 = 0$, x can be solved for as follows:

$$\cos x - 1 = 0$$
$$\cos x - 1 + 1 = 0 + 1$$
$$\cos x = 1$$

The solutions to the equation $\cos x = 1$ occur at $x = 0$ or 2π. Therefore, the solutions to the equation $(\sin x + 1)(\cos x - 1) = 0$ occur at $x = 0, \frac{3\pi}{2}$, or 2π.

7. a) Since $2\cos^2\theta + \cos\theta - 1 = 0$, $(2\cos\theta - 1)(\cos\theta + 1) = 0$. For this reason, either $2\cos\theta - 1 = 0$ or $\cos\theta + 1 = 0$ (or both). If $2\cos\theta - 1 = 0$, θ can be solved for as follows:

$$2\cos\theta - 1 = 0$$
$$2\cos\theta - 1 + 1 = 0 + 1$$
$$2\cos\theta = 1$$
$$\frac{2\cos\theta}{2} = \frac{1}{2}$$
$$\cos\theta = \frac{1}{2}$$

The solutions to the equation $\cos\theta = \frac{1}{2}$ occur at $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$. If $\cos\theta + 1 = 0$, θ can be solved for as follows:

$$\cos\theta + 1 = 0$$
$$\cos\theta + 1 - 1 = 0 - 1$$
$$\cos\theta = -1$$

The solution to the equation $\cos\theta = -1$ occurs at $\theta = \pi$. Therefore, the solutions to the equation $2\cos^2\theta + \cos\theta - 1 = 0$ occur at $\theta = \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3}$.

b) The equation $2\sin^2\theta = 1 - \sin\theta$ can be rewritten and factored as follows:

$$2\sin^2\theta = 1 - \sin\theta$$
$$2\sin^2\theta - 1 + \sin\theta = 1 - \sin\theta - 1 + \sin\theta$$
$$2\sin^2\theta + \sin\theta - 1 = 0$$
$$(2\sin\theta - 1)(\sin\theta + 1) = 0$$

Since $(2\sin\theta - 1)(\sin\theta + 1) = 0$, either $2\sin\theta - 1 = 0$ or $\sin\theta + 1 = 0$ (or both). If $2\sin\theta - 1 = 0$, θ can be solved for as follows:

$$2\sin\theta - 1 = 0$$
$$2\sin\theta - 1 + 1 = 0 + 1$$
$$2\sin\theta = 1$$
$$\frac{2\sin\theta}{2} = \frac{1}{2}$$
$$\sin\theta = \frac{1}{2}$$

The solutions to the equation $\sin\theta = \frac{1}{2}$ occur at $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. If $\sin\theta + 1 = 0$, θ can be solved for as follows:

$$\sin\theta + 1 = 0$$
$$\sin\theta + 1 - 1 = 0 - 1$$
$$\sin\theta = -1$$

The solution to the equation $\sin\theta = -1$ occurs at $x = \frac{3\pi}{2}$. Therefore, the solutions to the equation $2\sin^2\theta = 1 - \sin\theta$ occur at $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$, or $\frac{3\pi}{2}$.

c) The equation $\cos^2\theta = 2 + \cos\theta$ can be rewritten and factored as follows:

$$\cos^2\theta = 2 + \cos\theta$$
$$\cos^2\theta - 2 - \cos\theta = 2 + \cos\theta - 2 - \cos\theta$$
$$\cos^2\theta - \cos\theta - 2 = 0$$
$$(\cos\theta - 2)(\cos\theta + 1) = 0$$

Since $(\cos\theta - 2)(\cos\theta + 1) = 0$, either $\cos\theta - 2 = 0$ or $\cos\theta + 1 = 0$. If $\cos\theta - 2 = 0$, the equation can be rewritten as follows:

$$\cos\theta - 2 = 0$$
$$\cos\theta - 2 + 2 = 0 + 2$$
$$\cos\theta = 2$$

Since $\cos\theta$ can never equal 2, the factor $\cos\theta - 2$ can be ignored. If $\cos\theta + 1 = 0$, θ can be solved for as follows:

$$\cos\theta + 1 = 0$$
$$\cos\theta + 1 - 1 = 0 - 1$$
$$\cos\theta = -1$$

The solution to the equation $\cos\theta = -1$ occurs at $\theta = \pi$. Therefore, the solution to the equation $\cos^2\theta = 2 + \cos\theta$ occurs at $\theta = \pi$.

d) Since $2\sin^2\theta + 5\sin\theta - 3 = 0$, $(2\sin\theta - 1)(\sin\theta + 3) = 0$. For this reason, either $2\sin\theta - 1 = 0$ or $\sin\theta + 3 = 0$. If $2\sin\theta - 1 = 0$, θ can be solved for as follows:

$$2\sin\theta - 1 = 0$$
$$2\sin\theta - 1 + 1 = 0 + 1$$
$$2\sin\theta = 1$$
$$\frac{2\sin\theta}{2} = \frac{1}{2}$$
$$\sin\theta = \frac{1}{2}$$

The solutions to the equation $\sin\theta = \frac{1}{2}$ occur at $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. If $\sin\theta + 3 = 0$, the equation can be rewritten as follows:

$$\sin\theta + 3 = 0$$
$$\sin\theta + 3 - 3 = 0 - 3$$
$$\sin\theta = -3$$

Since $\sin\theta$ can never equal -3, the factor $\sin\theta + 3$ can be ignored. Therefore, the solutions to the equation $2\sin^2\theta + 5\sin\theta - 3 = 0$ occur at $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

e) The equation $3\tan^2\theta - 2\tan\theta = 1$ can be rewritten and factored as follows:

$$3\tan^2\theta - 2\tan\theta = 1$$
$$3\tan^2\theta - 2\tan\theta - 1 = 1 - 1$$
$$3\tan^2\theta - 2\tan\theta - 1 = 0$$
$$(3\tan\theta + 1)(\tan\theta - 1) = 0$$

Since $(3\tan\theta + 1)(\tan\theta - 1) = 0$, either $3\tan\theta + 1 = 0$ or $\tan\theta - 1 = 0$ (or both). If $3\tan\theta + 1 = 0$, θ can be solved for as follows:

$$3\tan\theta + 1 = 0$$
$$3\tan\theta + 1 - 1 = 0 - 1$$
$$3\tan\theta = -1$$
$$\frac{3\tan\theta}{3} = -\frac{1}{3}$$
$$\tan\theta = -\frac{1}{3}$$

The solutions to the equation $\tan\theta = -\frac{1}{3}$ occur at $\theta = 2.81$ or 5.96. If $\tan\theta - 1 = 0$, θ can be solved for as follows:

$$\tan\theta - 1 = 0$$
$$\tan\theta - 1 + 1 = 0 + 1$$
$$\tan\theta = 1$$

The solutions to the equation $\tan\theta = 1$ occur at $\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. Therefore, the solutions to the equation $3\tan^2\theta - 2\tan\theta = 1$ occur at $\theta = \frac{\pi}{4}, 2.82, \frac{5\pi}{4}$, or 5.96.

f) Since $12\sin^2\theta + \sin\theta - 6 = 0$, $(4\sin\theta + 3)(3\sin\theta - 2) = 0$. For this reason, either $4\sin\theta + 3 = 0$ or $3\sin\theta - 2 = 0$ (or both). If $4\sin\theta + 3 = 0$, θ can be solved for as follows:

$$4\sin\theta + 3 = 0$$
$$4\sin\theta + 3 - 3 = 0 - 3$$
$$4\sin\theta = -3$$
$$\frac{4\sin\theta}{4} = -\frac{3}{4}$$
$$\sin\theta = -\frac{3}{4}$$

The solutions to the equation $\sin\theta = -\frac{3}{4}$ occur at $\theta = 3.99$ or 5.44. If $3\sin\theta - 2 = 0$, θ can be solved for as follows:

$$3\sin\theta - 2 = 0$$
$$3\sin\theta - 2 + 2 = 0 + 2$$
$$3\sin\theta = 2$$
$$\frac{3\sin\theta}{2} = \frac{2}{3}$$
$$\sin\theta = \frac{2}{3}$$

The solutions to the equation $\sin\theta = \frac{2}{3}$ occur at $\theta = 0.73$ or 2.41. Therefore, the solutions to the equation $12\sin^2\theta + \sin\theta - 6 = 0$ occur at $\theta = 0.73, 2.41, 3.99$, or 5.44.

8. a) Since $\sec x\csc x - 2\csc x = 0$, $(\csc x)(\sec x - 2) = 0$. For this reason, either $\csc x = 0$ or $\sec x - 2 = 0$. Since $\csc x$ can never equal 0, the factor $\csc x$ can be ignored. If $\sec x - 2 = 0$, x can be solved for as follows:

$$\sec x - 2 = 0$$
$$\sec x - 2 + 2 = 0 + 2$$
$$\sec x = 2$$

The solutions to the equation $\sec x = 2$ occur at $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$. Therefore, the solutions to the equation $\sec x\csc x - 2\csc x = 0$ occur at $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.

b) Since $3\sec^2 x - 4 = 0$, $(\sqrt{3}\sec x - 2)(\sqrt{3}\sec x + 2) = 0$. For this reason, either $\sqrt{3}\sec x - 2 = 0$ or $\sqrt{3}\sec x + 2 = 0$ (or both). If $\sqrt{3}\sec x - 2 = 0$, x can be solved for as follows:

$$\sqrt{3}\sec x - 2 = 0$$
$$\sqrt{3}\sec x - 2 + 2 = 0 + 2$$
$$\sqrt{3}\sec x = 2$$
$$\frac{\sqrt{3}\sec x}{\sqrt{3}} = \frac{2}{\sqrt{3}}$$
$$\sec x = \frac{2}{\sqrt{3}}$$
$$\sec x = \frac{2\sqrt{3}}{3}$$

The solutions to the equation $\sec x = \frac{2\sqrt{3}}{3}$ occur at $x = \frac{\pi}{6}$ or $\frac{11\pi}{6}$. If $\sqrt{3}\sec x + 2 = 0$, x can be solved for as follows:

$$\sqrt{3}\sec x + 2 = 0$$
$$\sqrt{3}\sec x + 2 - 2 = 0 - 2$$
$$\sqrt{3}\sec x = -2$$
$$\frac{\sqrt{3}\sec x}{\sqrt{3}} = -\frac{2}{\sqrt{3}}$$
$$\sec x = -\frac{2}{\sqrt{3}}$$
$$\sec x = -\frac{2\sqrt{3}}{3}$$

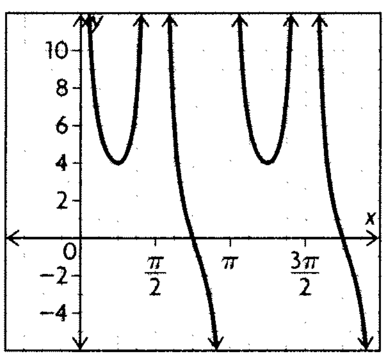
The solutions to the equation $\sec x = -\frac{2\sqrt{3}}{3}$ occur at $x = \frac{5\pi}{6}$ or $\frac{7\pi}{6}$. Therefore, the solutions to the equation $3\sec^2 x - 4 = 0$ occur at $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$.

c) Since $2\sin x \sec x - 2\sqrt{3}\sin x = 0$, $(\sin x)(2\sec x - 2\sqrt{3}) = 0$. For this reason, either $\sin x = 0$ or $2\sec x - 2\sqrt{3} = 0$ (or both). The solutions to the equation $\sin x = 0$ occur at $x = 0, \pi$, or 2π. If $2\sec x - 2\sqrt{3} = 0$, x can be solved for as follows:

$$2\sec x - 2\sqrt{3} = 0$$
$$2\sec x - 2\sqrt{3} + 2\sqrt{3} = 0 + 2\sqrt{3}$$
$$2\sec x = 2\sqrt{3}$$
$$\frac{2\sec x}{2} = \frac{2\sqrt{3}}{2}$$
$$\sec x = \sqrt{3}$$

The solutions to the equation $\sec x = \sqrt{3}$ occur at $x = 0.96$ or 5.33. Therefore, the solutions to the equation $2\sin x \sec x - 2\sqrt{3}\sin x = 0$ occur at $x = 0, 0.96, \pi, 5.33$, or 2π.

d) To solve the equation $2\cot x + \sec^2 x = 0$, graph the function $y = 2\cot x + \sec^2 x$ as follows:



Since the graph intersects the x-axis at $x = \frac{3\pi}{4}$ and $\frac{7\pi}{4}$, the solutions to the equation $2\cot x + \sec^2 x = 0$ occur at $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$.

e) The equation $\cot x \csc^2 x = 2\cot x$ can be rewritten and factored as follows:

$$\cot x \csc^2 x = 2\cot x$$
$$\cot x \csc^2 x - 2\cot x = 2\cot x - 2\cot x$$
$$\cot x \csc^2 x - 2\cot x = 0$$
$$(\cot x)(\csc x - \sqrt{2})(\csc x + \sqrt{2}) = 0$$

Since $(\cot x)(\csc x - \sqrt{2})(\csc x + \sqrt{2}) = 0$, either $\cot x = 0$, $\csc x - \sqrt{2} = 0$, or $\csc x + \sqrt{2} = 0$. The solutions to the equation $\cot x = 0$ occur at $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. If $\csc x - \sqrt{2} = 0$, x can be solved for as follows:

$$\csc x - \sqrt{2} = 0$$
$$\csc x - \sqrt{2} + \sqrt{2} = 0 + \sqrt{2}$$
$$\csc x = \sqrt{2}$$

The solutions to the equation $\csc x = \sqrt{2}$ occur at $x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$. If $\csc x + \sqrt{2} = 0$, x can be solved for as follows:

$$\csc x + \sqrt{2} = 0$$
$$\csc x + \sqrt{2} - \sqrt{2} = 0 - \sqrt{2}$$
$$\csc x = -\sqrt{2}$$

The solutions to the equation $\csc x = -\sqrt{2}$ occur at $x = \frac{5\pi}{4}$ or $\frac{7\pi}{4}$. Therefore, the solutions to the equation $\cot x \csc^2 x = 2\cot x$ occur at $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}$ or $\frac{7\pi}{4}$.

f) Since $3\tan^3 x - \tan x = 0$, $(\tan x)(\sqrt{3}\tan x - 1)(\sqrt{3}\tan x + 1) = 0$. For this reason, either $\tan x = 0$, $\sqrt{3}\tan x - 1 = 0$, or $\sqrt{3}\tan x + 1 = 0$. The solutions to the equation $\tan x = 0$ occur at $x = 0, \pi$, and 2π. If $\sqrt{3}\tan x - 1 = 0$, x can be solved for as follows:

$$\sqrt{3}\tan x - 1 = 0$$
$$\sqrt{3}\tan x - 1 + 1 = 0 + 1$$
$$\sqrt{3}\tan x = 1$$
$$\frac{\sqrt{3}\tan x}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$
$$\tan x = \frac{1}{\sqrt{3}}$$
$$\tan x = \frac{\sqrt{3}}{3}$$

The solutions to the equation $\tan x = \frac{\sqrt{3}}{3}$ occur at $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$. If $\sqrt{3}\tan x + 1 = 0$, x can be solved for as follows:

$$\sqrt{3}\tan x + 1 = 0$$
$$\sqrt{3}\tan x + 1 - 1 = 0 - 1$$
$$\sqrt{3}\tan x = -1$$
$$\frac{\sqrt{3}\tan x}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$$\tan x = -\frac{1}{\sqrt{3}}$$
$$\tan x = -\frac{\sqrt{3}}{3}$$

The solutions to the equation $\tan x = \frac{\sqrt{3}}{3}$ occur at $x = \frac{5\pi}{6}$ or $\frac{11\pi}{6}$. Therefore, the solutions to the equation $3\tan^3 x - \tan x = 0$ occur at $x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$, or 2π.

9. a) Since $\cos 2\theta = 2\cos^2\theta - 1$, the equation $5\cos 2x - \cos x + 3 = 0$ can be rewritten and factored as follows:

$$5\cos 2x - \cos x + 3 = 0$$
$$(5)(2\cos^2 x - 1) - \cos x + 3 = 0$$
$$10\cos^2 x - 5 - \cos x + 3 = 0$$
$$10\cos^2 x - \cos x - 2 = 0$$
$$(5\cos x + 2)(2\cos x - 1) = 0$$

For this reason, either $5\cos x + 2 = 0$ or $2\cos x - 1 = 0$ (or both). If $5\cos x + 2 = 0$, x can be solved for as follows:

$$5\cos x + 2 = 0$$
$$5\cos x + 2 - 2 = 0 - 2$$
$$5\cos x = -2$$
$$\frac{5\cos x}{5} = -\frac{2}{5}$$
$$\cos x = -\frac{2}{5}$$

The solutions to the equation $\cos x = -\frac{2}{5}$ occur at $x = 1.98$ or 4.30.

If $2\cos x - 1 = 0$, x can be solved for as follows:

$$2\cos x - 1 = 0$$
$$2\cos x - 1 + 1 = 0 + 1$$
$$2\cos x = 1$$
$$\frac{2\cos x}{2} = \frac{1}{2}$$
$$\cos x = \frac{1}{2}$$

The solutions to the equation $\cos x = \frac{1}{2}$ occur at $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$. Therefore, the solutions to the equation $5\cos 2x - \cos x + 3 = 0$ occur at $x = \frac{\pi}{3}$, 1.98, 4.30, or $\frac{5\pi}{3}$.

b) Since $\cos 2\theta = 2\cos^2\theta - 1$, the equation $10\cos 2x - 8\cos x + 1 = 0$ can be rewritten and factored as follows:

$$10\cos 2x - 8\cos x + 1 = 0$$
$$(10)(2\cos^2 x - 1) - 8\cos x + 1 = 0$$
$$20\cos^2 x - 10 - 8\cos x + 1 = 0$$
$$20\cos^2 x - 8\cos x - 9 = 0$$
$$(10\cos x - 9)(2\cos x + 1) = 0$$

For this reason, either $10\cos x - 9 = 0$ or $2\cos x + 1 = 0$ (or both). If $10\cos x - 9 = 0$, x can be solved for as follows:

$$10\cos x - 9 = 0$$
$$10\cos x - 9 + 9 = 0 + 9$$
$$10\cos x = 9$$
$$\frac{10\cos x}{10} = \frac{9}{10}$$
$$\cos x = \frac{9}{10}$$

The solutions to the equation $\cos x = \frac{9}{10}$ occur at $x = 0.45$ or 5.83. If $2\cos x + 1 = 0$, x can be solved for as follows:

$$2\cos x + 1 = 0$$
$$2\cos x + 1 - 1 = 0 - 1$$
$$2\cos x = -1$$
$$\frac{2\cos x}{2} = -\frac{1}{2}$$
$$\cos x = -\frac{1}{2}$$

The solutions to the equation $\cos x = -\frac{1}{2}$ occur at $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$. Therefore, the solutions to the equation $10\cos 2x - 8\cos x + 1 = 0$ occur at $x = 0.45, \frac{2\pi}{3}, \frac{4\pi}{3}$, or 5.83.

c) Since $\cos 2\theta = 1 - 2\sin^2\theta$, the equation $4\cos 2x + 10\sin x - 7 = 0$ can be rewritten and factored as follows:

$$4\cos 2x + 10\sin x - 7 = 0$$
$$(4)(1 - 2\sin^2 x) + 10\sin x - 7 = 0$$
$$4 - 8\sin^2 x + 10\sin x - 7 = 0$$
$$-8\sin^2 x + 10\sin x - 3 = 0$$
$$(-1)(-8\sin^2 x + 10\sin x - 3) = (-1)(0)$$
$$8\sin^2 x - 10\sin x + 3 = 0$$
$$(4\sin x - 3)(2\sin x - 1) = 0$$

For this reason, either $4\sin x - 3 = 0$ or $2\sin x - 1 = 0$ (or both). If $4\sin x - 3 = 0$, x can be solved for as follows:

$$4\sin x - 3 = 0$$
$$4\sin x - 3 + 3 = 0 + 3$$
$$4\sin x = 3$$
$$\frac{4\sin x}{4} = \frac{3}{4}$$
$$\sin x = \frac{3}{4}$$

The solutions to the equation $\sin x = \frac{3}{4}$ occur at $x = 0.85$ or 2.29. If $2 \sin x - 1 = 0$, x can be solved for as follows:

$$2 \sin x - 1 = 0$$
$$2 \sin x - 1 + 1 = 0 + 1$$
$$2 \sin x = 1$$
$$\frac{2 \sin x}{2} = \frac{1}{2}$$
$$\sin x = \frac{1}{2}$$

The solutions to the equation $\sin x = \frac{1}{2}$ occur at $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. Therefore, the solutions to the equation $4 \cos 2x + 10 \sin x - 7 = 0$ occur at $x = \frac{\pi}{6}$, 0.85, $\frac{5\pi}{6}$, or 2.29.

d) Since $\cos 2\theta = 1 - 2 \sin^2 \theta$, the equation $-2 \cos 2x = 2 \sin x$ can be rewritten and factored as follows:

$$-2 \cos 2x = 2 \sin x$$
$$(-2)(1 - 2 \sin^2 x) = 2 \sin x$$
$$-2 + 4 \sin^2 x = 2 \sin x$$
$$-2 + 4 \sin^2 x - 2 \sin x = 2 \sin x - 2 \sin x$$
$$4 \sin^2 x - 2 \sin x - 2 = 0$$
$$(2)(2 \sin x + 1)(\sin x - 1) = 0$$

For this reason, either $2 \sin x + 1 = 0$ or $\sin x - 1 = 0$ (or both). If $2 \sin x + 1 = 0$, x can be solved for as follows:

$$2 \sin x + 1 = 0$$
$$2 \sin x + 1 - 1 = 0 - 1$$
$$2 \sin x = -1$$
$$\frac{2 \sin x}{2} = -\frac{1}{2}$$
$$\sin x = -\frac{1}{2}$$

The solutions to the equation $\sin x = -\frac{1}{2}$ occur at $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$. If $\sin x - 1 = 0$, x can be solved for as follows:

$$\sin x - 1 = 0$$
$$\sin x - 1 + 1 = 0 + 1$$
$$\sin x = 1$$

The solution to the equation $\sin x = 1$ occurs at $x = \frac{\pi}{2}$. Therefore, the solutions to the equation $-2 \cos 2x = 2 \sin x$ occur at $x = \frac{\pi}{2}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$.

10. To solve the equation $8 \sin^2 x - 8 \sin x + 1 = 0$, first substitute θ for $\sin x$. The equation then becomes $8\theta^2 - 8\theta + 1$. Next, use the quadratic formula to solve for θ as follows:

$$\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(8)(1)}}{2(8)}$$
$$= \frac{8 \pm \sqrt{64 - 32}}{16}$$
$$= \frac{8 \pm \sqrt{32}}{16}$$
$$= \frac{8 \pm 4\sqrt{2}}{16}$$
$$= \frac{2 \pm \sqrt{2}}{4}$$
$$= 0.1464 \text{ or } 0.8536$$

Since $\theta = 0.1464$ or 0.8536, $\sin x = 0.1464$ or 0.8536. If $\sin x = 0.1464$, $x = 0.15$ or 2.99. If $\sin x = 0.8536$, $x = 1.02$ or 2.12. Therefore, the solutions to the equation $8 \sin^2 x - 8 \sin x + 1 = 0$ occur at $x = 0.15, 1.02, 2.12$, or 2.99.

11. Since the solutions of the quadratic trigonometric equation $\cot^2 x - b \cot x + c = 0$ are $\frac{\pi}{6}, \frac{\pi}{4}, \frac{7\pi}{6}$, and $\frac{5\pi}{4}$, the cotangent of these solutions must be found. The cotangent of both $\frac{\pi}{6}$ and $\frac{7\pi}{6}$ is $\sqrt{3}$, while the cotangent of both $\frac{\pi}{4}$ and $\frac{5\pi}{4}$ is 1. For this reason, $\cot x = \sqrt{3}$ and $\cot x = 1$, so $\cot x - \sqrt{3} = 0$ and $\cot x - 1 = 0$. If the factors $\cot x - \sqrt{3}$ and $\cot x - 1$ are multiplied together as follows, the quadratic trigonometric equation $\cot^2 x - b \cot x + c = 0$ is formed:

$$(\cot x - \sqrt{3})(\cot x - 1) = 0$$
$$\cot^2 x - \sqrt{3} \cot x - \cot x + (-1)(-\sqrt{3}) = 0$$
$$\cot^2 x - (1 + \sqrt{3}) \cot x + \sqrt{3} = 0$$

Therefore, $b = 1 + \sqrt{3}$ and $c = \sqrt{3}$.

12. It's clear from the graph that the quadratic trigonometric equation $\sin^2 x - c = 0$ has solutions at $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$, so the sine of these solutions must be found. The sine of both $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ is $\frac{\sqrt{2}}{2}$, while the sine of both $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ is $-\frac{\sqrt{2}}{2}$. For this reason, $\sin x = \frac{\sqrt{2}}{2}$ and $\sin x = -\frac{\sqrt{2}}{2}$, so $\sin x - \frac{\sqrt{2}}{2} = 0$ and $\sin x + \frac{\sqrt{2}}{2} = 0$. If the factors $\sin x - \frac{\sqrt{2}}{2}$ and $\sin x + \frac{\sqrt{2}}{2}$ are multiplied together as follows, the quadratic trigonometric equation $\sin^2 x - c = 0$ is formed:

$$\left(\sin x - \frac{\sqrt{2}}{2}\right)\left(\sin x + \frac{\sqrt{2}}{2}\right) = 0$$

$$\sin^2 x - \frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\sin x + \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = 0$$

$$\sin^2 x - \frac{2}{4} = 0$$

$$\sin^2 x - \frac{1}{2} = 0$$

Therefore, $c = \frac{1}{2}$.

13. To solve the problem, first the zeros of the function $h(d) = 4\cos^2 d - 1$ must be found as follows:

$$h(d) = 4\cos^2 d - 1$$

$$0 = 4\cos^2 d - 1$$

$$(2\cos d - 1)(2\cos d + 1) = 0$$

Since $(2\cos d - 1)(2\cos d + 1) = 0$, either $2\cos d - 1 = 0$ or $2\cos d + 1 = 0$ (or both). If $2\cos d - 1 = 0$, d can be solved for as follows:

$$2\cos d - 1 = 0$$

$$2\cos d - 1 + 1 = 0 + 1$$

$$2\cos d = 1$$

$$\frac{2\cos d}{2} = \frac{1}{2}$$

$$\cos d = \frac{1}{2}$$

The solutions to the equation $\cos d = \frac{1}{2}$ occur at $d = \frac{\pi}{3}$ or $\frac{5\pi}{3}$. If $2\cos d + 1 = 0$, d can be solved for as follows:

$$2\cos d + 1 = 0$$

$$2\cos d + 1 - 1 = 0 - 1$$

$$2\cos d = -1$$

$$\frac{2\cos d}{2} = -\frac{1}{2}$$

$$\cos d = -\frac{1}{2}$$

The solutions to the equation $\cos d = -\frac{1}{2}$ occur at $d = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$. Therefore, the zeros of the function $h(d) = 4\cos^2 d - 1$ are at $d = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$. With the zeros known, the 2π stretch of rolling hills can be broken down into the following regions: $d < \frac{\pi}{3}$; $\frac{\pi}{3} < d < \frac{2\pi}{3}$; $\frac{2\pi}{3} < d < \frac{4\pi}{3}$; $\frac{4\pi}{3} < d < \frac{5\pi}{3}$; $d > \frac{5\pi}{3}$. First, the region $d < \frac{\pi}{3}$ can be tested by finding $h\left(\frac{\pi}{6}\right)$ as follows:

$$h\left(\frac{\pi}{6}\right) = 4\cos^2\left(\frac{\pi}{6}\right) - 1$$

$$= 4\left(\frac{\sqrt{3}}{2}\right)^2 - 1$$

$$= 4\left(\frac{3}{4}\right) - 1 = 3 - 1 = 2$$

Since $h\left(\frac{\pi}{6}\right)$ is positive, the height of the rolling hills above sea level relative to Natasha's home is positive in the region $d < \frac{\pi}{3}$. Next, the region $\frac{\pi}{3} < d < \frac{2\pi}{3}$ can be tested by finding $h\left(\frac{\pi}{2}\right)$ as follows:

$$h\left(\frac{\pi}{2}\right) = 4\cos^2\left(\frac{\pi}{2}\right) - 1$$

$$= 4(0)^2 - 1 = 4(0) - 1$$

$$= 0 - 1 = -1$$

Since $h\left(\frac{\pi}{2}\right)$ is negative, the height of the rolling hills above sea level relative to Natasha's home is negative in the region $\frac{\pi}{3} < d < \frac{2\pi}{3}$. Next, the region $\frac{2\pi}{3} < d < \frac{4\pi}{3}$ can be tested by finding $h(\pi)$ as follows:

$$h(1) = 4\cos^2(\pi) - 1$$

$$= 4(-1)^2 - 1 = 4(1) - 1$$

$$= 4 - 1 = 3$$

Since $h(\pi)$ is positive, the height of the rolling hills above sea level relative to Natasha's home is positive in the region $\frac{2\pi}{3} < d < \frac{4\pi}{3}$. Next, the region $\frac{4\pi}{3} < d < \frac{5\pi}{3}$ can be tested by finding $h\left(\frac{3\pi}{2}\right)$ as follows:

$$h\left(\frac{3\pi}{2}\right) = 4\cos^2\left(\frac{3\pi}{2}\right) - 1$$

$$= 4(0)^2 - 1$$

$$= 4(0) - 1 = 0 - 1 = -1$$

Since $h\left(\frac{3\pi}{2}\right)$ is negative, the height of the rolling negative in the region $\frac{4\pi}{3} < d < \frac{5\pi}{3}$. Finally, the region $d > \frac{5\pi}{3}$ can be tested by finding $h\left(\frac{11\pi}{6}\right)$ as follows:

$$h\left(\frac{11\pi}{6}\right) = 4\cos^2\left(\frac{11\pi}{6}\right) - 1$$
$$= 4\left(\frac{\sqrt{3}}{2}\right)^2 - 1$$
$$= 4\left(\frac{3}{4}\right) - 1 = 3 - 1 = 2$$

Since $h\left(\frac{11\pi}{6}\right)$ is positive, the height of the rolling hills above sea level relative to Natasha's home is positive in the region $d > \frac{5\pi}{3}$. Therefore, the intervals at which the height of the rolling hills above sea level relative to Natasha's home is negative are $\frac{\pi}{3}$km $< d < \frac{2\pi}{3}$km and $\frac{4\pi}{3}$km $< d < \frac{5\pi}{3}$km.

14. Since $\sin^2\theta + \cos^2\theta = 1$, or $\sin^2\theta = 1 - \cos^2\theta$, the equation $6\sin^2 x = 17\cos x + 11$ can be rewritten and factored as follows:

$$6\sin^2 x = 17\cos x + 11;$$
$$6\sin^2 x - 17\cos x - 11 = 17\cos x + 11 - 17\cos x - 11;$$
$$6\sin^2 x - 17\cos x - 11 = 0;$$
$$6(1 - \cos^2 x) - 17\cos x - 11 = 0;$$
$$6 - 6\cos^2 x - 17\cos x - 11 = 0;$$
$$-6\cos^2 x - 17\cos x - 5 = 0;$$
$$(-1)(-6\cos^2 x - 17\cos x - 5) = (-1)(0);$$
$$6\cos^2 x + 17\cos x + 5 = 0;$$
$$(2\cos x + 5)(3\cos x + 1) = 0$$

For this reason, either $2\cos x + 5 = 0$ or $3\cos x + 1 = 0$. If $2\cos x + 5 = 0$, x can be solved for as follows:

$$2\cos x + 5 = 0$$
$$2\cos x + 5 - 5 = 0 - 5$$
$$2\cos x = -5$$
$$\frac{2\cos x}{2} = -\frac{5}{2}$$
$$\cos x = -\frac{5}{2}$$

Since $\cos x$ can never equal $-\frac{5}{2}$, the factor $2\cos x + 5$ can be ignored. If $3\cos x + 1 = 0$, x can be solved for as follows:

$$3\cos x + 1 = 0$$
$$3\cos x + 1 - 1 = 0 - 1$$
$$3\cos x = -1$$
$$\frac{3\cos x}{3} = -\frac{1}{3}$$
$$\cos x = -\frac{1}{3}$$

The solutions to the equation $\cos x = -\frac{1}{3}$ occur at $x = 1.91$ or 4.37. Therefore, the solutions to the equation $6\sin^2 x = 17\cos x + 11$ occur at $x = 1.91$ or 4.37.

15. a) Since $\sin^2\theta + \cos^2\theta = 1$, or $\sin^2\theta = 1 - \cos^2\theta$, the equation $\sin^2 x - \sqrt{2}\cos x = \cos^2 x + \sqrt{2}\cos x + 2$ can be rewritten and factored as follows:

$$\sin^2 x - \sqrt{2}\cos x = \cos^2 x + \sqrt{2}\cos x + 2;$$
$$1 - \cos^2 x - \sqrt{2}\cos x = \cos^2 x + \sqrt{2}\cos x + 2;$$
$$1 - \cos^2 x - \sqrt{2}\cos x - \cos^2 x - \sqrt{2}\cos x - 2$$
$$= \cos^2 x + \sqrt{2}\cos x + 2 - \cos^2 x - \sqrt{2}\cos x - 2;$$
$$1 - \cos^2 x - \sqrt{2}\cos x - \cos^2 x - \sqrt{2}\cos x - 2 = 0;$$
$$-2\cos^2 x - 2\sqrt{2}\cos x - 1 = 0;$$
$$\left(-\frac{1}{2}\right)(-2\cos^2 x - 2\sqrt{2}\cos x - 1) = 0;$$
$$\cos^2 x + \sqrt{2}\cos x + \frac{1}{2} = 0;$$
$$\left(\cos x + \frac{\sqrt{2}}{2}\right)^2 = 0$$

Since $\left(\cos x + \frac{\sqrt{2}}{2}\right)^2 = 0$, $\cos x + \frac{\sqrt{2}}{2} = 0$. For this reason, x can be solved for as follows:

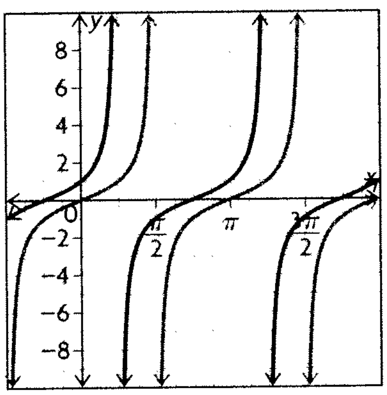
$$\cos x + \frac{\sqrt{2}}{2} = 0$$
$$\cos x + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 0 - \frac{\sqrt{2}}{2}$$
$$\cos x = -\frac{\sqrt{2}}{2}$$

The solutions to the equation $\cos x = -\frac{\sqrt{2}}{2}$ occur at $x = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$, so the solutions to the equation $\sin^2 x - \sqrt{2}\cos x = \cos^2 x + \sqrt{2}\cos x + 2$ occur at $x = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$.

b) Since the period of the cosine function is 2π, a general solution for the equation in part (a) is $x = \frac{3\pi}{4} + 2n\pi$ or $\frac{5\pi}{4} + 2n\pi$, where $n \in \mathbf{I}$.

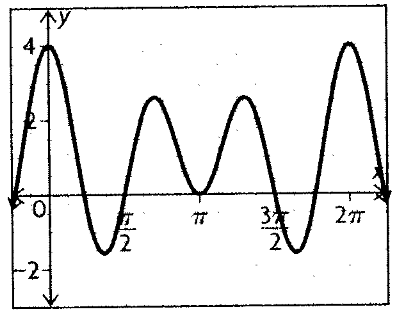
16. It is possible to have different numbers of solutions for quadratic trigonometric equations because, when factored, a quadratic trigonometric equation can be one expression multiplied by another expression or it can be a single expression squared. For example, the equation $\cos^2 x + \frac{3}{2}\cos x + \frac{1}{2}$ becomes $(\cos x + 1)(\cos x + \frac{1}{2})$ when factored, and it has the solutions $\frac{2\pi}{3}$, π, and $\frac{4\pi}{3}$ in the interval $0 \le x \le 2\pi$. In comparison, the equation $\cos^2 x + 2\cos x + 1 = 0$ becomes $(\cos x + 1)^2$ when factored, and it has only one solution, π, in the interval $0 \le x \le 2\pi$. Also, different expressions produce different numbers of solutions. For example, the expression $\cos x + \frac{1}{2}$ produces two solutions in the interval $0 \le x \le 2\pi \left(\frac{2\pi}{3}\right)$ and $\left(\frac{4\pi}{3}\right)$ because $\cos x = -\frac{1}{2}$ for two different values of x. The expression $\cos x + 1$, however, produces only one solution in the interval $0 \le x \le 2\pi (\pi)$, because $\cos x = -1$ for only one value of x.

17. To determine all the values of a such that $f(x) = \tan(x + a)$, first graph the functions $f(x) = \frac{\tan x}{1 - \tan x} - \frac{\cot x}{1 - \cot x}$ and $g(x) = \tan x$ on the same coordinate grid as follows:



It's apparent from the graphs that if the graph of the function $g(x) = \tan x$ were translated $\frac{\pi}{4}$ units to the left, it would be the same as the graph of the function $f(x) = \frac{\tan x}{1 - \tan x} - \frac{\cot x}{1 - \cot x}$. For this reason, the graph of the function $g(x) = \tan(x + \frac{\pi}{4})$ would be the same as the graph of the function $f(x) = \frac{\tan x}{1 - \tan x} - \frac{\cot x}{1 - \cot x}$. The same is true for $\frac{5\pi}{4}$. Therefore, the values of a such that $f(x) = \tan(x + a)$ are $a = \frac{\pi}{4}$ and $\frac{5\pi}{4}$.

18. To solve the equation $2\cos 3x + \cos 2x + 1 = 0$, graph the function $f(x) = 2\cos 3x + \cos 2x + 1$ on a coordinate grid as follows:



It's apparent from the graph that the x-intercepts are at $x = 0.72, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 5.56. Therefore, the solutions to the equation $2\cos 3x + \cos 2x + 1 = 0$ occur at $x = 0.72, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, or 5.56.

19. Since $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$, the equation $3\tan^2 2x = 1$ can be rewritten as follows:

$$3\tan^2 2x = 1;$$
$$3\left(\frac{2\tan x}{1 - \tan^2 x}\right)^2 = 1;$$
$$3\left(\frac{4\tan^2 x}{1 - \tan^2 x - \tan^2 x + \tan^4 x}\right) = 1;$$
$$\frac{12\tan^2 x}{1 - 2\tan^2 x + \tan^4 x} = 1;$$
$$1 - 2\tan^2 x + \tan^4 x = 12\tan^2 x;$$
$$1 - 2\tan^2 x + \tan^4 x - 12\tan^2 x = 12\tan^2 x - 12\tan^2 x;$$
$$\tan^4 x - 14\tan^2 x + 1 = 0$$

At this point the quadratic formula can be used to solve for $\tan^2 x$ as follows:

$$\tan^2 x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1)(1)}}{2(1)}$$
$$= \frac{14 \pm \sqrt{196 - 4}}{2}$$
$$= \frac{14 \pm \sqrt{192}}{2}$$
$$= \frac{14 \pm 8\sqrt{3}}{2}$$
$$= 7 \pm 4\sqrt{3}$$
$$= 0.0718 \text{ or } 13.9282$$

Since $\tan^2 x = 0.0718$ or 13.9282, $\tan x = \pm 0.2679$ or ± 3.7321. Therefore, four solutions for x are $x = 15°, 75°, 285°$, or $345°$. Also, since the value of $\tan x$ repeats itself every 180°, four more solutions for x are $x = 105°, 165°, 195°$, or $255°$.

20. To solve the equation $\sqrt{2}\sin\theta = \sqrt{3} - \cos\theta$, first square both sides of the equation as follows:

$$\sqrt{2}\sin\theta = \sqrt{3} - \cos\theta$$
$$(\sqrt{2}\sin\theta)^2 = (\sqrt{3} - \cos\theta)^2$$
$$2\sin^2\theta = 3 - \sqrt{3}\cos\theta - \sqrt{3}\cos\theta + \cos^2\theta$$
$$2\sin^2\theta = 3 - 2\sqrt{3}\cos\theta + \cos^2\theta$$

Since $\sin^2\theta + \cos^2\theta = 1$, or $\sin^2\theta = 1 - \cos^2\theta$, the equation $2\sin^2\theta = 3 - 2\sqrt{3}\cos\theta + \cos^2\theta$ can be rewritten as follows:

$$2\sin^2\theta = 3 - 2\sqrt{3}\cos\theta + \cos^2\theta;$$
$$2(1 - \cos^2\theta) = 3 - 2\sqrt{3}\cos\theta + \cos^2\theta;$$
$$2 - 2\cos^2\theta = 3 - 2\sqrt{3}\cos\theta + \cos^2\theta;$$
$$2 - 2\cos^2\theta - 2 + 2\cos^2\theta$$
$$= 3 - 2\sqrt{3}\cos\theta + \cos^2\theta - 2 + 2\cos^2\theta;$$
$$3\cos^2\theta - 2\sqrt{3}\cos\theta + 1 = 0$$

At this point the quadratic formula can be used to solve for $\cos\theta$ as follows:

$$\cos\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-2\sqrt{3}) \pm \sqrt{(-2\sqrt{3})^2 - 4(3)(1)}}{2(3)}$$
$$= \frac{2\sqrt{3} \pm \sqrt{12 - 12}}{6}$$
$$= \frac{2\sqrt{3} \pm \sqrt{0}}{6}$$
$$= \frac{2\sqrt{3} \pm 0}{6} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$$

Since $\cos\theta = \frac{\sqrt{3}}{3}$, $\theta = 0.96$

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1. a) Answers may vary. For example: Since $\sin(\pi - \theta) = \sin\theta$, $\sin\theta = \sin(\pi - \theta)$.

Therefore, $\sin\frac{3\pi}{10} = \sin\left(\pi - \frac{3\pi}{10}\right)$

$$= \sin\left(\frac{10\pi}{10} - \frac{3\pi}{10}\right) = \sin\frac{7\pi}{10}$$

b) Answers may vary. For example: Since $\cos(2\pi - \theta) = \cos\theta$, $\cos\theta = \cos(2\pi - \theta)$. Therefore,

$$\cos\frac{6\pi}{7} = \cos\left(2\pi - \frac{6\pi}{7}\right)$$
$$= \cos\left(\frac{14\pi}{7} - \frac{6\pi}{7}\right) = \cos\frac{8\pi}{7}$$

c) Answers may vary. For example: Since $-\sin\theta = \sin(\pi + \theta)$,

$$-\sin\frac{13\pi}{7} = \sin\left(\pi + \frac{13\pi}{7}\right)$$
$$= \sin\left(\frac{7\pi}{7} + \frac{13\pi}{7}\right) = \sin\frac{20\pi}{7}$$

Since $\sin\theta = \sin(\theta - 2\pi)$,

$$\sin\frac{20\pi}{7} = \sin\left(\frac{20\pi}{7} - 2\pi\right)$$
$$= \sin\left(\frac{20\pi}{7} - \frac{14\pi}{7}\right) = \sin\frac{6\pi}{7}$$

Therefore, $-\sin\frac{13\pi}{7} = \sin\frac{6\pi}{7}$.

d) Answers may vary. For example: Since $-\cos\theta = \cos(\pi + \theta)$,

$$-\cos\frac{8\pi}{7} = \cos\left(\pi + \frac{8\pi}{7}\right)$$
$$= \cos\left(\frac{7\pi}{7} + \frac{8\pi}{7}\right) = \cos\frac{15\pi}{7}$$

Since $\cos\theta = \cos(\theta - 2\pi)$,

$$\cos\frac{15\pi}{7} = \cos\left(\frac{15\pi}{7} - 2\pi\right)$$
$$= \cos\left(\frac{15\pi}{7} - \frac{14\pi}{7}\right) = \cos\frac{\pi}{7}$$

Therefore, $-\cos\frac{8\pi}{7} = \cos\frac{\pi}{7}$.

2. Since $\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$, the equation

$y = -5\sin\left(x - \frac{\pi}{2}\right) - 8$ can be rewritten

$y = -5\cos\left(x - \frac{\pi}{2} - \frac{\pi}{2}\right) - 8$

$= -5\cos(x - \pi) - 8$. Since a horizontal translation of π to the left or right is equivalent to a reflection in the x-axis, the equation $y = -5\cos(x - \pi) - 8$ can be rewritten $y = 5\cos x - 8$. Therefore, the equation $y = -5\sin x\left(-\frac{\pi}{2}\right) - 8$ can be rewritten $y = 5\cos x - 8$.

3. a) Since $\sin(a - b) = \sin a\cos b - \cos a\sin b$,

$$\sin\left(x - \frac{4\pi}{3}\right) = \sin x\cos\frac{4\pi}{3} - \cos x\sin\frac{4\pi}{3}$$
$$= (\sin x)\left(-\frac{1}{2}\right) - (\cos x)\left(-\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$$

b) Since $\cos(a+b) = \cos a \cos b - \sin a \sin b$,

$$\cos\left(x + \frac{3\pi}{4}\right) = \cos x \cos \frac{3\pi}{4} - \sin x \sin \frac{3\pi}{4}$$
$$= (\cos x)\left(-\frac{\sqrt{2}}{2}\right) - (\sin x)\left(\frac{\sqrt{2}}{2}\right)$$
$$= -\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x$$

c) Since $\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\tan\left(x + \frac{\pi}{3}\right) = \frac{\tan x + \tan\frac{\pi}{3}}{1 - \tan x \tan \frac{\pi}{3}}$$
$$= \frac{\tan x + \sqrt{3}}{1 - (\tan x)(\sqrt{3})}$$
$$= \frac{\tan x + \sqrt{3}}{1 - \sqrt{3}\tan x}$$

d) Since $\cos(a-b) = \cos a \cos b + \sin a \sin b$,

$$\cos\left(x - \frac{5\pi}{4}\right)$$
$$= (\cos x)\left(\cos\frac{5\pi}{4}\right) + (\sin x)\left(\sin\frac{5\pi}{4}\right)$$
$$= (\cos x)\left(-\frac{\sqrt{2}}{2}\right) + (\sin x)\left(-\frac{\sqrt{2}}{2}\right)$$
$$= -\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x$$

4. a) Since $\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\frac{\tan\frac{\pi}{12} + \tan\frac{7\pi}{4}}{1 - \tan\frac{\pi}{12}\tan\frac{7\pi}{4}} = \tan\left(\frac{\pi}{12} + \frac{7\pi}{4}\right)$$
$$= \tan\left(\frac{\pi}{12} + \frac{21\pi}{12}\right) = \tan\frac{22\pi}{12} = \tan\frac{11\pi}{6} = -\frac{\sqrt{3}}{3}$$

b) Since $\cos(a+b) = \cos a \cos b - \sin a \sin b$,

$$\cos\frac{\pi}{9}\cos\frac{19\pi}{18} - \sin\frac{\pi}{9}\sin\frac{19\pi}{18}$$
$$= \cos\left(\frac{\pi}{9} + \frac{19\pi}{18}\right) = \cos\left(\frac{2\pi}{18} + \frac{19\pi}{18}\right)$$
$$= \cos\frac{21\pi}{18} = \cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$$

5. a) Since $\sin 2\theta = 2\sin\theta\cos\theta$,

$$2\sin\frac{\pi}{12}\cos\frac{\pi}{12} = \sin\left((2)\left(\frac{\pi}{12}\right)\right)\sin\frac{2\pi}{12}$$
$$= \sin\frac{\pi}{6} = \frac{1}{2}$$

b) Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,

$$\cos^2\frac{\pi}{12} - \sin^2\frac{\pi}{12} = \cos\left((2)\left(\frac{\pi}{12}\right)\right)$$
$$= \cos\frac{2\pi}{12} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

c) Since $\cos 2\theta = 1 - 2\sin^2\theta$,

$$1 - 2\sin^2\frac{3\pi}{8} = \cos\left((2)\left(\frac{3\pi}{8}\right)\right)$$
$$= \cos\frac{6\pi}{8} = \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$$

d) Since $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$,

$$\frac{2\tan\frac{\pi}{6}}{1 - \tan^2\frac{\pi}{6}} = \tan 2\left(\frac{\pi}{6}\right) = \tan\frac{\pi}{3} = \sqrt{3}$$

6. a) Since $\sin x = \frac{3}{5}$, the leg opposite to the angle x in a right triangle has a length of 3, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$3^2 + y^2 = 5^2$$
$$9 + y^2 = 25$$
$$9 + y^2 - 9 = 25 - 9$$
$$y^2 = 16$$
$$y = 4, \text{ in quadrant I}$$

Since $\cos x = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos x = \dfrac{4}{5}$.

Therefore, since $\sin 2x = 2\sin x\cos x$,

$$\sin 2x = (2)\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}.$$

Also, since $\cos 2x = \cos^2 x - \sin^2 x$,

$$\cos 2x = \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}.$$

Since $\tan x = \dfrac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan x = \frac{3}{4}$.

Since $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$, $\tan 2x = \dfrac{(2)\left(\frac{3}{4}\right)}{1 - \left(\frac{3}{4}\right)^2}$

$$= \frac{\frac{3}{2}}{1 - \frac{9}{16}} = \frac{\frac{3}{2}}{\frac{16}{16} - \frac{9}{16}} = \frac{\frac{3}{2}}{\frac{7}{16}} = \frac{3}{2} \times \frac{16}{7} = \frac{24}{7}$$

b) Since $\cot x = -\frac{7}{24}$, the leg opposite the angle x in a right triangle has a length of 24, while the leg adjacent to the angle x has a length of 7. For this reason, the hypotenuse of the right triangle can be calculated as follows:

$$7^2 + 24^2 = c^2$$
$$49 + 576 = c^2$$

$625 = c^2$
$c = 25$

Since $\sin x = \frac{\text{opposite leg}}{\text{hypotenuse}}$, $\sin x = \frac{24}{25}$, and since $\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos x = -\frac{7}{25}$. (The reason the sign is negative is because angle x is in the second quadrant.) Therefore, since

$\sin 2x = 2\sin x\cos x$, $\sin 2x = (2)\left(\frac{24}{25}\right)\left(-\frac{7}{25}\right)$

$= -\frac{336}{625}$. Also, since $\cos 2x = \cos^2 x - \sin^2 x$,

$\cos 2x = \left(-\frac{7}{25}\right)^2 - \left(\frac{24}{25}\right)^2 = \frac{49}{625} - \frac{576}{625} = -\frac{527}{625}$.

Since $\cot x = -\frac{7}{24}$, $\tan x = -\frac{24}{7}$.

Since $\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$,

$$\tan 2x = \frac{(2)\left(-\frac{24}{7}\right)}{1 - \left(-\frac{24}{7}\right)^2} = \frac{-\frac{48}{7}}{1 - \frac{576}{49}} = \frac{-\frac{48}{7}}{\frac{49}{49} - \frac{576}{49}}$$

$$= \frac{-\frac{48}{7}}{-\frac{527}{49}} = -\frac{48}{7} \times \frac{49}{527} = \frac{336}{527}$$

c) Since $\cos x = \frac{12}{13}$, the leg adjacent to the angle x in a right triangle has a length of 12, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$
$$x^2 + 144 - 144 = 169 - 144$$
$$x^2 = 25$$
$$x = 5$$

Since $\sin x = \frac{\text{opposite leg}}{\text{hypotenuse}}$, $\sin x = -\frac{5}{13}$. (Sine is negative because x is in the fourth quadrant.) Therefore, since $\sin 2x = 2\sin x\cos x$,

$\sin 2x = (2)\left(-\frac{5}{13}\right)\left(\frac{12}{13}\right) = -\frac{120}{169}$. Also, since $\cos 2x = \cos^2 x - \sin^2 x$,

$$\cos 2x = \left(\frac{12}{13}\right)^2 - \left(-\frac{5}{13}\right)^2$$
$$= \frac{144}{169} - \frac{25}{169} = \frac{119}{169}$$

Finally, since $\tan x = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan x = -\frac{5}{12}$. (The reason the sign is negative is because angle x is in the fourth quadrant.)

Since $\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$,

$$\tan 2x = \frac{(2)\left(-\frac{5}{12}\right)}{1 - \left(-\frac{5}{12}\right)^2}$$
$$= \frac{-\frac{5}{6}}{1 - \frac{25}{144}} = \frac{-\frac{5}{6}}{\frac{144}{144} - \frac{25}{144}} = \frac{-\frac{5}{6}}{\frac{119}{144}}$$
$$= -\frac{5}{6} \times \frac{144}{119} = -\frac{120}{119}$$

7. a) Since $\sin 2\theta = 2\sin\theta\cos\theta$, and since $\cos 2\theta = 1 - 2\sin^2\theta$, $\tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{2\sin x\cos x}{1 - 2\sin^2 x}$. Therefore, $\tan 2x = \frac{2\sin x\cos x}{1 - 2\sin^2 x}$ is a trigonometric identity.

b) Since $1 + \tan^2 x = \sec^2 x$, $\sec^2 x - \tan^2 x = 1$. Therefore, since $\sec^2 x - \tan^2 x = 1$ is a trigonometric identity, $\sec^2 x - \tan^2 x = \cos x$ must be a trigonometric equation, because $\cos x$ does not always equal 1.

c) Since $1 + \cot^2 x = \csc^2 x$, $\csc^2 x - \cot^2 x = 1$. Therefore, since $\sin^2 x + \cos^2 x = 1$, $\csc^2 x - \cot^2 x = \sin^2 x + \cos^2 x$ is a trigonometric identity.

d) Since $\tan^2 x = 1$, $\tan x = \pm 1$. Therefore, since $\tan x$ does not always equal -1 or 1, $\tan^2 x = 1$ must be a trigonometric equation.

8. The trigonometric identity $\frac{1 - \sin^2 x}{\cot^2 x} = 1 - \cos^2 x$ can be proven as follows:

$$\frac{1 - \sin^2 x}{\cot^2 x} = 1 - \cos^2 x$$
$$\frac{\cos^2 x}{\cot^2 x} = 1 - \cos^2 x$$
$$\frac{\cos^2 x}{\frac{\cos^2 x}{\sin^2 x}} = 1 - \cos^2 x$$
$$\frac{(\cos^2 x)(\sin^2 x)}{\cos^2 x} = 1 - \cos^2 x$$
$$\sin^2 x = 1 - \cos^2 x$$
$$1 - \cos^2 x = 1 - \cos^2 x$$

9. The trigonometric identity $\frac{2\sec^2 x - 2\tan^2 x}{\csc x} = \sin 2x\sec x$ can be proven as follows:

$$\frac{2\sec^2 x - 2\tan^2 x}{\csc x} = \sin 2x\sec x$$
$$\frac{2(\sec^2 x - \tan^2 x)}{\csc x} = \sin 2x\sec x$$

$$\frac{2(1)}{\csc x} = \sin 2x \sec x$$
$$\frac{2}{\csc x} = \sin 2x \sec x$$
$$2 \sin x = \sin 2x \sec x$$
$$\frac{2 \sin x \cos x}{\cos x} = \sin 2x \sec x$$
$$\frac{\sin 2x}{\cos x} = \sin 2x \sec x$$
$$\sin 2x \sec x = \sin 2x \sec x$$

10. a) The trigonometric equation $\frac{2}{\sin x} + 10 = 6$ can be solved as follows:

$$\frac{2}{\sin x} + 10 = 6$$
$$\frac{2}{\sin x} + 10 - 10 = 6 - 10$$
$$\frac{2}{\sin x} = -4$$
$$-4 \sin x = 2$$
$$\frac{-4 \sin x}{-4} = \frac{2}{-4}$$
$$\sin x = -\frac{1}{2}$$

The solutions to the equation $\sin x = -\frac{1}{2}$ occur at $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$.

b) The trigonometric equation $-\frac{5 \cot x}{2} + \frac{7}{3} = -\frac{1}{6}$ can be solved as follows:

$$-\frac{5 \cot x}{2} + \frac{7}{3} = -\frac{1}{6}$$
$$-\frac{15 \cot x}{6} + \frac{14}{6} = -\frac{1}{6}$$
$$-15 \cot x + 14 = -1$$
$$-15 \cot x + 14 - 14 = -1 - 14$$
$$-15 \cot x = -15$$
$$\frac{-15 \cot x}{-15} = \frac{-15}{-15}$$
$$\cot x = 1$$

The solutions to the equation $\cot x = 1$ occur at $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

c) The trigonometric equation $3 + 10 \sec x - 1 = -18$ can be solved as follows:

$$3 + 10 \sec x - 1 = -18$$
$$2 + 10 \sec x = -18$$
$$2 + 10 \sec x - 2 = -18 - 2$$
$$10 \sec x = -20$$
$$\frac{10 \sec x}{10} = -\frac{20}{10}$$
$$\sec x = -2$$

The solutions to the equation $\sec x = -2$ occur at $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

11. a) The equation $y^2 - 4 = 0$ can be solved as follows:

$$y^2 - 4 = 0$$
$$(y - 2)(y + 2) = 0$$

Since $(y - 2)(y + 2) = 0$, either $y - 2 = 0$ or $y + 2 = 0$ (or both). If $y - 2 = 0$, y can be solved for as follows:

$$y - 2 = 0$$
$$y - 2 + 2 = 0 + 2$$
$$y = 2$$

If $y + 2 = 0$, y can be solved for as follows:

$$y + 2 = 0$$
$$y + 2 - 2 = 0 - 2$$
$$y = -2$$

Therefore, the solutions to the equation $y^2 - 4 = 0$ are $y = 2$ or $y = -2$.

b) The equation $\csc^2 x - 4 = 0$ can be solved as follows:

$$\csc^2 x - 4 = 0$$
$$(\csc x - 2)(\csc x + 2) = 0$$

Since $(\csc x - 2)(\csc x + 2) = 0$, either $\csc x - 2 = 0$ or $\csc x + 2 = 0$ (or both). If $\csc x - 2 = 0$, x can be solved for as follows:

$$\csc x - 2 = 0$$
$$\csc x - 2 + 2 = 0 + 2$$
$$\csc x = 2$$

The solutions to the equation $\csc x = 2$ occur at $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. If $\csc x + 2 = 0$, x can be solved for as follows:

$$\csc x + 2 = 0$$
$$\csc x + 2 - 2 = 0 - 2$$
$$\csc x = -2$$

The solutions to the equation $\csc x = -2$ occur at $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$. Therefore, the solutions to the equation $\csc^2 x - 4 = 0$ occur at $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$.

12. a) The equation $2 \sin^2 x - \sin x - 1 = 0$ can be solved as follows:

$$2 \sin^2 x - \sin x - 1 = 0$$
$$(2 \sin x + 1)(\sin x - 1) = 0$$

Since $(2\sin x + 1)(\sin x - 1) = 0$, either $2\sin x + 1 = 0$ or $\sin x - 1 = 0$ (or both). If $2\sin x + 1 = 0$, x can be solved for as follows:

$$2\sin x + 1 = 0$$
$$2\sin x + 1 - 1 = 0 - 1$$
$$2\sin x = -1$$
$$\frac{2\sin x}{2} = -\frac{1}{2}$$
$$\sin x = -\frac{1}{2}$$

The solutions to the equation $\sin x = -\frac{1}{2}$ occur at $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$. If $\sin x - 1 = 0$, x can be solved for as follows:

$$\sin x - 1 = 0$$
$$\sin x - 1 + 1 = 0 + 1$$
$$\sin x = 1$$

The solution to the equation $\sin x = 1$ occurs at $x = \frac{\pi}{2}$. Therefore, the solutions to the equation $2\sin^2 x - \sin x - 1 = 0$ occur at $x = \frac{\pi}{2}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$.

b) The equation $\tan^2 x \sin x - \frac{\sin x}{3} = 0$ can be solved as follows:

$$\tan^2 x \sin x - \frac{\sin x}{3} = 0$$
$$(\sin x)\left(\tan^2 x - \frac{1}{3}\right) = 0$$
$$(\sin x)\left(\tan x - \frac{\sqrt{3}}{3}\right)\left(\tan x + \frac{\sqrt{3}}{3}\right) = 0$$

Since $(\sin x)\left(\tan x - \frac{\sqrt{3}}{3}\right)\left(\tan x + \frac{\sqrt{3}}{3}\right) = 0$, either $\sin x = 0$, $\tan x - \frac{\sqrt{3}}{3} = 0$, or $\tan x + \frac{\sqrt{3}}{3} = 0$. The solutions to the equation $\sin x = 0$ occur at $x = 0$, π, or 2π. If $\tan x - \frac{\sqrt{3}}{3} = 0$, x can be solved for as follows:

$$\tan x - \frac{\sqrt{3}}{3} = 0$$
$$\tan x - \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3} = 0 + \frac{\sqrt{3}}{3}$$
$$\tan x = \frac{\sqrt{3}}{3}$$

The solutions to the equation $\tan x = \frac{\sqrt{3}}{3}$ occur at $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$. If $\tan x + \frac{\sqrt{3}}{3} = 0$, x can be solved for as follows:

$$\tan x + \frac{\sqrt{3}}{3} = 0$$
$$\tan x + \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3} = 0 - \frac{\sqrt{3}}{3}$$
$$\tan x = -\frac{\sqrt{3}}{3}$$

The solutions to the equation $\tan x = -\frac{\sqrt{3}}{3}$ occur at $x = \frac{5\pi}{6}$ or $\frac{11\pi}{6}$. Therefore, the solutions to the equation $\tan^2 x \sin x - \frac{\sin x}{3} = 0$ occur at $x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$, or 2π.

c) The equation

$$\cos^2 x + \left(\frac{1 - \sqrt{2}}{2}\right)\cos x - \frac{\sqrt{2}}{4} = 0$$

can be solved as follows:

$$\cos^2 x + \left(\frac{1 - \sqrt{2}}{2}\right)\cos x - \frac{\sqrt{2}}{4} = 0$$
$$\left(\cos x - \frac{\sqrt{2}}{2}\right)\left(\cos x + \frac{1}{2}\right) = 0$$

Since $\left(\cos x - \frac{\sqrt{2}}{2}\right)\left(\cos x + \frac{1}{2}\right) = 0$, either $\cos x - \frac{\sqrt{2}}{2} = 0$ or $\cos x + \frac{1}{2} = 0$ (or both). If $\cos x - \frac{\sqrt{2}}{2} = 0$, x can be solved for as follows:

$$\cos x - \frac{\sqrt{2}}{2} = 0$$
$$\cos x - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0 + \frac{\sqrt{2}}{2}$$
$$\cos x = \frac{\sqrt{2}}{2}$$

The solutions to the equation $\cos x = \frac{\sqrt{2}}{2}$ occur at $x = \frac{\pi}{4}$ or $\frac{7\pi}{4}$. If $\cos x + \frac{1}{2} = 0$, x can be solved for as follows:

$$\cos x + \frac{1}{2} = 0$$
$$\cos x + \frac{1}{2} - \frac{1}{2} = 0 - \frac{1}{2}$$
$$\cos x = -\frac{1}{2}$$

The solutions to the equation $\cos x = -\frac{1}{2}$ occur at $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$. Therefore, the solutions to the equation $\cos^2 x + \left(\frac{1 - \sqrt{2}}{2}\right)\cos x - \frac{\sqrt{2}}{4} = 0$ occur at $x = \frac{\pi}{4}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{7\pi}{4}$.

d) The equation $25\tan^2 x - 70\tan x = -49$ can be solved as follows:

$$25\tan^2 x - 70\tan x = -49$$
$$25\tan^2 x - 70\tan x + 49 = -49 + 49$$
$$25\tan^2 x - 70\tan x + 49 = 0$$
$$(5\tan x - 7)^2 = 0$$

Since $(5\tan x - 7)^2 = 0$, $5\tan x - 7 = 0$. For this reason, x can be solved for as follows:

$$5\tan x - 7 = 0$$
$$5\tan x - 7 + 7 = 0 + 7$$
$$5\tan x = 7$$
$$\frac{5\tan x}{5} = \frac{7}{5}$$
$$\tan x = \frac{7}{5}$$

The solutions to the equation $\tan x = \frac{7}{5}$ occur at $x = 0.95$ and 4.09. Therefore, the solutions to the equation $25\tan^2 x - 70\tan x = -49$ occur at $x = 0.95$ or 4.09.

13. Since $1 + \tan^2 x = \sec^2 x$, the equation $\frac{1}{1 + \tan^2 x} = -\cos x$ can be rewritten and factored as follows:

$$\frac{1}{1 + \tan^2 x} = -\cos x$$
$$\frac{1}{\sec^2 x} = -\cos x$$
$$\cos^2 x = -\cos x$$
$$\cos^2 x + \cos x = -\cos x + \cos x$$
$$\cos^2 x + \cos x = 0$$
$$(\cos x)(\cos x + 1) = 0$$

Since $(\cos x)(\cos x + 1) = 0$, either $\cos x = 0$ or $\cos x + 1 = 0$ (or both). The solutions to the equation $\cos x = 0$ occur at $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. If $\cos x + 1 = 0$, x can be solved for as follows:

$$\cos x + 1 = 0$$
$$\cos x + 1 - 1 = 0 - 1$$
$$\cos x = -1$$

The solution to the equation $\cos x = -1$ occurs at $x = \pi$. Therefore, the solutions to the equation $\frac{1}{1 + \tan^2 x} = -\cos x$ occur at $x = \frac{\pi}{2}, \pi$, or $\frac{3\pi}{2}$.

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1. The identity

$$\frac{1 - 2\sin^2 x}{\cos x + \sin x} + 2\sin\frac{x}{2}\cos\frac{x}{2} = \cos x$$

can be proven as follows:

$$\frac{1 - 2\sin^2 x}{\cos x + \sin x} + 2\sin\frac{x}{2}\cos\frac{x}{2} = \cos x$$
$$\frac{1 - 2\sin^2 x}{\cos x + \sin x} + \sin x = \cos x$$
$$\frac{1 - 2\sin^2 x}{\cos x + \sin x} + \sin x - \sin x = \cos x - \sin x$$
$$\frac{1 - 2\sin^2 x}{\cos x + \sin x} = \cos x - \sin x$$
$$1 - 2\sin^2 x = (\cos x - \sin x) \times (\cos x + \sin x)$$
$$\cos 2x = (\cos x - \sin x) \times (\cos x + \sin x)$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\cos 2x = \cos 2x$$

2. Since $\cos 2\theta = 1 - 2\sin^2\theta$, the equation $\cos 2x + 2\sin^2 x - 3 = -2$ can be rewritten and solved as follows:

$$\cos 2x + 2\sin^2 x - 3 = -2$$
$$1 - 2\sin^2 x + 2\sin^2 x - 3 = -2$$
$$-2 = -2$$

Since -2 always equals -2, the equation $\cos 2x + 2\sin^2 x - 3 = -2$ is an identity and is true for all real numbers x, where $0 \le x \le 2\pi$.

3. a) The solutions to the equation $\cos x = \frac{\sqrt{3}}{2}$ occur at $x = \frac{\pi}{6}$ or $\frac{11\pi}{6}$.

b) The solutions to the equation $\tan x = -\sqrt{3}$ occur at $x = \frac{2\pi}{3}$ or $\frac{5\pi}{3}$.

c) The solutions to the equation $\sin x = -\frac{\sqrt{2}}{2}$ occur at $x = \frac{5\pi}{4}$ or $\frac{7\pi}{4}$.

4. Since the quadratic trigonometric equation $a\cos^2 x + b\cos x - 1 = 0$ has the solutions $\frac{\pi}{3}$, π, and $\frac{5\pi}{3}$, the left side of the equation must have factors of $\cos x - \frac{1}{2}$ and $\cos x + 1$. This is because the cosine of $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ is $\frac{1}{2}$, and the cosine of π is -1. For this reason, the quadratic trigonometric equation can be found as follows:

$$\left(\cos x - \frac{1}{2}\right)(\cos x + 1) = 0$$
$$\cos^2 x - \frac{1}{2}\cos x + \cos x - \frac{1}{2} = 0$$
$$\cos^2 x + \frac{1}{2}\cos x - \frac{1}{2} = 0$$

$$(2)\left(\cos^2 x + \frac{1}{2}\cos x - \frac{1}{2}\right) = (2)(0)$$
$$2\cos^2 x + \cos x - 1 = 0$$
Therefore, $a = 2$ and $b = 1$.

5. Since the depth of the ocean in metres can be modelled by the function $d(t) = 4 + 2\sin\left(\frac{\pi}{6}t\right)$, when the depth is 3 metres, $3 = 4 + 2\sin\left(\frac{\pi}{6}t\right)$.

The equation $3 = 4 + 2\sin\left(\frac{\pi}{6}t\right)$ can be solved as follows:

$$3 = 4 + 2\sin\left(\frac{\pi}{6}t\right)$$
$$3 - 4 = 4 + 2\sin\left(\frac{\pi}{6}t\right) - 4$$
$$-1 = 2\sin\left(\frac{\pi}{6}t\right)$$
$$-\frac{1}{2} = \frac{2\sin\left(\frac{\pi}{6}t\right)}{2}$$
$$\sin\left(\frac{\pi}{6}t\right) = -\frac{1}{2}$$
$$\sin^{-1}\left(\sin\left(\frac{\pi}{6}t\right)\right) = \sin^{-1}\left(-\frac{1}{2}\right)$$
$$\frac{\pi}{6}t = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$$

If $\frac{\pi}{6}t = \frac{7\pi}{6}$, t can be solved for as follows:
$$\frac{\pi}{6}t = \frac{7\pi}{6}$$
$$\left(\frac{6}{\pi}\right)\left(\frac{\pi}{6}t\right) = \left(\frac{6}{\pi}\right)\left(\frac{7\pi}{6}\right)$$
$$t = 7$$

If $\frac{\pi}{6}t = \frac{11\pi}{6}$, t can be solved for as follows:
$$\frac{\pi}{6}t = \frac{11\pi}{6}$$
$$\left(\frac{6}{\pi}\right)\left(\frac{\pi}{6}t\right) = \left(\frac{6}{\pi}\right)\left(\frac{11\pi}{6}\right)$$
$$t = 11$$

Therefore, two times when the depth of the water is 3 metres are $t = 7$ h or 11 h. Also, since the period of the function $d(t) = 4 + 2\sin\left(\frac{\pi}{6}t\right)$ is $\frac{2\pi}{\frac{\pi}{6}} = (2\pi)\left(\frac{6}{\pi}\right) = 12$ h, the depth of the water is also at 3 metres at $t = 7 + 12 = 19$ h or at $t = 11 + 12 = 23$ h.

6. Nina can find the cosine of $\frac{11\pi}{4}$ by using the formula $\cos(x + y) = \cos x \cos y - \sin x \sin y$. The cosine of π is -1, and the cosine of $\frac{7\pi}{4}$ is $\frac{\sqrt{2}}{2}$. Also, the sine of π is 0, and the sine of $\frac{7\pi}{4}$ is $-\frac{\sqrt{2}}{2}$. Therefore, $\cos\frac{11\pi}{4} = \cos\left(\pi + \frac{7\pi}{4}\right)$
$$= \left(-1 \times \frac{\sqrt{2}}{2}\right) - \left(0 \times -\frac{\sqrt{2}}{2}\right)$$
$$= -\frac{\sqrt{2}}{2} - 0 = -\frac{\sqrt{2}}{2}$$

7. The equation $3\sin x + 2 = 1.5$ can be solved as follows:
$$3\sin x + 2 = 1.5$$
$$3\sin x + 2 - 2 = 1.5 - 2$$
$$3\sin x = -0.5$$
$$\frac{3\sin x}{3} = -\frac{0.5}{3}$$
$$\sin x = -0.1667$$
The solutions to the equation $\sin x = -0.1667$ occur at $x = 3.31$ or 6.12.

8. Since $\tan\alpha = 0.75$, the leg opposite the angle α in a right triangle has a length of 3, while the leg adjacent to angle α has a length of 4. For this reason, the hypotenuse of the right triangle can be calculated as follows:
$$3^2 + 4^2 = z^2$$
$$9 + 16 = z^2$$
$$25 = z^2$$
$$z = 5$$
Since $\cos\alpha = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos\alpha = \frac{4}{5}$. Also, since $\sin\alpha = \frac{\text{opposite leg}}{\text{hypotenuse}}$, $\sin\alpha = \frac{3}{5}$. In addition, since $\tan\beta = 2.4$, the leg opposite the angle β in a right triangle has a length of 12, while the leg adjacent to angle β has a length of 5. For this reason, the hypotenuse of the right triangle can be calculated as follows:
$$12^2 + 5^2 = z^2$$
$$144 + 25 = z^2$$
$$169 = z^2$$
$$z = 13$$
Since $\cos\beta = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos\beta = \frac{5}{13}$. Also, since $\sin\beta = \frac{\text{opposite leg}}{\text{hypotenuse}}$, $\sin\beta = \frac{12}{13}$. Therefore, since

$\sin(a - b) = \sin a \cos b - \cos a \sin b$,
$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$
$= \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) = \frac{15}{65} - \frac{48}{65} = -\frac{33}{65}$
Also, since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,
$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$
$= \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) = \frac{20}{65} - \frac{36}{65} = -\frac{16}{65}$

9. a) Since $\sin^2 x = \frac{4}{9}$, and since the angle x is in the second quadrant, $\sin x = \sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$.
Since $\sin x = \frac{2}{3}$, the leg opposite the angle x in a right triangle has a length of 2, while the hypotenuse of the right triangle has a length of 3. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + (2)^2 = 3^2$$
$$x^2 + 4 = 9$$
$$x^2 + 4 - 4 = 9 - 4$$
$$x^2 = 5$$
$$x = \sqrt{5}$$

Since $\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, and since the angle x is in the second quadrant, $\cos x = -\frac{\sqrt{5}}{3}$. Since $\sin 2\theta = 2 \sin \theta \cos \theta$,
$\sin 2x = 2 \sin x \cos x$
$= (2)\left(\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right)$
$= -\frac{4\sqrt{5}}{9}$.

b) Since $\sin^2 x = \frac{4}{9}$, and since the angle x is in the second quadrant, $\sin x = \sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$. Since $\sin x = \frac{2}{3}$, the leg opposite the angle x in a right triangle has a length of 2, while the hypotenuse of the right triangle has a length of 3. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + (2)^2 = 3^2$$
$$x^2 + 4 = 9$$
$$x^2 + 4 - 4 = 9 - 4$$
$$x^2 = 5$$
$$x = \sqrt{5}$$

Since $\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, and since the angle x is in the second quadrant, $\cos x = -\frac{\sqrt{5}}{3}$. Since $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$,
$\cos 2x = \cos^2 x - \sin^2 x$
$= \left(-\frac{\sqrt{5}}{3}\right)^2 - \left(\frac{2}{3}\right)^2$
$= \frac{5}{9} - \frac{4}{9} = \frac{1}{9}$
(The formulas $\cos 2\theta = 2\cos^2 \theta - 1$ and $\cos 2\theta = 1 - 2\sin^2 \theta$ could also have been used.)

c) First note that because x is in the second quadrant, $\frac{x}{2}$ is in the first quadrant, where the cosine is positive. Since $\sin^2 x = \frac{4}{9}$, and since the angle x is in the second quadrant, $\sin x = \sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$.
Since $\sin x = \frac{2}{3}$, the leg opposite the angle x in a right triangle has a length of 2, while the hypotenuse of the right triangle has a length of 3. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + (2)^2 = 3^2$$
$$x^2 + 4 = 9$$
$$x^2 + 4 - 4 = 9 - 4$$
$$x^2 = 5$$
$$x = \sqrt{5}$$

Since $\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, and since the angle x is in the second quadrant, $\cos x = -\frac{\sqrt{5}}{3}$. Since $\cos 2\theta = 2\cos^2 \theta - 1$, $\cos \theta = 2\cos^2 \frac{\theta}{2} - 1$, and since $\cos x = -\frac{\sqrt{5}}{3}$, $-\frac{\sqrt{5}}{3} = 2\cos^2 \frac{x}{2} - 1$. The value of $\cos \frac{x}{2}$ can now be determined as follows:

$$-\frac{\sqrt{5}}{3} = 2\cos^2 \frac{x}{2} - 1$$
$$-\frac{\sqrt{5}}{3} + 1 = 2\cos^2 \frac{x}{2} - 1 + 1$$
$$\frac{3 - \sqrt{5}}{3} = 2\cos^2 \frac{x}{2}$$
$$\frac{\frac{(3 - \sqrt{5})}{3}}{2} = \frac{2\cos^2 \frac{x}{2}}{2}$$
$$\cos^2 \frac{x}{2} = \frac{(3 - \sqrt{5})}{6}$$
$$\cos \frac{x}{2} = \sqrt{\left[\frac{(3 - \sqrt{5})}{6}\right]}$$

d) Since $\sin^2 x = \frac{4}{9}$, and since the angle x is in the second quadrant, $\sin x = \sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$.
Since $\sin x = \frac{2}{3}$, the leg opposite the angle x in a right triangle has a length of 2, while the hypotenuse of the right triangle has a length of 3.

For this reason, the other leg of the right triangle can be calculated as follows:

$x^2 + (2)^2 = 3^2$

$x^2 + 4 = 9$

$x^2 + 4 - 4 = 9 - 4$

$x^2 = 5$

$x = \sqrt{5}$

Since $\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, and since the angle x is in the second quadrant, $\cos x = -\frac{\sqrt{5}}{3}$. Since $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$,

$\sin 3x = 3\cos^2 x \sin x - \sin^3 x$

$= (3)\left(-\frac{\sqrt{5}}{3}\right)^2\left(\frac{2}{3}\right) - \left(\frac{2}{3}\right)^3$

$= (3)\left(\frac{5}{9}\right)\left(\frac{2}{3}\right) - \frac{8}{27}$

$= \frac{30}{27} - \frac{8}{27} = \frac{22}{27}$

10. a) The equation $2 - 14\cos x = -5$ can be solved as follows:

$2 - 14\cos x = -5$

$2 - 14\cos x - 2 = -5 - 2$

$-14\cos x = -7$

$\frac{-14\cos x}{-14} = \frac{-7}{-14}$

$\cos x = \frac{1}{2}$

From the graph, the solutions to the equation $\cos x = \frac{1}{2}$ occur at $x = -\frac{5\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}$, or $\frac{5\pi}{3}$.

b) The equation $9 - 22\cos x - 1 = 19$ can be solved as follows:

$9 - 22\cos x - 1 = 19$

$8 - 22\cos x = 19$

$8 - 22\cos x - 8 = 19 - 8$

$-22\cos x = 11$

$\frac{-22\cos x}{-22} = \frac{11}{-22}$

$\cos x = -\frac{1}{2}$

From the graph, the solutions to the equation $\cos x = -\frac{1}{2}$ occur at $x = -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}$, or $\frac{4\pi}{3}$.

c) The equation $2 + 7.5\cos x = -5.5$ can be solved as follows:

$2 + 7.5\cos x = -5.5$

$2 + 7.5\cos x - 2 = -5.5 - 2$

$7.5\cos x = -7.5$

$\frac{7.5\cos x}{7.5} = \frac{-7.5}{7.5}$

$\cos x = -1$

From the graph, the solutions to the equation $\cos x = -1$ occur at $x = -\pi$ and π.